Cluster synchronization of diffusively-coupled nonlinear systems: A contraction based approach

Zahra Aminzare

Joint work with B. Dey, E. N. Davison, N. E. Leonard

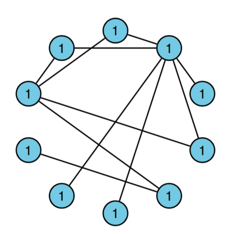
PRINCETON UNIVERSITY

Summary

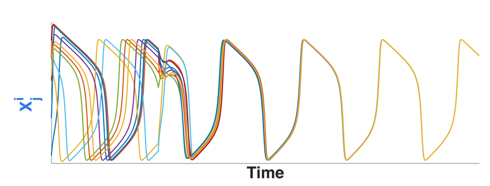
- Synchronized activity is crucial for brain function:
  - Occurs at multiple levels (neuronal or regional)
  - Related to many pathological conditions (Parkinson's disease & epilepsy)
- In realistic networks, complex patterns of synchronization emerge:
  - (Complete) synchronization
  - Cluster synchronization (e.g. phase lock)
- Finding the conditions that foster synchronization is critical to understanding biological systems. Useful tools:
  - Contraction theory & Algebraic graph theory

Synchronization definitions

- **Synchronization:** N coupled systems, $\mathbf{x}^1, \dots, \mathbf{x}^N \in \mathbb{R}^n$, converge to a forward-invariant manifold, called *synchronization manifold*:

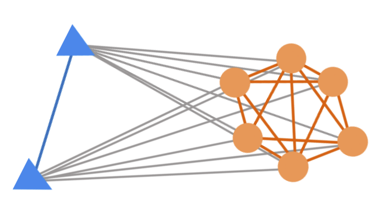


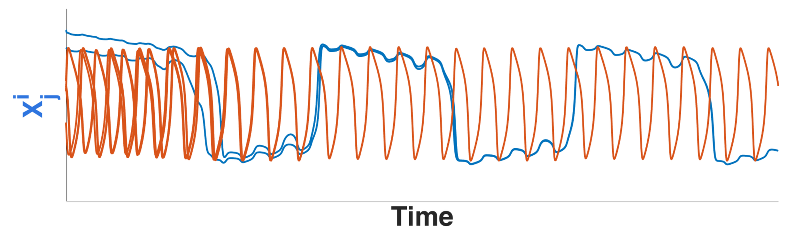
$$\mathscr{S}_1 := \left\{ \mathbf{x} \in \mathbb{R}^{nN} \,\middle|\, \mathbf{x}^1 = \dots = \mathbf{x}^N,\ \mathbf{x}^i \in \mathbb{R}^n \right\}$$



- **Cluster synchronization:** N coupled systems, $\mathbf{x}^1, \dots, \mathbf{x}^N \in \mathbb{R}^n$, partitioned in $K \le N$ groups, converge to a forward-invariant manifold, called *cluster synchronization manifold*:

$$\mathscr{S}_K := \left\{ \mathbf{x} \in \mathbb{R}^{nN} \,\middle|\, \underbrace{\mathbf{x}^1 = \dots = \mathbf{x}^{c_1}}_{\text{cluster 1}}, \dots, \underbrace{\mathbf{x}^{N-c_K+1} = \dots = \mathbf{x}^N}_{\text{cluster K}},\ \mathbf{x}^i \in \mathbb{R}^n \right\}$$





Network model

- Dynamics of a network of N nodes, $\mathbf{x}^i \in \mathbb{R}^n$, $i = 1, \dots, N$:

$$\dot{\mathbf{x}}^i(t) = \mathbf{f}^i\left(\mathbf{x}^i(t), t\right) + \sum_{j \in \mathcal{N}_i} \gamma^{ij} D \left(\mathbf{x}^j(t) - \mathbf{x}^i(t)\right) \qquad \mathcal{N}_i : \text{neighbors of } \mathbf{x}^i$$

- Nonlinear intrinsic dynamics of each node: $\mathbf{f}^i$
- Linear diffusive coupling: Diffusion matrix $D \in \mathbb{R}^{n \times n}$ is diagonal and indicates which dimensions of the dynamics are coupled.
- Laplacian matrix with eigenvalues $0 = \lambda^{(1)} \le \lambda^{(2)} \le \dots \le \lambda^{(N)}$:

$$\mathcal{L}_{ij} = \begin{cases} \sum_{k \in \mathcal{N}_i} \gamma^{ik} & i = j \\ -\gamma^{ij} & i \ne j, j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$$

References

1. Lewis (1949) & Dahlquist, Lozinskii (1958) & Hartman (1961) & Yoshizawa (1966) & Desoer, Haneda (1972), Deimling (1985) & Slotine, Lohmiller (1998)
2. Arcak. (2011), Theorem 4 (modified)
3. Aminzare, Sontag (2014)
4. Sorrentino, Ott (2007)
5. Stewart, Golubitsky, Pivato (2003)
6. Belykh, Osipov, Petrov, Suykens, Vandewalle (2008)

Contraction theory [1]

- **Contractive system:** $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is *contractive*, if any two trajectories $\mathbf{u}$ & $\mathbf{v}$ converge to each other exponentially:

$$\|\mathbf{u}(t) - \mathbf{v}(t)\| \le e^{\mu t} \|\mathbf{u}(0) - \mathbf{v}(0)\|, \ \mu < 0$$

- **Matrix measures (Logarithmic norms)**
$(X, \|\cdot\|_X)$: a finite-dim normed vector space over $\mathbb{R}$ or $\mathbb{C}$
$\mathcal{L}(X, X)$: normed space of linear transformations $A \colon X \to X$ with the induced operator norm

$$\|A\|_{X \to X} = \sup_{x \ne 0} \frac{\|Ax\|_X}{\|x\|_X}$$

matrix measure $\mu_X[\cdot]$ induced by $\|\cdot\|_X$ is defined as the directional derivative of the matrix norm:

$$\mu_X[A] = \lim_{h \to 0^+} \frac{1}{h} \left(\|I + hA\|_{X \to X} - 1\right)$$

- **A sufficient condition for contractivity** If $\exists \, \|\cdot\|_X$ s.t. $\sup_{\mathbf{x}} \mu_X[J_{\mathbf{f}}(\mathbf{x})] < 0$, then $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is contractive. ($J_{\mathbf{f}}$: Jacobian)

Sufficient conditions for synchronization[2] with homogenous $\mathbf{f}^i = \mathbf{f}$

- Q: a positive definite matrix s.t. $Q^2 D + DQ^2$ is positive definite.
- $\|x\|_{2,Q} := \|Qx\|_2$: a Q weighted L^2 norm
- $\mu_{2,Q}[A]$: matrix measure induced by $\|\cdot\|_{2,Q}$
- $\mu := \sup_{(x,t)} \mu_{2,Q}\left[J_{\mathbf{f}}(x,t) - \lambda^{(2)} D\right]$

Then for any solution $\mathbf{x}$, $\exists$ a solution $\bar{\mathbf{x}}$ such that

$$\|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\|_{2, I_N \otimes Q^2} \le e^{\mu t} \|\mathbf{x}(0) - \bar{\mathbf{x}}(0)\|_{2, I_N \otimes Q^2}$$

$\otimes$: Kronecker product

For $\mu < 0$, the dynamics synchronize $\mathbf{x}^i(t) - \mathbf{x}^j(t) \to 0 \quad$ as $t \to \infty$.

For some graphs (complete, star, linear), this result is true for any L^p norms[3].

Cluster-input-equivalence condition & invariant cluster synchronization manifold

- Cluster synchronization arises in heterogeneous networks [4].
  1. $K \le N$ groups $\mathscr{C}_1, \dots, \mathscr{C}_K$ with identical intrinsic dynamics: $\mathbf{f}_{\mathscr{C}_r}$.
  2. **Cluster-input-equivalence (CIE):**[5,6] for any $x^r_{\mathscr{C}_i}, x^s_{\mathscr{C}_i}$ in $\mathscr{C}_i$,

$$\sum_{k \in \mathcal{N}_r^{\mathscr{C}_j}} \gamma^{rk} = \sum_{k \in \mathcal{N}_s^{\mathscr{C}_j}} \gamma^{sk}$$

  ($\mathcal{N}_r^{\mathscr{C}_j}$ – indices of neighbors of r in group $\mathscr{C}_j$.)
- Then, the *cluster synchronization manifold* $\mathscr{S}_K$ is *forward-invariant* (necessary condition for cluster synchronization).

Contact

aminzare@math.princeton.edu http://www.math.princeton.edu/~aminzare/

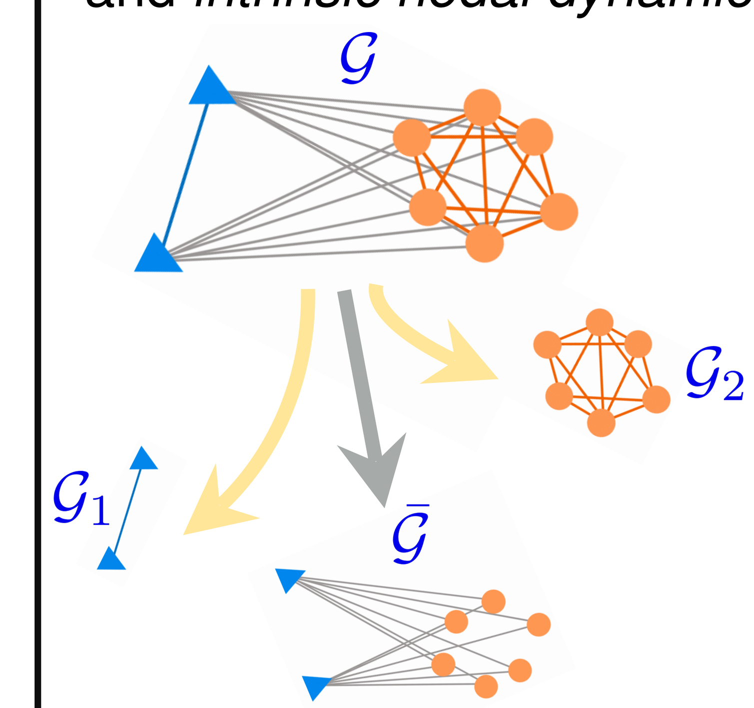
Main result

Consider a network with dynamics

$$\dot{\mathbf{x}}^k_{\mathscr{C}_r}(t) = \mathbf{f}_{\mathscr{C}_r}\left(\mathbf{x}^k_{\mathscr{C}_r}(t), t\right) + \sum_{j \in \mathcal{N}^k} \gamma^{kj} D \left(\mathbf{x}^j_{\mathscr{C}_r}(t) - \mathbf{x}^k_{\mathscr{C}_r}(t)\right) \qquad k = 1, \dots, c_i$$

(K clusters with homogenous intrinsic dynamics) that satisfies CIE.

Objective: Find conditions on *network structure*, *coupling weights*, and *intrinsic nodal dynamics*, that guarantee convergence to $\mathscr{S}_K$.



Graph decomposition:

- $\mathcal{G}$: interconnection graph
- $\mathcal{G}_r$: subgraph describing connections within cluster r, $\lambda^{(2)}_{\mathscr{C}_r}$: 2nd eigenvalue
- $\bar{\mathcal{G}}$: subgraph describing connections among clusters, $\bar{\lambda}^{(2)}$: 2nd eigenvalue

Sufficient conditions for cluster synchronization with heterogeneous $\mathbf{f}^i$

Let $\mu := \max_r \sup_{(x,t)} \mu_{2,Q}\left[J_{\mathbf{f}_{\mathscr{C}_r}}(x,t) - \left(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}\right) D\right]$. Then for any solution $\mathbf{x}$, $\exists$ a solution $\bar{\mathbf{x}}$ s.t.

$$\|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\|_{2, I_N \otimes Q^2} \le e^{\mu t} \|\mathbf{x}(0) - \bar{\mathbf{x}}(0)\|_{2, I_N \otimes Q^2}$$

For $\mu < 0$, the dynamics in each cluster synchronize: $\forall i, j \in \mathscr{C}_r$,

$$\mathbf{x}^i_{\mathscr{C}_r}(t) - \mathbf{x}^j_{\mathscr{C}_r}(t) \to 0 \quad \text{as } t \to \infty$$

Neuronal network: Fitzhugh-Nagumo & Hindmarsh-Rose

Network of Fitzhugh-Nagumo (FN) and Hindmarsh-Rose (HR) oscillators:

2D reduction of Hodgkin-Huxley:

$$\dot{y} = y - y^3/3 - z - a + I$$
$$\dot{z} = \epsilon(y - bz)$$

2D version of Hindmarsh-Rose:

$$\dot{y} = y^3 + cy^2 + z + I$$
$$\dot{z} = \delta(1 - 5y^2 - z)$$

Sufficient condition for synchronization with diffusion matrix $D = \text{diag}(\gamma, 0)$:

$$\gamma > \frac{1}{\lambda^{(2)}_{\mathscr{C}_i} + \bar{\lambda}^{(2)}} \max_{0 < p < \sqrt{\frac{3}{25\delta_{\mathscr{C}_i}}}} \left\{ \frac{-(2c_{\mathscr{C}_i} - 5)^2}{4(25\delta_{\mathscr{C}_i} p^2 - 3)} + \frac{1}{4\delta_{\mathscr{C}_i} p}, \frac{c^2_{\mathscr{C}_i}}{3} - \delta_{\mathscr{C}_i}, 1 + \alpha_j \right\}, \ \alpha_j = \frac{\left(\epsilon_{\mathscr{C}_j} p - 1/p\right)^2}{4 b_{\mathscr{C}_j} \epsilon_{\mathscr{C}_j}}$$