J Nonlinear Sci
https://doi.org/10.1007/s00332-018-9457-y

Journal of Nonlinear Science

Cluster Synchronization of Diffusively Coupled Nonlinear Systems: A Contraction-Based Approach

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Received: 5 July 2017 / Accepted: 21 March 2018

Abstract Finding the conditions that foster synchronization in networked nonlinear systems is critical to understanding a wide range of biological and mechanical systems. However, the conditions proved in the literature for synchronization in nonlinear systems with linear coupling, such as has been used to model neuronal networks, are in general not strict enough to accurately determine the system behavior. We leverage contraction theory to derive new sufficient conditions for cluster synchronization in terms of the network structure, for a network where the intrinsic nonlinear dynamics of each node may differ. Our result requires that network connections satisfy a cluster-input-equivalence condition, and we explore the influence of this requirement on network dynamics. For application to networks of nodes with FitzHugh–Nagumo dynamics, we show that our new sufficient condition is tighter than those found in previous analyses that used smooth or nonsmooth Lyapunov functions. Improving the analytical conditions for when cluster synchronization will occur based on network

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Communicated by Danielle S. Bassett.

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configuration is a significant step toward facilitating understanding and control of complex networked systems.

Keywords Cluster synchronization · Contraction theory for stability · Diffusively coupled nonlinear networks · Neuronal oscillators

1 Introduction

Synchronization has been observed and studied in diverse fields. Its presence has been characterized in symmetric networks of identical mechanical systems or identical biological systems, as well as those with differing types of individual components and nonuniform coupling (Pikovsky et al. 2003). The role of synchronization has been studied in a multitude of both natural and engineered settings including collective motion (Sepulchre et al. 2008), power-grid networks (Motter et al. 2013), robotics (Nair and Leonard 2008), sensor networks (Sivrikaya and Yener 2004), circadian rhythms (Winfree 1967), bioluminescence in fireflies (Smith 1935), pacemaker cells in the heart (Mirollo and Strogatz 1990), neuronal ensembles (Chow and Kopell 2000), and numerous others. In the human brain, synchronization at the neuronal or regional level can be beneficial, allowing for production of a vast range of behaviors (Dumas et al. 2010; MacLeod and Laurent 1996), or detrimental, causing disorders such as Parkinson's disease (Chen et al. 2007) and epilepsy (Lehnertz et al. 2009). Applications for control of neural dynamics may involve regulating patterns of synchronized phenomena among nodes or subsystems that have different intrinsic dynamics and are connected in an arbitrary network (Abrams et al. 2016; Wilson and Moehlis 2015). Most generally, nodes can be agents in a multi-agent system, compartments in a compartmental system, or other units that interact with one another in a pairwise framework. Characterizing the emergence and persistence of synchronization in a system with multiple heterogeneous nodes is the first step toward effective control of desired behavior.

Heterogeneous nodes and nonuniform coupling structure in a network often lead to complex patterns of synchronization. Under certain conditions, it is possible to partition the network into clusters of nodes that are synchronized within clusters but not across clusters. In a cluster synchronized network, nodes in the same cluster will have similar behavior after a transient. The cluster synchronized network can thus be reduced to a network where each node corresponds to a cluster, commonly referred to as the *quotient network* (Chung et al. 2007; Russo and Slotine 2010; Schaub et al. 2016). The simplified dynamics represent a powerful tool for facilitating analysis of the dynamics of cluster synchronized systems.

Cluster synchronization has been defined in various ways in the literature. According to one common definition for phase oscillators, clusters are subgroups of oscillators that share common phases (e.g., Brown et al. 2003; Orosz et al. 2009; Belykh et al. 2015; Tiberi et al. 2017). Another definition is based on approximate cluster synchronization, wherein nodes within a given cluster can have slightly different behaviors (Sorrentino and Pecora 2016; Pham and Slotine 2007; Favaretto et al. 2017a, b). In the present work, we define cluster synchronization as convergence to an invariant manifold, called the cluster synchronization manifold, on which the states of all nodes in a cluster evolve identically (Belykh et al. 2008; Sorrentino and Ott 2007).

A necessary condition for cluster synchronization is the existence of an invariant manifold. In this work, we assume "cluster-input-equivalence", which ensures existence of such a manifold. Cluster-input-equivalence was proposed by Stewart et al. (2003), Golubitsky et al. (2005), under the name "balanced equivalence", and by Belykh et al. (2008) it is used under the name "cluster partition manifold" (Eq. 13). Subsequently this condition was used to show existence of an invariant manifold for cluster synchronization (Ferreira and Arcak 2013; Schaub et al. 2016; Sorrentino et al. 2016).

Another important problem is the establishment of sufficient conditions that guarantee stability of a cluster synchronization manifold. The problem has been well studied for networks where the dynamics can be described by reduced phase oscillators, e.g., Brown et al. (2003), Orosz et al. (2009). The problem has also been studied for networks of more general nonlinear dynamics. For example, Lu et al. (2010), Wang et al. (2009), and Fiore et al. (2017) have explored conditions that rely on intra-cluster network structure. Specialized network graphs have been considered in Pecora et al. (2014). Xia and Cao (2011) have explored time-delay and negative coupling as mechanisms to realize cluster synchronization in a network with homogeneous dynamics.

In the present paper, we propose a new sufficient condition for cluster synchronization that applies to general network structure and heterogeneous nonlinear dynamics. The method leverages contraction theory, which has been used to analyze the stability of invariant dynamics, including cluster synchronization (Pham and Slotine 2007). Here, we use contraction theory to find a sufficient condition for cluster synchronization that incorporates a novel measure of connectivity between clusters not found in previous work on the subject.

Contraction theory is a powerful tool for understanding synchronization phenomena in networked systems. The proper tool for characterizing contractivity for nonlinear systems is provided by the logarithmic norms, or matrix measures (Desoer and Vidyasagar 1975), of the Jacobian of the vector field, evaluated at all possible states. This idea is a classical one, and can be traced back at least to work of Lewis (1949). Dahlquist's 1958 thesis under Hörmander used matrix measures to show contractivity of differential equations, and more generally of differential inequalities, the latter applied to the analysis of convergence of numerical schemes for solving differential equations (Dahlquist 1958). Several authors have independently rediscovered the basic ideas. For example, in the 1960s, Demidovič (1961, 1967) established basic convergence results with respect to Euclidean norms, as did Hartman (1961) and Yoshizawa (1966, 1975). In control theory, the field attracted much attention after the work of Lohmiller and Slotine (1998). We refer the reader especially to the careful historical analysis given in Jouffroy (2005). Other useful historical references are Pavlov et al. (2004) and the survey Soderlind (2006). An introductory tutorial to basic results in contraction theory for nonlinear control systems is given by Aminzare and Sontag (2014a). Results on synchronization using contraction-based techniques are described, for example, in Russo and Slotine (2010), Arcak (2011), Lohmiller and Slotine (2005), Russo and Bernardo (2009), Wang and Slotine (2005), Aminzare et al. (2014).

The main contributions of the present paper are as follows. We extend contraction theory to a setting where the nodal dynamics may have heterogeneous intrinsic dynamics and the network satisfies the cluster-input-equivalence condition. Using this extension of contraction theory, we prove new sufficient conditions for cluster synchronization in a network of heterogeneous nodal dynamics. We improve upon our earlier analysis of synchronization in networks of homogeneous FitzHugh–Nagumo (FN) oscillators (Davison et al. 2016), and show that the proposed result yields a tighter bound on the algebraic connectivity of the associated undirected graph. The bound is a significant advance over previous results because it incorporates terms that reflect inter- and intra- cluster network structure.

The paper proceeds as follows. In Sect. 2, we review relevant concepts and results from the contraction theory literature. In Sect. 3, we present our main result, an extension of the existing theory to a cluster synchronized setting. In Sect. 4, we consider networks of neuronal oscillators, modeled by FitzHugh–Nagumo and Hindmarsh–Rose dynamics, and demonstrate how the proposed approach provides sufficient conditions for cluster synchronization. We conclude in Sect. 5.

2 A Review of Contraction Theory

In what follows, we review notations, definitions, and main results in contraction theory that will be applied in later sections.

Definition 1 (Logarithmic norm, Soderlind 2006) For any matrix $A \in \mathbb{R}^{n\times n}$ and any given norm $\|\cdot\|$ on $\mathbb{R}^n$, the logarithmic norm (also called the matrix measure) of A induced by the norm $\|\cdot\|$ is defined by

$$\mu[A] = \lim_{h\to 0^+} \sup_{x\neq 0\in\mathbb{R}^n} \frac{1}{h}\left(\frac{\|(I+hA)x\|}{\|x\|} - 1\right), \tag{1}$$

where I is the identity matrix of size n.

Notation 1 *For any* $1 \le p \le \infty$ *and any* $n\times n$ *positive definite matrix* Q*, let* $\|\cdot\|_p$ *denote the* L^p *norm on* $\mathbb{R}^n$*, and* $\|\cdot\|_{p,Q}$ *denote the* $Q-$*weighted* L^p *norm on* $\mathbb{R}^n$ *defined by* $\|x\|_{p,Q} := \|Qx\|_p$*. By* $\mu_p[A]$*, we mean the logarithmic norm of A induced by* $\|\cdot\|_p$ *and by* $\mu_{p,Q}[A]$*, we mean the logarithmic norm of A induced by* $\|\cdot\|_{p,Q}$*. Note that* $\mu_{p,Q}[A] = \mu_p[QAQ^{-1}]$.

Notation 2 *For any matrix A, denote A positive semidefinite as* $A \ge 0$.

Remark 1 In Table 1, the algebraic expression of logarithmic norms induced by the L^p norm for $p = 1, 2,$ and ∞ are shown. For proofs, see for instance (Desoer and Vidyasagar 1975).

Definition 2 (Contraction) Consider the following nonlinear dynamical system on $V \times [0, \infty]$, where V is a convex subset of $\mathbb{R}^n$. Consider appropriate conditions on vector field G (e.g., $G(x, t)$ Lipschitz on x and continuous on (x, t)) that guarantee existence and uniqueness of solutions of

Table 1 Standard matrix measures for a real $n \times n$ matrix, $A = [a_{ij}]$

Vector norm, $\|\cdot\|$	Induced matrix measure, $M[A]$
$\|x\|_1 = \sum_{i=1}^{n} \|x_i\|$	$\mu_1[A] = \max_j \left(a_{jj} + \sum_{i \neq j} \|a_{ij}\| \right)$
$\|x\|_2 = \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}}$	$\mu_2[A] = \max_{\lambda \in \mathrm{spec} \frac{1}{2}(A+A^T)} \lambda$
$\|x\|_\infty = \max_{1 \leq i \leq n} \|x_i\|$	$\mu_\infty[A] = \max_i \left(a_{ii} + \sum_{i \neq j} \|a_{ij}\| \right)$

$$\dot{x}(t) = G(x(t), t). \tag{2}$$

Equation (2) is *contractive* if there exist $c < 0$ and a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that, for any two solutions x and y of Eq. (2), the following inequality holds for any $t \geq 0$:

$$\|x(t) - y(t)\| \leq e^{ct} \|x(0) - y(0)\|. \tag{3}$$

Proposition 1 (Theorem 1, Aminzare and Sontag 2014a)
Consider Eq. (2) and assume that G *is a continuously differentiable function on its first variable. Let* $c := \sup_{(x,t)} \mu[J_G(x, t)]$, *where* μ *is the logarithmic norm induced by an arbitrary norm on* $\mathbb{R}^n$, *and* J_G *is the Jacobian of* G. *Then for any two solutions* x *and* y *of Eq. (2), and* $t \geq 0$,

$$\|x(t) - y(t)\| \leq e^{ct} \|x(0) - y(0)\|.$$

In particular, when $c < 0$, *Eq. (2) satisfies Eq. (3) and is contractive.*

Throughout the paper, we denote the Jacobian of the vector field $f(x, t)$ evaluated at (x, t) as $J_f(x, t)$, i.e., $J_f(x, t) = \frac{\partial f}{\partial x}(x, t)$.

We consider a network of N nodes, with states $\{X^1, \ldots, X^N\}$ and intrinsic dynamics F^i:

$$\dot{X}^i(t) = F^i\left(X^i(t), t\right).$$

Here, X^i and F^i have dimension $n \geq 1$. For a fixed convex subset $V \subset \mathbb{R}^n$, $F^i : V \times [0, \infty) \to \mathbb{R}^n$, defined by $F^i = F^i(z, t)$, is Lipschitz on z and continuous on (z, t). We also assume that the nodes are diffusively connected through an undirected weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and describe the dynamics of the network as follows:

$$\dot{X}^i(t) = F^i\left(X^i(t), t\right) + \sum_{j \in \mathcal{N}^i} \gamma^{ij} D\left(X^j(t) - X^i(t)\right) \qquad i = 1, \ldots, N. \tag{4}$$

The indices in $\mathcal{N}^i$ represent the neighbors of node i. Without loss of generality[1], we can assume that the *diffusion matrix* D is a nonzero diagonal matrix of size n, $D = \operatorname{diag}(d_1, \ldots, d_n)$, where $d_i \geq 0$. The positive constants γ^{ij} represent the edge weights of $\mathcal{G}$. The products of the elements in D and the edge weights γ^{ij} represent the coupling strengths between the nodes. This allows representation of all possible diffusive coupling structures by manipulation of the diagonal elements of D and the edge weights.

Let $\mathcal{L} = (\mathcal{L}_{ij})$ be the Laplacian matrix of $\mathcal{G}$:

$$\mathcal{L}_{ij} = \begin{cases} \sum_{k\in\mathcal{N}^i} \gamma^{ik} & i = j, \\ -\gamma^{ij} & i \neq j, j \in \mathcal{N}^i, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

We denote the eigenvalues of $\mathcal{L}$ as $0 = \lambda^{(1)} \leq \lambda^{(2)} \leq \cdots \leq \lambda^{(N)}$. The second smallest eigenvalue, $\lambda^{(2)}$, is called the *algebraic connectivity* of the graph. This number helps to quantify "how connected" the graph is. The number of the zero eigenvalues is equal to the number of connected components of $\mathcal{G}$.

Using the notation of the Laplacian matrix, Eq. (4) can be written in closed form:

$$\dot{X}(t) = \mathcal{F}(X(t), t) - (\mathcal{L} \otimes D)X(t), \tag{6}$$

where $X = \left(X^{1^T}, \ldots, X^{N^T}\right)^T$, $\mathcal{F} = \left(F^{1^T}, \ldots, F^{N^T}\right)^T$, and $\otimes$ represents the Kronecker product.

Definition 3 (Complete synchronization) Let

$$\mathscr{S}_1 := \left\{X \in \mathbb{R}^{nN} \mid X^1 = \cdots = X^N,\ X^i \in \mathbb{R}^n\right\}.$$

The dynamics given in Eq. (4) *synchronize completely* if any solution of Eq. (4) converges to $\mathscr{S}_1$ in an appropriate norm. In other words, let X be a solution of Eq. (4). Then there exists a solution $\bar{X} \in \mathscr{S}_1$ such that, in an appropriate norm,

$$X(t) - \bar{X}(t) \to 0 \quad \text{as } t \to \infty.$$

$\mathscr{S}_1$ is called the *synchronization manifold.*

We will use synchronization and complete synchronization alternatively.

Definition 4 (Cluster synchronization) For any $1 \leq K \leq N$ and any $1 \leq c_1, \ldots, c_K \leq N$ such that $c_1 + \cdots + c_K = N$, let

$$\mathscr{S}_K := \left\{X \in \mathbb{R}^{nN} \mid X^1 = \cdots = X^{c_1},\ \ldots,\ X^{N-c_K+1} = \cdots = X^N,\ X^i \in \mathbb{R}^n\right\}.$$

[1] If D is not diagonal, an appropriate change of coordinate can render it diagonal.

The dynamics given in Eq. (4) *synchronize in clusters* if there exists $1 \leq K \leq N$ such that all solutions of Eq. (4) converge to $\mathscr{S}_K$ in an appropriate norm.
$\mathscr{S}_K$ is called the $K-$*cluster synchronization manifold*.

The $1-$cluster synchronization manifold is the same as the synchronization manifold (Definition 3).

In the following two propositions, we consider Eq. (4) with homogeneous $F^i = F$, and state two sufficient conditions that guarantee that Eq. (4) synchronizes.

Proposition 2 (Proposition 1, Aminzare and Sontag 2014b) *Consider Eq.* (4) *with homogeneous* $F^i = F$. *Assume that there exists a norm on* $\mathbb{R}^n$ *such that*

$$\sup_{(x,t)} \mu[J_F(x,t)] < 0. \tag{7}$$

Then Eq. (4) *synchronizes.*

In Russo and Slotine (2010), Proposition 2 has been generalized[2] to F^i with heterogeneous elements. The work shows that, under some conditions on the weights of the interconnected graph, if each node has contractive dynamics, then Eq. (4) synchronizes in clusters. In Sect. 4, we provide an example (with FitzHugh–Nagumo and Hindmarsh–Rose oscillators) that synchronizes in clusters and supports our theory derived in the next section but does not satisfy the condition provided in Russo and Slotine (2010).

Note that the sufficient condition provided in Proposition 2 depends only on the dynamics of each isolated node, namely J_F. The next proposition from Arcak (2011) provides a sufficient condition for complete synchronization less restrictive than Eq. (7), which depends on J_F, the diffusion matrix D, and the graph $\mathcal{G}$. It is based on the weighted L^2 norms. For some special graphs, the result has been generalized to weighted L^p norms (Aminzare and Sontag 2014b).

Proposition 3 (Theorem 4 (modified), Arcak 2011) *Consider Eq.* (4) *with homogeneous* $F^i = F$. *Assume that there exists a positive definite matrix* P *such that* $P^2D + DP^2$ *is also positive definite, and let*

$$c := \sup_{(x,t)\in V\times[0,\infty)} \mu_{2,P}\left[J_F(x,t) - \lambda^{(2)}D\right].$$

Then for any solution X *of Eq.* (4) *that remains in* V^N, *there exists a solution* $\bar{X}$ *such that*

$$\|X(t) - \bar{X}(t)\|_{2,I_N\otimes P^2} \leq e^{ct}\|X(0) - \bar{X}(0)\|_{2,I_N\otimes P^2}.$$

Moreover, if $c < 0$, *then Eq.* (4) *synchronizes, i.e., for any pair* $i, j \in \{1,\ldots,N\}$,

$$X^i(t) - X^j(t) \to 0 \quad as\ t \to \infty.$$

2 The statement of Theorem 3 in Russo and Slotine (2010) is correct; however, the proof needs revision to be complete.

In the following section, we present the main result of this work – we generalize Proposition 3 to heterogeneous F^i and provide sufficient conditions for cluster synchronization.

3 Main Result: Cluster Synchronization

In this section, we provide sufficient conditions on heterogeneous intrinsic dynamics F^i, the graph $\mathcal{G}$, and the diffusion matrix D, that guarantee cluster synchronization of the network described in Eq. (4).

Assumption 1 In the network described by Eq. (4), we assume that

1. There exist $K \le N$ and $c_1, \dots, c_K \ge 2$, such that $c_1 + \cdots + c_K = N$, and

$$F^{i_1} = \cdots = F^{i_{c_1}} =: F_{\mathscr{C}_1}, \; \dots, \; F^{i_{N-c_K+1}} = \cdots = F^{i_N} =: F_{\mathscr{C}_K},$$

   where $\{i_1, \dots, i_N\}$ is a permutation of $\{1, \dots, N\}$. Without loss of generality, we can assume:

$$F^1 = \cdots = F^{c_1} =: F_{\mathscr{C}_1}, \; \dots, \; F^{N-c_K+1} = \cdots = F^N =: F_{\mathscr{C}_K}.$$

   Let $\mathscr{C}_1, \dots, \mathscr{C}_K$ denote K clusters of nodes. The nodes in cluster $\mathscr{C}_1$ are defined by $X^1, \dots, X^{c_1}$ and they all have dynamics $F_{\mathscr{C}_1}$, the nodes in cluster $\mathscr{C}_2$ are defined by $X^{c_1+1}, \dots, X^{c_1+c_2}$ and they all have dynamics $F_{\mathscr{C}_2}$, etc. For ease of notation in our calculations, we let

$$\begin{aligned} X^1_{\mathscr{C}_1} &= X^1, \dots, X^{c_1}_{\mathscr{C}_1} = X^{c_1}, \\ X^1_{\mathscr{C}_2} &= X^{c_1+1}, \dots, X^{c_2}_{\mathscr{C}_2} = X^{c_1+c_2}, \\ &\vdots \\ X^1_{\mathscr{C}_K} &= X^{N-c_K+1}, \dots, X^{c_K}_{\mathscr{C}_K} = X^N. \end{aligned} \tag{8}$$

2. The *cluster-input-equivalence* condition, defined by Belykh et al. (2008), holds. This implies that the following edge weight sums are equal: for any two nodes $X^i_{\mathscr{C}_r}, X^j_{\mathscr{C}_r}, (i, j) \in \mathscr{C}_r$,

$$\eta_{\mathscr{C}_r\mathscr{C}_s} := \sum_{k \in \mathscr{N}^i_{\mathscr{C}_s}} \gamma^{ik} = \sum_{k \in \mathscr{N}^j_{\mathscr{C}_s}} \gamma^{jk}, \tag{9}$$

   where $\mathscr{N}^i_{\mathscr{C}_s}$ denotes the indices of the neighbors of node i which are in cluster $\mathscr{C}_s$.

Assumption 1 ensures that the $K-$cluster synchronization manifold is invariant, which is a necessary condition for cluster synchronization.

Next we provide sufficient conditions to show that $\mathscr{S}_K$ is (globally) stable, i.e., any solution of Eq. (4) converges to $\mathscr{S}_K$.

Recall that the network graph is $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Denote the subgraph for the nodes in $\mathscr{C}_r$ by $\mathcal{G}_{\mathscr{C}_r} = (\mathcal{V}_{\mathscr{C}_r}, \mathcal{E}_{\mathscr{C}_r})$. The set $\mathcal{V}_{\mathscr{C}_r}$ consists of all the nodes in $\mathscr{C}_r$ and the set $\mathcal{E}_{\mathscr{C}_r}$ consists of all edges that have both end points in $\mathcal{V}_{\mathscr{C}_r}$. Then

$$\mathcal{G} = \left(\bigcup_{r=1}^{K} \mathcal{G}_{\mathscr{C}_r} \right) \bigcup \bar{\mathcal{G}},$$

where $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E} \setminus \cup_r \mathcal{E}_{\mathscr{C}_r})$ is the graph describing connections among the clusters $\mathscr{C}_r$.

Let $\mathcal{L}_{\mathscr{C}_r}$ denote the Laplacian matrix of $\mathcal{G}_{\mathscr{C}_r}$ with eigenvalues $0 = \lambda_{\mathscr{C}_r}^{(1)} \leq \lambda_{\mathscr{C}_r}^{(2)} \leq \cdots \leq \lambda_{\mathscr{C}_r}^{(c_r)}$ and $\bar{\mathcal{L}}$ denote the Laplacian matrix of $\bar{\mathcal{G}}$ with eigenvalues $0 = \bar{\lambda}^{(1)} \leq \bar{\lambda}^{(2)} \leq \cdots \leq \bar{\lambda}^{(N)}$. In the special case of $K = 1$, we set $\bar{\lambda}^{(2)} = 0$. Then $\mathcal{L}$, the Laplacian matrix of $\mathcal{G}$ can be written as follows:

$$\mathcal{L} = \mathcal{L}_{\mathscr{C}} + \bar{\mathcal{L}}, \tag{10}$$

where $\mathcal{L}_{\mathscr{C}}$ is a block diagonal matrix with the form:

$$\mathcal{L}_{\mathscr{C}} = \begin{pmatrix} \mathcal{L}_{\mathscr{C}_1} & & \\ & \ddots & \\ & & \mathcal{L}_{\mathscr{C}_K} \end{pmatrix}. \tag{11}$$

With these definitions, Eq. (6) can be written as

$$\dot{X}(t) = \mathcal{F}(X(t), t) - (\mathcal{L}_{\mathscr{C}} \otimes D)X(t) - (\bar{\mathcal{L}} \otimes D)X(t). \tag{12}$$

Theorem 1 *Consider Eq. (4), or equivalently Eq. (12), with Assumption 1, and let*

$$\mu := \max_{r=1,\ldots,K} \sup_{(x,t)\in V\times[0,\infty)} \mu_{2,P} \left[J_{F_{\mathscr{C}_r}}(x,t) - \left(\lambda_{\mathscr{C}_r}^{(2)} + \bar{\lambda}^{(2)} \right) D \right], \tag{13}$$

where $P \in \mathbb{R}^{n\times n}$ is a positive definite matrix chosen such that $P^2 D + D P^2$ is positive semidefinite. Then, for any solution X of Eq. (4) that remains in V^N, there exists $\bar{X}(t)$ such that

$$\|X(t) - \bar{X}(t)\|_{2,\mathcal{P}} \leq e^{\mu t} \|X(0) - \bar{X}(0)\|_{2,\mathcal{P}}, \tag{14}$$

where $\mathcal{P} = I_N \otimes P^2$ and $\|\cdot\|_{2,\mathcal{P}}$ is a $\mathcal{P}$-weighted L^2 norm on $\mathbb{R}^{nN}$, defined by

$$\|x\|_{2,\mathcal{P}} := \left\| \left(\left\| P^2 x^1 \right\|_2, \ldots, \left\| P^2 x^N \right\|_2 \right)^T \right\|_2,$$

for any $x = \left({x^1}^T, \ldots, {x^N}^T \right)^T \in \mathbb{R}^{nN}$. In particular, if $\mu < 0$, then for any pair of nodes $i, j \in \mathscr{C}_r$, $X_{\mathscr{C}_r}^i$ and $X_{\mathscr{C}_r}^j$ satisfy

$$X_{\mathscr{C}_r}^i(t) - X_{\mathscr{C}_r}^j(t) \to 0 \quad \text{as } t \to \infty.$$

Remark 2 Theorem 1 provides a sufficient condition for cluster synchronization that depends on the dynamics of each isolated cluster $J_{F_{\mathscr{C}_r}}$, the diffusion matrix D, the structure $\lambda^{(2)}_{\mathscr{C}_r}$ of each subgraph $\mathcal{G}_{\mathscr{C}_r}$ describing connections among the nodes in cluster $\mathscr{C}_r$, and the structure $\bar{\lambda}^{(2)}$ of the subgraph $\bar{\mathcal{G}}$ describing connections among the clusters. Proposition 3 is a special case of Theorem 1 when $K = 1$ and $\bar{\lambda}^{(2)} = 0$. One can still apply Proposition 3 to $K > 1$ clusters to show cluster synchronization. However, Theorem 1 provides a less restrictive sufficient condition for cluster synchronization because it makes use of coupling structure both within and between clusters.

Remark 3 Theorem 1 provides a sufficient condition for the cluster synchronization manifold to be globally attractive. However, this result can be made less conservative by restricting the domain over which we take the supremum in Eq. (13). In particular, if we take the supremum over a neighborhood around the cluster synchronization manifold, it should give us a condition for the manifold to be only locally attractive.

Remark 4 For systems that satisfy $\mu < 0$, the rate of convergence to the $K-$cluster synchronization manifold can be approximated by μ. In addition to the dependence on the dynamics in each cluster $J_{F_{\mathscr{C}_r}}$ and the diffusion matrix D, the rate of convergence depends on the structure of the coupling within ($\lambda^{(2)}_{\mathscr{C}_r}$) and between ($\bar{\lambda}^{(2)}$) clusters.

In the proof of Theorem 1, we need the following key lemmas. We first state the Courant-Fischer minimax Theorem (Horn and Johnson 1991).

Lemma 1 *Let L be a positive semidefinite matrix in $\mathbb{R}^{l\times l}$. Let $\lambda^{(1)} \le \cdots \le \lambda^{(l)}$ be l eigenvalues with $e^1, \ldots, e^l$ corresponding normalized orthogonal eigenvectors. For any $v \in \mathbb{R}^l$, if $v^T e^j = 0$ for $1 \le j \le k-1$, $1 \le k \le l$, then*

$$v^T L v \ge \lambda^{(k)} v^T v.$$

Lemma 2 (Lemma 3, Aminzare and Sontag 2014a) *Suppose that P is a positive definite matrix and A is an arbitrary matrix. If $\mu_{2,P}[A] = \mu$, then $P^2 A + A^T P^2 \le 2\mu P^2$.*

Proof of Theorem 1 Let $w := X - \bar{X}$, where

$$X = \left(X^{1\;T}_{\mathscr{C}_1}, \ldots, X^{c_1\;T}_{\mathscr{C}_1}, \ldots, X^{1\;T}_{\mathscr{C}_K}, \ldots, X^{c_K\;T}_{\mathscr{C}_K}\right)^T,$$

is a solution of (4) and

$$\bar{X} = \left((\mathbf{1}_{c_1} \otimes x_1)^T, \ldots, (\mathbf{1}_{c_K} \otimes x_K)^T\right)^T,$$

with $x_r := \frac{1}{c_r}\sum_{i=1}^{c_r} X^i_{\mathscr{C}_r}$ and $\mathbf{1}_{c_r} \in \mathbb{R}^{c_r}$ is a vector of ones. Let $w = \left(w_1^T, \ldots, w_K^T\right)^T$, where $w_r := \left((X^1_{\mathscr{C}_r} - x_r)^T, \ldots, (X^{c_r}_{\mathscr{C}_r} - x_r)^T\right)^T \in \mathbb{R}^{c_r n}$, and define

$$\Phi(w) \;:=\; \frac{1}{2} w^T \mathcal{P} w = \frac{1}{2}\sum_{r=1}^{K} w_r^T \left(I_{c_r} \otimes P^2\right) w_r \;.$$

Since $\Phi(w) = \frac{1}{2}\|\mathcal{P}w\|_2^2$, to prove (14), it suffices to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(w) \leq 2\mu\Phi(w).$$

Let

$$\mathcal{F}(X,t) = \left(F_{\mathscr{C}_1}^T(X_{\mathscr{C}_1}^1,t),\ldots,F_{\mathscr{C}_1}^T(X_{\mathscr{C}_1}^{c_1},t),\ldots,F_{\mathscr{C}_K}^T(X_{\mathscr{C}_K}^1,t),\ldots,F_{\mathscr{C}_K}^T(X_{\mathscr{C}_K}^{c_K},t)\right)^T,$$

and

$$\bar{\mathcal{F}}(X,t) = \left(\left(\mathbf{1}_{c_1}\otimes y_1\right)^T,\ldots,\left(\mathbf{1}_{c_K}\otimes y_K\right)^T\right)^T \quad \text{where} \quad y_r = \frac{1}{c_r}\sum_{i=1}^{c_r} F_{\mathscr{C}_r}(X_{\mathscr{C}_r}^i,t).$$

Standard calculations show that the derivative of Φ is as follows:

$$\begin{aligned}
\frac{\mathrm{d}\Phi}{\mathrm{d}t}(w) &= w^T\mathcal{P}\left(\mathcal{F}(X,t)-\bar{\mathcal{F}}(X,t)\right) - w^T\mathcal{P}(\mathcal{L}_{\mathscr{C}}\otimes D)w - w^T\mathcal{P}(\bar{\mathcal{L}}\otimes D)w \\
&= w^T\mathcal{P}\left(\mathcal{F}(X,t)-\mathcal{F}(\bar{X},t)\right) + w^T\mathcal{P}\left(\mathcal{F}(\bar{X},t)-\bar{\mathcal{F}}(X,t)\right) \\
&\quad - w^T\mathcal{P}(\mathcal{L}_{\mathscr{C}}\otimes D)w - w^T\mathcal{P}(\bar{\mathcal{L}}\otimes D)w \\
&= w^T\mathcal{P}\left(\mathcal{F}(X,t)-\mathcal{F}(\bar{X},t)\right) - w^T\mathcal{P}(\mathcal{L}_{\mathscr{C}}\otimes D)w - w^T\mathcal{P}(\bar{\mathcal{L}}\otimes D)w\,.
\end{aligned} \tag{15}$$

In the second equation, we added and subtracted $w^T\mathcal{P}\mathcal{F}(\bar{X},t)$, where $\mathcal{F}(\bar{X},t)$ is written as

$$\mathcal{F}(\bar{X},t) = \left(\left(\mathbf{1}_{c_1}\otimes F_{\mathscr{C}_1}(x_1,t)\right)^T,\ldots,\left(\mathbf{1}_{c_K}\otimes F_{\mathscr{C}_K}(x_K,t)\right)^T\right)^T.$$

The last equality holds because $w_r^T(\mathbf{1}_{c_r}\otimes I_n) = 0$ implies that

$$\begin{aligned}
w^T\mathcal{P}\left(\mathcal{F}(\bar{X},t)-\bar{\mathcal{F}}(X,t)\right) &= \sum_{r=1}^K w_r^T\left(I_{c_r}\otimes P^2\right)\left(\mathbf{1}_{c_r}\otimes\left(F_{\mathscr{C}_r}^T(x_r,t)-y_r^T\right)\right) \\
&= \sum_{r=1}^K w_r^T\left(\mathbf{1}_{c_r}\otimes P^2\left(F_{\mathscr{C}_r}^T(x_r,t)-y_r^T\right)\right) \\
&= \sum_{r=1}^K w_r^T\left(\mathbf{1}_{c_r}\otimes I_n\right)P^2\left(F_{\mathscr{C}_r}^T(x_r,t)-y_r^T\right) \\
&= 0.
\end{aligned}$$

Step 1. We show that

$$-w^T\mathcal{P}(\mathcal{L}_{\mathscr{C}}\otimes D)w \leq -\sum_{r=1}^K \lambda_{\mathscr{C}_r}^{(2)} w_r^T\left(I_{c_r}\otimes P^2 D\right)w_r\,. \tag{16}$$

Since $P^2D + DP^2$ is positive semidefinite, Cholesky decomposition yields an upper triangular matrix M such that $P^2D + DP^2 = 2M^TM$. For any $r = 1, \ldots, K$,

$$\begin{aligned}
-w_r^T\left(I_{c_r}\otimes P^2\right)(\mathcal{L}_{\mathscr{C}_r}\otimes D)w_r &= -\,w_r^T\left(\mathcal{L}_{\mathscr{C}_r}\otimes P^2D\right)w_r\\
&= -\,\frac{1}{2}w_r^T\left(\mathcal{L}_{\mathscr{C}_r}\otimes\left(P^2D+DP^2\right)\right)w_r\\
&= -\,w_r^T\left(\mathcal{L}_{\mathscr{C}_r}\otimes\left(M^TM\right)\right)w_r\\
&= -\,w_r^T\left(I_{c_r}\otimes M^T\right)\left(\mathcal{L}_{\mathscr{C}_r}\otimes I_n\right)\left(I_{c_r}\otimes M\right)w_r\\
&\le -\,\lambda^{(2)}_{\mathscr{C}_r}\left((I_{c_r}\otimes M)w_r\right)^T(I_{c_r}\otimes M)w_r\\
&= -\,\lambda^{(2)}_{\mathscr{C}_r}w_r^T\left(I_{c_r}\otimes M^TM\right)w_r\\
&= -\,\lambda^{(2)}_{\mathscr{C}_r}w_r^T\left(I_{c_r}\otimes P^2D\right)w_r\ .
\end{aligned}$$

Note that the inequality holds by Lemma 1. To apply Lemma 1, we need to show that

$$\left(\left(I_{c_r}\otimes M\right)w_r\right)^T(\mathbf{1}_{c_r}\otimes I_n) = 0.$$

By definition of w_r, $w_r^T\mathbf{1}_{nc_r} = 0$ and hence

$$\begin{aligned}
&\left(\left(I_{c_r}\otimes M\right)w_r\right)^T(\mathbf{1}_{c_r}\otimes I_n) = w_r^T\left(I_{c_r}\otimes M^T\right)(\mathbf{1}_{c_r}\otimes I_n) = w_r^T\left(\mathbf{1}_{c_r}\otimes M^T\right)\\
&\quad = \sum_{i=1}^{c_r}(X^i_{\mathscr{C}_r}-x_r)^TM^T = \left(\sum_{i=1}^{c_r}(X^i_{\mathscr{C}_r}-x_r)^T\right)M^T = 0.
\end{aligned}$$

Both $\mathcal{P}$ and $\mathcal{L}_{\mathscr{C}}$ are block diagonal with blocks of same sizes, $c_1, \ldots, c_K$, so we have:

$$-w^T\mathcal{P}(\mathcal{L}_{\mathscr{C}}\otimes D)w = -\sum_{r=1}^{K}w_r^T\left(I_{c_r}\otimes P^2\right)(\mathcal{L}_{\mathscr{C}_r}\otimes D)w_r \le -\sum_{r=1}^{K}\lambda^{(2)}_{\mathscr{C}_r}w_r^T\left(I_{c_r}\otimes P^2D\right)w_r.$$

Step 2. We show that

$$-\,w^T\mathcal{P}(\bar{\mathcal{L}}\otimes D)w \le -\sum_{r=1}^{K}\bar{\lambda}^{(2)}w_r^T\left(I_{c_r}\otimes P^2D\right)w_r\ . \tag{17}$$

The proof is analogous to the previous step.

$$\begin{aligned}
-w^T\mathcal{P}(\bar{\mathcal{L}}\otimes D)w &= -\,w^T\left(I_N\otimes P^2\right)(\bar{\mathcal{L}}\otimes D)w\\
&= -\,w^T\left(\bar{\mathcal{L}}\otimes P^2D\right)w
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2} w^T \left(\bar{\mathcal{L}} \otimes \left(P^2 D + D P^2\right)\right) w \\
&= -w^T \left(\bar{\mathcal{L}} \otimes M^T M\right) w \\
&= -w^T \left(I_N \otimes M^T\right) \left(\bar{\mathcal{L}} \otimes I_n\right) \left(I_N \otimes M\right) w \\
&\leq -\bar{\lambda}^{(2)} w^T \left(I_N \otimes M^T\right) \left(I_N \otimes M\right) w \\
&= -\bar{\lambda}^{(2)} w^T \left(I_N \otimes M^T M\right) w \\
&= -\bar{\lambda}^{(2)} w^T \left(I_N \otimes P^2 D\right) w \\
&= -\sum_{r=1}^{K} \bar{\lambda}^{(2)} w_r^T \left(I_{c_r} \otimes P^2 D\right) w_r .
\end{aligned}
$$

Step 3. We show that

$$
\begin{aligned}
w^T \mathcal{P}(\mathcal{F}(X,t) - \mathcal{F}(\bar{X},t)) = &\sum_{r=1}^{K} \sum_{i=1}^{c_r} \int_0^1 (X_{\mathscr{C}_r}^i - x_r)^T P^2 J_{F_{\mathscr{C}_r}} \left(x_r + \tau (X_{\mathscr{C}_r}^i - x_r)\right) \\
&\times (X_{\mathscr{C}_r}^i - x_r) \, \mathrm{d}\tau. \qquad (18)
\end{aligned}
$$

Note that $w^T \mathcal{P}(\mathcal{F}(X,t) - \mathcal{F}(\bar{X},t)) = \sum_{r=1}^{K} w_r^T \left(I_{c_r} \otimes P^2\right) \tilde{\mathcal{F}}_r(X_{\mathscr{C}_r})$, where

$$
\tilde{\mathcal{F}}_r(X_{\mathscr{C}_r}) = \left(F_{\mathscr{C}_r}^T(X_{\mathscr{C}_r}^1, t) - F_{\mathscr{C}_r}^T(x_r, t), \ldots, F_{\mathscr{C}_r}^T(X_{\mathscr{C}_r}^{c_r}, t) - F_{\mathscr{C}_r}^T(x_r, t)\right)^T .
$$

By the Mean Value Theorem for integrals, for any $r = 1, \ldots, K$,

$$
\begin{aligned}
w_r^T \left(I_{c_r} \otimes P^2\right) \tilde{\mathcal{F}}_r(X_{\mathscr{C}_r}) &= \sum_{i=1}^{c_r} (X_{\mathscr{C}_r}^i - x_r)^T P^2 \left(F_{\mathscr{C}_r}(X_{\mathscr{C}_r}^i, t) - F_{\mathscr{C}_r}(x_r, t)\right) \\
&= \sum_{i=1}^{c_r} \int_0^1 (X_{\mathscr{C}_r}^i - x_r)^T P^2 J_{F_{\mathscr{C}_r}} \left(x_r + \tau (X_{\mathscr{C}_r}^i - x_r)\right) \\
&\quad \times (X_{\mathscr{C}_r}^i - x_r) \, \mathrm{d}\tau.
\end{aligned}
$$

Adding over r, $r = 1, \ldots, K$, we obtain Eq. (18).

Note that the sum of the left-hand side of Eqs. (16)–(18), is equal to $\frac{d\Phi}{dt}$. Combining Steps 1–3, we have shown that

$$
\frac{\mathrm{d}\Phi}{\mathrm{d}t} \leq \sum_{r=1}^{K} \phi_r ,
$$

where for any $r = 1, \ldots, K$,

$$\begin{aligned}
\phi_r &:= w_r^T \left(I_{c_r} \otimes P^2\right) \tilde{\mathcal{F}}_r(X_{\mathscr{C}_r}) - w_r^T \left(I_{c_r} \otimes P^2\right) \left(I_{c_r} \otimes \lambda_{\mathscr{C}_r}^{(2)} D\right) w_r \\
&\quad - w_r^T \left(I_{c_r} \otimes P^2\right) \left(I_{c_r} \otimes \bar{\lambda}^{(2)} D\right) w_r \\
&= \sum_{i=1}^{c_r} \int_0^1 (X_{\mathscr{C}_r}^i - x_r)^T P^2 \left[J_{F_{\mathscr{C}r}} \left(x_r + \tau(X_{\mathscr{C}_r}^i - x_r)\right) - \lambda_{\mathscr{C}_r}^{(2)} D - \bar{\lambda}^{(2)} D\right] \\
&\quad (X_{\mathscr{C}_r}^i - x_r) \, \mathrm{d}\tau \\
&\le \sum_{i=1}^{c_r} \frac{2\mu}{2} \int_0^1 (X_{\mathscr{C}_r}^i - x_r)^T P^2 (X_{\mathscr{C}_r}^i - x_r) \, \mathrm{d}\tau \\
&= \frac{2\mu}{2} \, w_r^T \left(I_{c_r} \otimes P^2\right) w_r \, .
\end{aligned} \tag{19}$$

The inequality holds by applying Lemma 2 to Eq. (13): we obtain, for any $r = 1, \ldots, K$, and any $(x, t) \in V \times [0, \infty)$,

$$P^2 \left[J_{F_{\mathscr{C}r}}(x, t) - \left(\lambda_{\mathscr{C}_r}^{(2)} + \bar{\lambda}^{(2)}\right) D\right] + \left[J_{F_{\mathscr{C}r}}^T(x, t) - \left(\lambda_{\mathscr{C}_r}^{(2)} + \bar{\lambda}^{(2)}\right) D\right] P^2 \le 2\mu P^2.$$

Summing both sides of Eq. (19) over r, for $r = 1, \ldots, K$, we obtain the desired result, $\frac{d\Phi}{dt}(w) \le 2\mu\Phi(w)$. □

4 Applications and Numerical Examples

In this section, we apply Theorem 1 to two types of nonlinear neuronal oscillator dynamics: FitzHugh–Nagumo dynamics and Hindmarsh–Rose dynamics. We then present numerical simulations for heterogeneous networks that include nodal dynamics of both types. In a second numerical example we show partial cluster synchronization, which results when γ takes an intermediate value below the bound.

4.1 Application to Networks Of Heterogeneous FitzHugh–Nagumo Neuronal Oscillators

Here, we apply Theorem 1 to a network of N FitzHugh–Nagumo (FN) neuronal oscillators with graph $\mathcal{G}$. Let $(y^i, z^i)^T \in \mathbb{R}^2$ be the state of oscillator i and I^i be the external input to oscillator i, for $i = 1, \ldots, N$. y^i and z^i represent the membrane potential and the recovery variable, respectively. The input current for oscillator i is I^i. The FN dynamics are

$$\begin{aligned}
\dot{y}^i &= f^i(y^i) - z^i + I^i + \gamma \sum_{j \in \mathcal{N}^i} \gamma^{ij}(y^j - y^i), \\
\dot{z}^i &= \epsilon^i (y^i - b^i z^i),
\end{aligned} \tag{20}$$

where f^i is a cubic function, $f^i(y) = y - \frac{y^3}{3} - a^i$, $\gamma > 0, a^i > 0, 0 < b^i < 1$, $0 < \epsilon^i \ll 1$ are constant, and $\mathcal{N}^i$ denotes the set of all the neighbors of node i in the network. In the FN model, ϵ^i represents the time-scale separation between y^i and z^i, which affects oscillation frequency. The model parameter b^i controls the shape of the spike by changing the ratio of the duration of the spike to the refractory period. Using the notation of Theorem 1, $n = 2$, $X^i = (y^i, z^i)^T$, $F^i(X^i, t) = \left(f^i(y^i) - z^i + I^i, \epsilon^i(y^i - b^i z^i)\right)^T$, $D = \text{diag}\,(\gamma, 0)$ is the diffusion matrix, and the γ^{ij} are the edge weights on the graph $\mathcal{G}$.

Assume that there exist $K \geq 1$ clusters $\mathscr{C}_1, \ldots, \mathscr{C}_K$ of FN oscillators such that $a^i = a_{\mathscr{C}_r}$, $b^i = b_{\mathscr{C}_r}$, $\epsilon^i = \epsilon_{\mathscr{C}_r}$, and $I^i = I_{\mathscr{C}_r}$ for all FN oscillators $i \in \mathscr{C}_r$ and all clusters $r = 1, \ldots, K$.

In what follows we show that, for $K = 1$ cluster, if $\gamma\lambda^{(2)} > 1$, then Eq. (20) synchronizes, and, more generally, if $K > 1$, and for all $r = 1, \ldots, K$, $\epsilon_{\mathscr{C}_r} = \epsilon$, and $\gamma\left(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}\right) > 1$, then Eq. (20) converges to its K-cluster synchronization manifold.

Corollary 1 *Consider Eq. (20), with Assumption 1. For all $r = 1, \ldots, K$, let*

$$\gamma > \frac{1 + \alpha_r}{\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}},$$

where $\alpha_r = \frac{(\epsilon_{\mathscr{C}_r} p - 1/p)^2}{4 b_{\mathscr{C}_r} \epsilon_{\mathscr{C}_r}}$ and $p > 0$ constant. Then for any pair of FN oscillators $\{(y^i, z^i)^T, (y^j, z^j)^T\}$ such that $(i, j) \in \mathscr{C}_r$,

$$y^i(t) - y^j(t) \to 0, \quad z^i(t) - z^j(t) \to 0, \quad \text{as } t \to \infty.$$

In particular, if $p = \max_r \frac{1}{\sqrt{\epsilon_{\mathscr{C}_r}}}$, then α_r is minimized.

Proof To apply Theorem 1, we find a positive definite matrix P such that $P^2 D + D P^2$ is positive semidefinite and

$$\mu := \max_r \sup_{(y,z)^T \in \mathbb{R}^2} \mu_{2,P}\left[J_{F_{\mathscr{C}_r}}(y, z) - \left(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}\right) D\right] < 0.$$

Let $P = \text{diag}\,(1, p)$ so that $P^2 D + D P^2 = \text{diag}\,(2\gamma, 0)$, which is positive semidefinite. Then

$$\begin{aligned} &\mu_{2,P}\left[J_{F_{\mathscr{C}_r}}(y, z) - \left(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}\right) D\right] \\ &= \mu_2\left[P\left(J_{F_{\mathscr{C}_r}}(y, z) - \left(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}\right) D\right) P^{-1}\right] \\ &= \lambda_{\max}\left[\begin{pmatrix} 1 - y^2 - \gamma\lambda^{(2)}_{\mathscr{C}_r} - \gamma\bar{\lambda}^{(2)} & \frac{\epsilon_{\mathscr{C}_r} p}{2} - \frac{1}{2p} \\ \frac{\epsilon_{\mathscr{C}_r} p}{2} - \frac{1}{2p} & -b_{\mathscr{C}_r}\epsilon_{\mathscr{C}_r} \end{pmatrix}\right]. \end{aligned} \tag{21}$$

To see this recall that $\mu_{2,P}[A] = \mu_2[PAP^{-1}]$, and, by Remark 1, $\mu_2[A] = \lambda_{\max}\left[\frac{A+A^T}{2}\right]$, where $\lambda_{\max}[B]$ denotes the largest eigenvalue of B. Note that the matrix shown in the second line, call it $\mathcal{B}$, is the symmetric part of $P\left(J_{F_{\mathscr{C}_r}}(y, z) - \left(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}\right) D\right) P^{-1}$. Standard calculations show that if $\gamma > \frac{1+\alpha_r}{\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}} \geq \frac{1}{\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}}$ then the trace and the determinant of $\mathcal{B}$ satisfy

$$\mathrm{Tr}[\mathcal{B}] = 1 - y^2 - \gamma\lambda^{(2)}_{\mathscr{C}_r} - \gamma\bar{\lambda}^{(2)} - b_{\mathscr{C}_r}\epsilon_{\mathscr{C}_r} < 0,$$
$$\mathrm{Det}[\mathcal{B}] = -b_{\mathscr{C}_r}\epsilon_{\mathscr{C}_r}\left(1 - y^2 - \gamma\lambda^{(2)}_{\mathscr{C}_r} - \gamma\bar{\lambda}^{(2)} + \alpha_r\right) > 0.$$

Therefore, $\lambda_{\max}[\mathcal{B}] < 0$ and Theorem 1 yields the desired result. □

In Corollary 1, the parameter γ can be interpreted as the diffusion matrix, D, that represents the overall strength of graph coupling. For a system of Fitzhugh–Nagumo oscillators, the sufficient condition depends on a parameter, ϵ, that controls the frequency of oscillations through the time-scale separation between the voltage variable and gating variable. In general, as the value of ϵ for a given cluster is increased, the value of γ needed to guarantee synchronization in that cluster is also increased. Furthermore, for values of ϵ in a biologically relevant range (0.02, 0.2), increasing the minimum ϵ over all clusters also increases the value of γ required for cluster synchronization. This indicates that systems with a lower frequency of oscillation synchronize more rapidly than those with higher oscillation frequencies. The other parameter that influences the sufficient condition for cluster synchronization is b that controls the ratio of the time over which the neuron is spiking to the refractory period. As this parameter is increased (for biologically realistic results, it is required to stay in the range (0, 1)), a smaller overall graph coupling is required to guarantee cluster synchronization for the entire network.

Remark 5 In Corollary 1:

1. If we assume that, for all $r = 1, \ldots, K$, $\epsilon_{\mathscr{C}_r} = \epsilon$, then $\alpha_r = 0$ and we obtain a smaller lower bound for γ, namely

$$\gamma > \frac{1}{\lambda^{(2)}_{\mathscr{C}_i} + \bar{\lambda}^{(2)}}.$$

2. Nondiagonal P does not give a smaller lower bound for γ. If P is not diagonal, the condition for positive determinant is quadratic in terms of γ. This contradicts the positiveness of γ and so cannot be used to improve the bound for diagonal P.
3. Theorem 1 does not require constant system parameters, so it can be used to derive an analogous condition for a network of FN oscillators with time-varying parameters.

Remark 6 In the previous work (Davison et al. 2016), we showed that for $K = 1$, if $\gamma \geq \frac{1+\epsilon+\beta^2/3}{\lambda^{(2)}_{\mathscr{C}_1}}$, where β is the ultimate bound for the y variable, then Eq. (20) synchronizes. By Corollary 1 we have found a smaller lower bound for γ, $\gamma > \frac{1}{\lambda^{(2)}_{\mathscr{C}_1}}$, that guarantees synchronization.

4.2 Application to Networks Of Heterogeneous Hindmarsh–Rose Neuronal Oscillators

Here, we apply Theorem 1 to a network of N two-dimensional modified Hindmarsh–Rose (HR) neuronal oscillators with graph $\mathcal{G}$. Let $(y^i, z^i)^T \in \mathbb{R}^2$ be the state of oscillator i for $i = 1, \ldots, N$. y^i and z^i represent the membrane potential and the recovery variable, respectively. The input current for oscillator i is I^i. The two-dimensional HR dynamics are

$$\begin{aligned} \dot{y}^i &= g^i(y^i) + z^i + I^i + \gamma \sum_{j \in \mathcal{N}^i} \gamma^{ij}(y^j - y^i), \\ \dot{z}^i &= \delta^i(1 - 5{y^i}^2 - z^i), \end{aligned} \tag{22}$$

where $g^i(y) = -{y^i}^3 + c^i {y^i}^2$, $\gamma, c^i > 0$, $0 < \delta^i \ll 1$ is a parameter that determines the time-scale separation between the fast and slow dynamics, and $\mathcal{N}^i$ denotes the set of all the neighbors of node i in the network. Using the notation of Theorem 1, $n = 2$, $X^i = (y^i, z^i)^T$, $F^i(X^i, t) = \left(g^i(y^i) + z^i + I^i, \delta^i(1 - 5{y^i}^2 - z^i)\right)^T$, $D = \mathrm{diag}\,(\gamma, 0)$ is the diffusion matrix, and the γ^{ij} are the edge weights on the graph $\mathcal{G}$.

Assume there exist $K \geq 1$ clusters $\mathscr{C}_1, \ldots, \mathscr{C}_K$ of HR oscillators such that $c^i = c_{\mathscr{C}_r}$, $\delta^i = \delta_{\mathscr{C}_r}$, and $I^i = I_{\mathscr{C}_r}$ for all HR oscillators $i \in \mathscr{C}_r$ and all clusters $r = 1, \ldots, K$.

Corollary 2 *Consider Eq. (22), under Assumption 1. For all $r = 1, \ldots, K$, let*

$$\gamma > \frac{1}{\lambda^{(2)}_{\mathscr{C}_i} + \bar{\lambda}^{(2)}} \max \left\{ \frac{-(2c_{\mathscr{C}_i} - 5)^2}{4(25\delta_{\mathscr{C}_i} p^2 - 3)} + \frac{1}{4\delta_{\mathscr{C}_i} p}, \frac{c^2_{\mathscr{C}_i}}{3} - \delta_{\mathscr{C}_i} \right\}, \tag{23}$$

where p is a constant that satisfies $0 < p < \sqrt{\frac{3}{25\delta_{\mathscr{C}_i}}}$. Then for any pair of HR oscillators $\{(y^i, z^i)^T, (y^j, z^j)^T\}$ such that $(i, j) \in \mathscr{C}_r$,

$$y^i(t) - y^j(t) \to 0, \quad z^i(t) - z^j(t) \to 0, \quad \textit{as } t \to \infty.$$

In particular, if $p = \max_r \frac{3}{5\delta_{\mathscr{C}_r}(5 + |2c_{\mathscr{C}_r} - 5|)}$, then the first argument of the max operator in Eq. (23) is minimized and takes value $\frac{(5 + |2c_{\mathscr{C}_r} - 5|)^2}{12}$.

Proof To apply Theorem 1, we find a positive definite matrix P such that $P^2 D + D P^2$ is positive semidefinite and

$$\mu := \max_r \sup_{(y,z)^T \in \mathbb{R}^2} \mu_{2,P} \left[J_{F_{\mathscr{C}_r}}(y, z) - \left(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}\right) D \right] < 0.$$

Let $P = \text{diag}\,(1,\,p)$ so that $P^2D+DP^2 = \text{diag}\,(2\gamma, 0)$, which is positive semidefinite. Then

$$\begin{aligned}
&\mu_{2,P}\left[J_{F_{\mathscr{C}\,r}}(y,z) - \left(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}\right)D\right] \\
&\quad = \mu_2\left[P\left(J_{F_{\mathscr{C}\,r}}(y,z) - \left(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}\right)D\right)P^{-1}\right] \\
&\quad = \lambda_{\max}\left[\begin{pmatrix} -3y^2 + 2c_{\mathscr{C}_r}y - \gamma\lambda^{(2)}_{\mathscr{C}_r} - \gamma\bar{\lambda}^{(2)} & \frac{1}{2p} - 5\delta_{\mathscr{C}_r}py \\ \frac{1}{2p} - 5\delta_{\mathscr{C}_r}py & -\delta_{\mathscr{C}_r} \end{pmatrix}\right].
\end{aligned} \tag{24}$$

We denote this matrix as $\mathcal{C}$.

If $\gamma(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}) > \frac{c^2_{\mathscr{C}\,r}}{3} - \delta_{\mathscr{C}_r}$, then we have

$$\gamma\left(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}\right) + \delta_{\mathscr{C}_r} > \frac{c^2_{\mathscr{C}_r}}{3} > \frac{c^2_{\mathscr{C}_r}}{3} - 3\left(y - \frac{c_{\mathscr{C}_r}}{3}\right)^2 = -3y^2 + 2c_{\mathscr{C}_r}y.$$

Therefore, the trace of $\mathcal{C}$ satisfies

$$\text{Tr}[\mathcal{C}] = -3y^2 + 2c_{\mathscr{C}_r}y - \gamma\lambda^{(2)}_{\mathscr{C}_r} - \gamma\bar{\lambda}^{(2)} - \delta_{\mathscr{C}_r} < 0.$$

Further, if $\gamma(\lambda^{(2)}_{\mathscr{C}_r} + \bar{\lambda}^{(2)}) > \frac{-(2c_{\mathscr{C}\,r}-5)^2}{4(25\delta_{\mathscr{C}\,r}p^2-3)} + \frac{1}{4\delta_{\mathscr{C}\,r}p}$, then, under the condition that $p^2 < \frac{3}{25\delta_{\mathscr{C}\,r}}$, the determinant of $\mathcal{C}$ satisfies

$$\text{Det}[\mathcal{C}] = -\delta_{\mathscr{C}_r}\left(-3y^2 + 2c_{\mathscr{C}_r}y - \gamma\lambda^{(2)}_{\mathscr{C}_r} - \gamma\bar{\lambda}^{(2)}\right) - \left(\frac{1}{2p} - 5\delta_{\mathscr{C}_r}py\right)^2 > 0.$$

Therefore, $\lambda_{\max}[\mathcal{C}] < 0$ and Theorem 1 yields the desired result. □

4.3 Numerical Examples

Example 1 In this example, we consider the network of 17 neuronal oscillators shown in the left panel of Fig. 1. This network can be grouped into three different clusters based on the individual nodal dynamics:

(i) Cluster $\mathscr{C}_1$ (orange circles): six FN oscillators; $a_{\mathscr{C}_1} = 0.5$, $b_{\mathscr{C}_1} = 0.1$, $I_{\mathscr{C}_1} = -2$, and $\epsilon_{\mathscr{C}_1} = 0.08$;
(ii) Cluster $\mathscr{C}_2$ (green squares): seven HR oscillators; $c_{\mathscr{C}_2} = 2$, $I_{\mathscr{C}_2} = 2$, and $\delta_{\mathscr{C}_2} = 0.02$;
(iii) Cluster $\mathscr{C}_3$ (blue triangles): four HR oscillators; $c_{\mathscr{C}_3} = 3$, $I_{\mathscr{C}_3} = 4$, and $\delta_{\mathscr{C}_3} = 0.01$.

The second smallest eigenvalues of the Laplacian of the three intra-cluster subgraphs and the inter-cluster subgraph are $\lambda^{(2)}_{\mathscr{C}_1} = 1.83$, $\lambda^{(2)}_{\mathscr{C}_2} = \lambda^{(2)}_{\mathscr{C}_3} = 2$, and $\bar{\lambda}^{(2)} = 0.262$, respectively. It follows directly from Corollaries 1 and 2 that the clusters will synchronize if γ satisfies the following inequality:

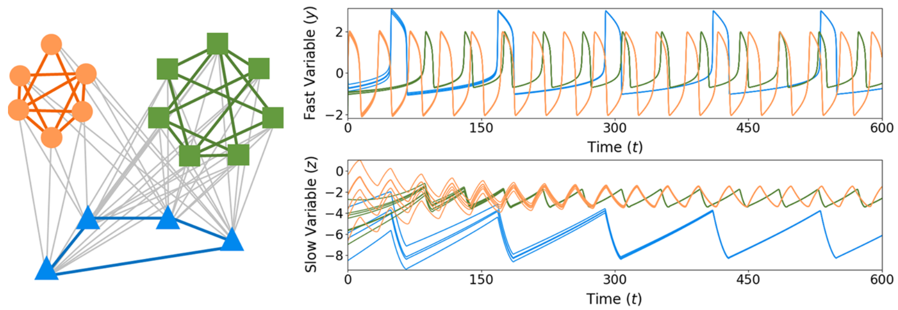

Fig. 1 Cluster synchronization in a network of 17 heterogeneous neuronal oscillators shown on the left: six FitzHugh–Nagumo oscillators (orange circles) with $a_{\mathscr{C}_1} = 0.5$, $b_{\mathscr{C}_1} = 0.1$, $I_{\mathscr{C}_1} = -2$, $\epsilon_{\mathscr{C}_1} = 0.08$, seven Hindmarsh–Rose oscillators (green squares) with $c_{\mathscr{C}_2} = 2$, $I_{\mathscr{C}_2} = 2$, $\delta_{\mathscr{C}_2} = 0.02$, and four Hindmarsh–Rose oscillators (blue triangles) with $c_{\mathscr{C}_3} = 3$, $I_{\mathscr{C}_3} = 4$, $\delta_{\mathscr{C}_3} = 0.01$. The states converge to the 3-cluster synchronization manifold for $\gamma = 4.7$ (Color figure online)

$$\gamma > \max_{p\in(0,\sqrt{6})} \left\{ \frac{1+31.25(\frac{p}{12.5}-\frac{1}{p})^2}{\lambda^{(2)}_{\mathscr{C}_1}+\bar{\lambda}^{(2)}}, \frac{\frac{12.5}{p}-\frac{1}{2p^2-12}}{\lambda^{(2)}_{\mathscr{C}_2}+\bar{\lambda}^{(2)}}, \frac{1.3133}{\lambda^{(2)}_{\mathscr{C}_2}+\bar{\lambda}^{(2)}}, \frac{\frac{25}{p}-\frac{1}{p^2-12}}{\lambda^{(2)}_{\mathscr{C}_3}+\bar{\lambda}^{(2)}}, \frac{2.99}{\lambda^{(2)}_{\mathscr{C}_3}+\bar{\lambda}^{(2)}} \right\}. \tag{25}$$

For $p = 2.4$, $\gamma > 4.6$ provides a sufficient condition for cluster synchronization. As shown in Fig. 1, the network indeed stabilizes to three synchronized clusters when $\gamma = 4.7$.

Example 2 In this example, we consider a large network of 200 FN oscillators illustrated in the left panel of Fig. 2. The network is obtained through interconnection of two clusters:

(i) Cluster $\mathscr{C}_1$ (magenta squares): A complete graph of 100 FN oscillators; $a_{\mathscr{C}_1} = 0.9$, $b_{\mathscr{C}_1} = 0.5$, $I_{\mathscr{C}_1} = 2.0$, and $\epsilon_{\mathscr{C}_1} = 0.08$;
(ii) Cluster $\mathscr{C}_2$ (green triangles): A star graph of 100 FN oscillators; $a_{\mathscr{C}_2} = 0.7$, $b_{\mathscr{C}_2} = 0.8$, $I_{\mathscr{C}_2} = 0.3$, and $\epsilon_{\mathscr{C}_2} = 0.08$.

Each node in the first cluster is connected to a unique node in the second cluster with coupling strength 0.25. Note that the cluster-input-equivalence condition holds in this case. For this network $\lambda^{(2)}_{\mathscr{C}_1} = 100$, $\lambda^{(2)}_{\mathscr{C}_2} = 1$ and $\bar{\lambda}^{(2)} = 0$. By choosing $\gamma = 0.02$ such that $\gamma > 1/\big(\lambda^{(2)}_{\mathscr{C}_1} + \bar{\lambda}^{(2)}\big)$ but $\gamma < 1/\big(\lambda^{(2)}_{\mathscr{C}_2} + \bar{\lambda}^{(2)}\big)$ we do not obey the sufficient condition. However, numerical simulation (Fig. 2) shows that the magenta cluster ($\mathscr{C}_1$) synchronizes nevertheless as suggested by the fact that $\gamma > 1/\big(\lambda^{(2)}_{\mathscr{C}_1}+\bar{\lambda}^{(2)}\big)$ is satisfied. Our future work will explore more along this direction.

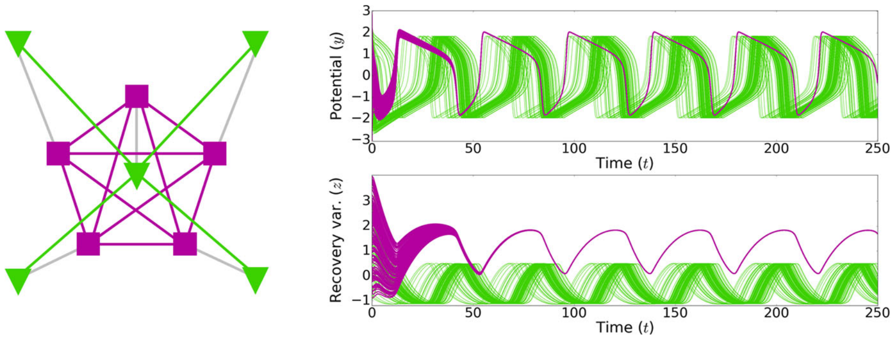

Fig. 2 Synchronization of only one (magenta) of two clusters in a large network of heterogeneous FN oscillators when the coupling strength takes an intermediate value. There are 100 oscillators in one cluster connected through a star graph (green triangles) and 100 oscillators in a second cluster connected through a complete graph (magenta squares). The network on the left illustrates the connections between clusters (in gray) in the case of 5 oscillators in each cluster (Color figure online)

5 Conclusion

In this paper, we consider the patterns of synchronization that emerge in networks where individual nodes may have different intrinsic nonlinear dynamics. We leverage the cluster-input-equivalence condition, developed by Stewart et al. (2003), Belykh et al. (2008) and extended with a useful graph-theoretical perspective in Schaub et al. (2016), to provide a starting framework for proving sufficient conditions for synchronization within clusters based on properties of the nodes and network structure. By adopting an approach based on contraction theory (Aminzare 2015), our work proves a new sufficient condition for cluster synchronization, and provides its characterization in terms of the intra-cluster network structure and the inter-cluster network structure. The inter-cluster network structure has not been explicitly used in previous works on finding sufficient conditions for cluster synchronization; our work improves on sufficient conditions by incorporating significantly more information about network structure.

Another key contribution of our work is an improvement on previous sufficient conditions for cluster synchronization (Davison et al. 2016) in networks with heterogeneous intrinsic dynamics. We have detailed an approach to finding sufficient conditions for synchronization independent of nonlinear model and network structure. However, the strict requirements imposed by studying complete synchronization within clusters that manifest in the cluster-input-equivalence condition limit the amount of heterogeneity in the nodal dynamics and asymmetry in the network that can be addressed. Future generalizations of our results should include relaxations of the complete synchronization requirement which would allow for more complex and realistic network configurations. A concrete first relaxation would be to combine the robustness result for contracting systems, as in Pham and Slotine (2007), with the results from this work to study a system perturbed from a cluster-input-equivalence state by Brownian noise.

Acknowledgements This work was jointly supported by the National Science Foundation under NSF-CRCNS grant DMS-1430077 and the Office of Naval Research under ONR Grant N00014-14-1-0635. This material is also based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant DGE-1656466. Any opinion, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. The authors thank the anonymous reviewers for their thoughtful and detailed comments.

References

Abrams, D.M., Pecora, L.M., Motter, A.E.: Introduction to focus issue: patterns of network synchronization. Chaos **26**(9), 094601 (2016)

Aminzare, Z.: On Synchronous Behavior in Complex Nonlinear Dynamical Systems. Dissertation, Rutgers University, The State University of New Jersey (2015)

Aminzare, Z., Sontag, E.D.: Contraction methods for nonlinear systems: A brief introduction and some open problems. In: Proceedings of the 53rd IEEE Conference on Decision and Control (CDC), pp. 3835–3847 (2014a)

Aminzare, Z., Sontag, E.D.: Synchronization of diffusively-connected nonlinear systems: results based on contractions with respect to general norms. IEEE Trans. Netw. Sci. Eng. **1**(2), 91–106 (2014b)

Aminzare, Z., Shafi, Y., Arcak, M., Sontag, E.D.: Guaranteeing spatial uniformity in reaction-diffusion systems using weighted L_2 -norm contractions. In: Kulkarni, V., Stan, G.-B., Raman, K. (eds.) A Systems Theoretic Approach to Systems and Synthetic Biology I: Models and System Characterizations, pp. 73–101. Springer, Berlin (2014)

Arcak, M.: Certifying spatially uniform behavior in reaction-diffusion PDE and compartmental ODE systems. Automatica **47**(6), 1219–1229 (2011)

Belykh, V.N., Osipov, G.V., Petrov, V.S., Suykens, J.A.K., Vandewalle, J.: Cluster synchronization in oscillatory networks. Chaos **18**(3), 037106 (2008)

Belykh, V.N., Petrov, S., Osipov, G.V.: Dynamics of the finite-dimensional Kuramoto model: global and cluster synchronization. Regular Chaotic Dyn. **20**(1), 37–48 (2015)

Brown, E., Holmes, P., Moehlis, J.: Globally coupled oscillator networks. In: Perspectives and Problems in Nonlinear Science: A Celebratory Volume in Honor of Larry Sirovich, pp 183–215 (2003)

Chen, C.C., Litvak, V., Gilbertson, T., Kühn, A., Lu, C.S., Lee, S.T., Tsai, C.H., Tisch, S., Limousin, P., Hariz, M., et al.: Excessive synchronization of basal ganglia neurons at 20 Hz slows movement in Parkinson's disease. Exp. Neurol. **205**(1), 214–221 (2007)

Chow, C.C., Kopell, N.: Dynamics of spiking neurons with electrical coupling. Neural Computat. **12**(7), 1643–1678 (2000)

Chung, S.-J., Slotine, J.-J.E., Miller, D.W.: Nonlinear model reduction and decentralized control of tethered formation flight. J. Guid. Control Dyn. **30**(2), 390–400 (2007)

Dahlquist, G.: Stability and Error Bounds in the Numerical Integration of Ordinary Differential Equations. Inaugural dissertation, University of Stockholm, Almqvist & Wiksells Boktryckeri AB, Uppsala (1958)

Davison, E.N., Dey, B., Leonard, N.E.: Synchronization bound for networks of nonlinear oscillators. In: Proceedings of the 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pp 1110–1115 (2016)

Demidovič, B.P.: On the dissipativity of a certain non-linear system of differential equations. I. Vestnik Moskov University Series in Mathematical. Methods **1961**(6), 19–27 (1961)

Demidovič, B.P.: Lektsii po matematicheskoi teorii ustoichivosti. Izdat. Nauka, Moscow (1967)

Desoer, C.A., Vidyasagar, M.: Feedback Systems: Input-Output Properties. Electrical Science. Academic Press [Harcourt Brace Jovanovich, Publishers], New York (1975)

Dumas, G., Nadel, J., Soussignan, R., Martinerie, J., Garnero, L.: Inter-brain synchronization during social interaction. PLoS ONE **5**(8), 1–10 (2010)

Favaretto, C., Bassett, D.S., Cenedese, A., Pasqualetti, F.: Bode meets kuramoto: synchronized clusters in oscillatory networks. In: Proceedings of American Control Conference (ACC), pp. 2799–2804 (2017a)

Favaretto, C., Cenedese, A., Pasqualetti, F.: Cluster Synchronization in Networks of Kuramoto Oscillators. In: Proceedings of the IFAC 2017 World Congress, pp. 2485–2490 (2017b)

Ferreira, A.S.R., Arcak, M.: A graph partitioning approach to predicting patterns in lateral inhibition systems. SIAM J. Appl. Dyn. Syst. **12**(4), 2012–2031 (2013)

Fiore, D., Russo, G., di Bernardo, M.: Exploiting nodes symmetries to control synchronization and consensus patterns in multiagent systems. IEEE Control Syst. Lett. **1**(2), 364–369 (2017)

Golubitsky, M., Stewart, I., Török, A.: Patterns of synchrony in coupled cell networks with multiple arrows. SIAM J. Appl. Dyn. Syst. **4**(1), 78–100 (2005)
Hartman, P.: On stability in the large for systems of ordinary differential equations. Can. J. Math. **13**, 480–492 (1961)
Horn, R.A., Johnson, C.R.: Topics in Matrix Analysis. Cambridge University Press, Cambridge (1991)
Jouffroy, J.: Some ancestors of contraction analysis. In: Proceedings of the 44th IEEE Conference on Decision and Control 2005 and European Control Conference 2005, pp. 5450–5455 (Dec 2005)
Lehnertz, K., Bialonski, S., Horstmann, M.-T., Krug, D., Rothkegel, A., Staniek, M., Wagner, T.: Synchronization phenomena in human epileptic brain networks. J. Neurosci. Methods **183**(1), 42–48 (2009)
Lewis, D.C.: Metric properties of differential equations. Am. J. Math. **71**, 294–312 (1949)
Lohmiller, W., Slotine, J.-J.E.: On contraction analysis for non-linear systems. Automatica **34**(6), 683–696 (1998)
Lohmiller, W., Slotine, J.: Contraction analysis of nonlinear distributed systems. Int. J. Control **78**, 678–688 (2005)
Lu, W., Liu, B., Chen, T.: Cluster synchronization in networks of coupled nonidentical dynamical systems. Chaos **20**(1), 013120 (2010)
MacLeod, K., Laurent, G.: Distinct mechanisms for synchronization and temporal patterning of odor-encoding neural assemblies. Science **274**(5289), 976–979 (1996)
Mirollo, R.E., Strogatz, S.H.: Synchronization of pulse-coupled biological oscillators. SIAM J. Appl. Math. **50**(6), 1645–1662 (1990)
Motter, A.E., Myers, S.A., Anghel, M., Nishikawa, T.: Spontaneous synchrony in power-grid networks. Nature Phys. **9**, 191–197 (2013)
Nair, S., Leonard, N.E.: Stable synchronization of mechanical system networks. SIAM J. Control Optim. **47**(2), 661–683 (2008)
Orosz, G., Moehlis, J., Ashwin, P.: Designing the dynamics of globally coupled oscillators. Prog. Theor. Phys. **122**(3), 611–630 (2009)
Pavlov, A., Pogromvsky, A., van de Wouv, N., Nijmeijer, H.: Convergent dynamics, a tribute to Boris Pavlovich Demidovich. Syst. Control Lett. **52**, 257–261 (2004)
Pecora, L.M., Sorrentino, F., Hagerstrom, A.M., Murphy, T.E., Roy, R.: Cluster synchronization and isolated desynchronization in complex networks with symmetries. Nat. Commun. **5**, 4079 (2014)
Pham, Q.-C., Slotine, J.-J.: Stable concurrent synchronization in dynamic system networks. Neural Netw. **20**(1), 62–77 (2007)
Pikovsky, A., Rosenblum, M., Kurths, J.: Synchronization: A Universal Concept in Nonlinear Sciences, volume 12 of Cambridge Nonlinear Science Series. Cambridge University Press, Cambridge (2003)
Russo, G., Di Bernardo, M.: Contraction theory and master stability function: linking two approaches to study synchronization of complex networks. IEEE Trans. Circuits Syst. **56**(2), 177–181 (2009)
Russo, G., Slotine, J.-J.E.: Global convergence of quorum-sensing networks. Phys. Rev. E **82**(4), 041919 (2010)
Schaub, M.T., O'Clery, N., Billeh, Y.N., Delvenne, J.-C., Lambiotte, R., Barahona, M.: Graph partitions and cluster synchronization in networks of oscillators. Chaos Interdiscip. J. Nonlinear Sci. **26**(9), 094821 (2016)
Sepulchre, R., Paley, D., Leonard, N.E.: Stabilization of planar collective motion with limited communication. IEEE Trans. Autom. Control **53**(3), 706–719 (2008)
Sivrikaya, F., Yener, B.: Time synchronization in sensor networks: a survey. IEEE Netw. **18**(4), 45–50 (2004)
Smith, H.M.: Synchronous flashing of fireflies. Science **82**(2120), 151–152 (1935)
Soderlind, G.: The logarithmic norm. History and modern theory. BIT Numer. Math. **46**(3), 631–652 (2006)
Sorrentino, F., Ott, E.: Network synchronization of groups. Phys. Rev. E **76**(5), 056114 (2007)
Sorrentino, F., Pecora, L.: Approximate cluster synchronization in networks with symmetries and parameter mismatches. Chaos **26**(9), 094823 (2016)
Sorrentino, F., Pecora, L.M., Hagerstrom, A.M., Murphy, T.E., Roy, R.: Complete characterization of the stability of cluster synchronization in complex dynamical networks. Sci. Adv. **2**(4), e1501737 (2016)
Stewart, I., Golubitsky, M., Pivato, M.: Symmetry groupoids and patterns of synchrony in coupled cell networks. SIAM J. Appl. Dyn. Syst. **2**(4), 609–646 (2003)
Tiberi, L., Favaretto, C., Innocenti, M., Bassett, D.S., Pasqualetti, F.: Synchronization patterns in networks of Kuramoto oscillators: A geometric approach for analysis and control. In: Proceedings of the 56th IEEE Conference on Decision and Control (CDC), pp. 481–486 (2017)

Wang, W., Slotine, J.J.E.: On partial contraction analysis for coupled nonlinear oscillators. Biol. Cybern. **92**, 38–53 (2005)

Wang, K., Fu, X., Li, K.: Cluster synchronization in community networks with nonidentical nodes. Chaos **19**(2), 023106 (2009)

Wilson, D., Moehlis, J.: Clustered desynchronization from high-frequency deep brain stimulation. PLoS Comput. Biol. **11**(12), e1004673 (2015)

Winfree, A.T.: Biological rhythms and the behavior of populations of coupled oscillators. J. Theor. Biol. **16**(1), 15–42 (1967)

Xia, W., Cao, M.: Clustering in diffusively coupled networks. Automatica **47**(11), 2395–2405 (2011)

Yoshizawa, T.: Stability Theory by Liapunov's Second Method. Publications of the Mathematical Society of Japan, No. 9. The Mathematical Society of Japan, Tokyo (1966)

Yoshizawa, T.: Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions. Applied Mathematical Sciences, vol. 14. Springer, New York (1975)