Induction Proofs

Class/Discussion Section Activities -Homework

There are many problems in which you want to verify that a certain pattern or formula is correct 'for all numbers'. Below are four examples.

Examples that arise in geometry such as the figurate number patterns:

1. Triangular numbers and other problems
Sum of the first n natural numbers
Verify that

$$1 + 2 + 3 + \ldots + n = n * (n + 1) / 2$$

is correct where n is **any** natural number 1, 2, 3, 4, …

Here we can call the expression
$1 + 2 + 3 + \ldots + n$
the *left* hand side of the equation and
$n * (n + 1) / 2$
the *right* hand side of the equation.

2. Pentagonal numbers

Verify that

$$1 + 4 + 7 + 10 + \ldots + (1 + 3*(n - 1)) = (1/2)\, n * (3n - 1)$$

is correct where n is **any** natural number 1, 2, 3, 4, …

Here we can call the expression
$1 + 4 + 7 + 10 + \ldots + (1 + 3*(n - 1))$
the *left* hand side of the equation and
$(1/2)\, n * (3n - 1)$
the *right* hand side of the equation.

Examples that arise in number patterns:

3. Verify that

" $11^n - 3^n$ is divisible by 8 "

is correct where n is **any** natural number 1, 2, 3, 4, …

If we replace the statement is “divisible by 8” by “ = 8*m for some natural number m”, then

" $11^n - 3^n$ is divisible by 8 " can be written as an equation:

" $11^n - 3^n = 8*m$ for some natural number *m* "

And the original statement problem-

Verify that

" $11^n - 3^n$ is divisible by 8 "

is correct where *n* is **any** natural number 1, 2, 3, 4, …

can be rewritten as the problem:
for any natural number *n* there is a natural number *m* such that

$11^n - 3^n = 8*m$ for some natural number *m*

Here we can call the expression
$11^n - 3^n$
the *left* hand side of the equation and
$8*m$ (for some natural number *m*)
the *right* hand side of the equation.

Examples that arise in number patterns:
4. Sum of the squares of the first n natural numbers:

Verify that

$$1^2 + 2^2 + 3^2 + 4^2 \ldots + n^2 = n * (n + 1)*(2 n + 1)/6$$

is correct where n is **any** natural number 1, 2, 3, 4, …

(if n is less than 4 you stop the summing on the left side at that n)

Here we can call the expression

$$1^2 + 2^2 + 3^2 + 4^2 \ldots + n^2$$

the *left* hand side of the equation and

$$n * (n + 1)*(2 n + 1)/6$$

the *right* hand side of the equation.

In all these examples, and many others, you have a formula or statement that involves an 'unknown' natural number labeled n,

Example: Verify $1 + 2 + 3 + \ldots + n = n * (n + 1) / 2$

and your task is to verify that the formula is correct no matter what the natural number n is (n could be 1, 2, 3, 4, … or any other natural number).

ONE WAY TO COMPLETE THIS TASK IN A MATHEMATICALLY RIGOROUS WAY IS TO DO THE FOLLOWING TWO STEPS CORRECTLY. COMPLETING THESE TWO STEPS CORRECTLY IS CALLED
PROOF BY INDUCTION.
The reason why completing these two steps correctly does constitute a mathematically rigorous argument is discussed later in this document.

STEP 1. Verify 'by hand' or by 'direct computation' or algebra or arithmetic or geometry that when you when you plug in 1 for n you get a valid statement or a correct equation.

So in **STEP 1** you are checking the correctness of the hypothetically correct equation or statement for the specific value of $n = 1$ and not worrying about letting n stand for all possible natural numbers.

For an equation this means checking that when you plug in 1 for n into the expression on the **left** side of the equation, you get the same result as when you plug in 1 for n into the expression on the **right** side of the equation.

Example 1: $1 + 2 + 3 + \ldots + n = n * (n + 1) / 2$

Plug in 1 for n on the left you get 1. That is because you are supposed to assume that to do the computation on the left, you add natural numbers until you get to the last number *n*. When $n = 1$, the whole sum is just the number 1. This kind of assumption might not have been clear from the notation : $1 + 2 + 3 + \ldots + n$ but this is the conventional understanding of the notation when $n = 1$.

Plug in 1 for *n* on the right you get $1* (1 +1) / 2 = 1$.

So the left and right sides both equal 1 and hence agree so you know

$1 + 2 + 3 + \ldots + n = n * (n + 1) / 2$

is correct when $n = 1$.

For a problem involving a statement instead of an equation with '*n*'s on both sides, this STEP 1 means that the when you when you plug in 1 for *n* you get a valid statement.

Example $11^n - 3^n$ is divisible by 8 for every natural number $n =1, 2, 3, \ldots$

Plug in 1 for n and this statement becomes:
$11^1 - 3^1$ is divisible by 8 .
Since $11^1 - 3^1 = 8$, and 8 is divisible by 8, this statement is correct when $n = 1$.

Note: it's ok to check the equation or statement is also true for other specific values of *n* like $n = 2, 3, 56,..$ or any other choices. But the only one you are *required* to check is $n = 1$ (the 'starting' number) if the conclusion is going to be a true statement for *all* natural numbers.

STEP 2. *Assume* that the hypothetically correct equation or statement *is* correct with the given natural number *n*. This is called the *induction hypothesis.*

Now take *just the left hand side* of the hypothetically correct equation and *replace all occurrences of n with n + 1.* Call this the *new* left hand side.

Then take *just the right hand side* of the hypothetically correct equation and *replace all occurrences of n with n + 1.* Call this the *new* right hand side.

Finally, verify that this *new* left hand side is equal to this *new* right hand side by using algebra, geometry, arithmetic and/or any other mathematically correct operation or operations. Usually the verification that the *new* left hand side is equal to the *new* right hand side will in some way use the induction hypothesis.

STEP 2.

Example : $1 + 2 + 3 + \ldots + n = n * (n + 1) / 2$

Assume the above equation *is correct* (this is called the induction hypothesis)

Now take just the left hand side of the hypothetically correct equation and replace all occurrences of n with n + 1. Call this the new left hand side.

So the left hand side

$1 + 2 + 3 + \ldots + n$ becomes $1 + 2 + 3 + \ldots + (n + 1)$

This new sum starts at 1 and adds up all the consecutive numbers up to n+1.

Now take just the right hand side of the hypothetically correct equation and replace all occurrences of n with n + 1. Call this the new right hand side.

$n * (n + 1) / 2$ becomes

$(n + 1) * ((n + 1) + 1) / 2$

or

$(n + 1) * (n + 2) / 2$

We want to try to verify that $1 + 2 + 3 + \ldots + (n + 1)$ is equal $(n + 1) * (n + 2) / 2$.

We can assume

$1 + 2 + 3 + \ldots + n = n * (n + 1) / 2$

is correct by the induction hypothesis.

We now go about using some algebra and the induction hypothesis to verify (*).

(*) $1 + 2 + 3 + \ldots + (n + 1)$

$= 1 + 2 + 3 + \ldots + n + (n + 1)$

$= [\ 1 + 2 + 3 + \ldots + n\ ] + (n + 1)$

we can now substitute $n * (n + 1) / 2$ for $[\ 1 + 2 + 3 + \ldots + n\ ]$ b/c of the induction hypothesis, so (*) becomes:

(*) $1 + 2 + 3 + \ldots + (n + 1)$

$= [\ 1 + 2 + 3 + \ldots + n\ ] + (n + 1)$

$= n * (n + 1) / 2 + (n + 1)$ (do you see why this last step is correct?)

We have now shown that the new left-hand side
$1 + 2 + 3 + \ldots + (n + 1)$ is the same as $n * (n + 1) / 2 + (n + 1)$

Finally do some algebra on this last expression

$n * (n + 1) / 2 + (n + 1)$ (which we know is the same as the new left hand side).

We will factor out the (n+1) to the right in the expression $n * (n + 1) / 2 + (n + 1)$

$= [n / 2 + 1] (n + 1)$

$= [(n + 2) / 2] (n + 1) =$

$(n + 1) * (n + 2) / 2$

This is exactly the new right hand side!

So we have now concluded

(*) $1 + 2 + 3 + \ldots + (n + 1) = (n + 1) * (n + 2) / 2$

using the induction hypothesis! We have successfully completed step 2 of the induction proof and can now assert that

$1 + 2 + 3 + \ldots + n = n * (n + 1) / 2$

is correct where *n* is any natural number 1, 2, 3, 4, …

Below is **STEP 2** for another example.

Assume

$11^n - 3^n$ is divisible by 8

is a correct statement (this is the induction hypothesis). You can check by 'hand' that this statement is correct when $n = 1$ (that is step 1 of the induction argument).

To do step 2 for this statement we proceed as instructed above. *Take just the left hand side of the hypothetically correct equation and replace all occurrences of n with n + 1. Call this the new left hand side.* The left hand side in this example is

$$11^n - 3^n$$

So replacing *all occurrences of n with n + 1 in the expression*
$11^n - 3^n$
means we replace $11^n - 3^n$

with $11^{n+1} - 3^{n+1}$

We now want to verify that the following statement is true:

$11^{n+1} - 3^{n+1}$ is divisible by 8 [call this statement (**)].

We will verify that this *new* equation or statement *(with the n + 1) is correct* using the induction hypothesis, together with a 'by hand' or by 'direct computation' using some algebra and arithmetic.

We now go about using some algebra and the induction hypothesis to verify (**).

$$(**)\ 11^{n+1} - 3^{n+1}$$

$$= 11^n * 11 - 3^n * 3$$

now replace the second "11" with 8 + 3

(we might think of doing this b/c we want to check for multiples of 8 so this is a way to 'get' another 8 into this expression)

$$= 11^n * (8 + 3) - 3^n * 3$$

$$= 11^n * 8 + 11^n * 3 - 3^n * 3$$

$$= 11^n * 8 + (11^n - 3^n) * 3$$

Now by the induction hypothesis, $(11^n - 3^n)$ is divisible by 8.

So it is also true that $(11^n - 3^n) * 3$ is divisible by 8.

And the first expression $11^n * 8$ is divisible by 8.

Therefore the sum

$$11^n * 8 + (11^n - 3^n) * 3$$

$$= 11^{n+1} - 3^{n+1}$$

is also divisible by 8.

We have successfully completed step 2 of the induction proof and can now assert that

$11^n - 3^n$ is divisible by 8 for every natural number n =1, 2, 3, …

In **STEP 2** you are checking that if you know the equation or statement is correct for a given natural number called *n*, then the equation or statement is automatically true for the *next* natural number $n + 1$.

Successful completion of **STEP 1** and **STEP 2** together give a proof 'by induction' that the equation or statement is correct for *all* natural numbers called $n = 1, 2, 3$,..

EXERCISES: VERIFY THAT ALL THE OTHER EXAMPLES ARE CORRECT USING THIS PROOF BY INDUCTION METHOD.

ADDITIONAL DISCUSSION OF THE MATHEMATICAL ISSUES INVOLVED IN PROOFS BY INDUCTION-WHY DO THEY WORK?

We stated earlier that in all the above examples, verifying that any of the above statements like

$1 + 2 + 3 + \ldots + n = n * (n + 1) / 2$ for every natural number n =1, 2, 3, …

Example $11^n - 3^n$ is divisible by 8 for every natural number n =1, 2, 3, …

$1 + 4 + 7 + 10 + \ldots + (1 + 3*(n - 1)) = (1/2)\, n * (3n - 1)$ for every natural number n =1, 2, 3, …

$2 + 4 + 6 + 8 + \ldots + (2 * n) = n * (n + 1)$ for every natural number n =1, 2, 3, …

are correct for all natural numbers *n* can be accomplished as follows

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The reason why completing these two steps correctly does constitute a mathematically rigorous argument is discussed later in this document.

Here is the discussion. The way one constructs the natural numbers N = { 1, 2, 3, …} guarantees that the following Principle of Induction is true:

PRINCIPLE OF INDUCTION

Suppose S is a subset of the natural numbers N = { 1, 2, 3, …}, $S \subseteq N$.
Suppose S has the following two properties.

Property 1. 1 is an element of S, i.e. $1 \in S$ and

Property 2. Whenever a natural number *n* is in S, $n \in S$, then it is automatically true that the next natural number $n+1$ is *also* in S. In other words, $n \in S$ implies $n + 1 \in S$.

THEN S = N. That is, *S must contain every natural number 1,2,3,…*

Recall the arguments verifying that

$$1 + 2 + 3 + \ldots + n = n * (n + 1) / 2 \text{ for every natural number } n = 1, 2, 3, \ldots$$

and

$$11^n - 3^n \text{ is divisible by 8 for every natural number } n = 1, 2, 3, \ldots$$

What we did there was to verify, in the two steps, that Property 1 and Property 2 were valid for the following sets of natural numbers:

$$S = \{ n \in N \mid 1 + 2 + 3 + \ldots + n = n * (n + 1) / 2 \}$$

and

$$S = \{ n \in N \mid 11^n - 3^n \text{ is divisible by 8} \}$$

By verifying that Property 1 and Property 2 were valid for the set S, we are allowed to conclude (by the PRINCIPLE OF INDUCTION) that in each case S must be *all* of N.

In other words, by verifying that Property 1 and Property 2 were valid for these sets S, we are allowed to conclude (by the PRINCIPLE OF INDUCTION) that in each the desire equality is true for *all* natural numbers.

The main property of the natural numbers which makes the principle of induction correct is this one: *every set of natural numbers has a smallest element.* A reference where these questions are studied is the book

Hamilton, A. (1982). *Numbers, set and axioms: the apparatus of mathematics.* Cambridge: Cambridge University Press (see pages 3-5 and Chapter 4).