The Teaching and Learning of Mathematics at the University Level

Crucial Questions for Contemporary Research in Education

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For more than thirty years, research in mathematics education has tried to clarify, through work both theoretical and experimental, the processes of learning in mathematics. It has tried also to develop strategies for teaching that take into account the knowledge that it progressively constructed, and then to test it. Research was first carried out on the earliest stages of learning, those of elementary school, with work concerning high school and the university, beyond the required schooling, being relatively marginal. But the great increase in the number of students taking mathematics at these more advanced levels today poses educational problems that again constitute new challenges for research. In this article we will be interested in these problems and will try to make more precise the potential as well as the limitations of work conducted so far.

The discussion is from a personal point of view, marked by my own European and French culture; other researchers would doubtless have a considerably different overview. In effect, it is not as easy in education as in mathematics to give an overall picture of the state of development of research in a given area and to identify clearly the results that have been obtained there. This results initially from the fact that this field of research is not at all unified. Diverse approaches coexist, making generalizations difficult, as the recent ICMI study edited by A. Sierpinska and J. Kilpatrick and dedicated to mathematics education as a research domain attests. This diversity is doubtless tied to the relative youth of the field, but it results also from the complexity of the studied phenomena; a single point of view seems insufficient to encompass this complexity. The diversity results also from the fact that the processes of teaching and learning are partially dependent on the social and cultural environments in which they develop; the results of the research thus depend on the social and cultural environments also, and it is not always easy to make precise the field of validity for them. The diversity results finally from the fact that although the benchmarks acquired by research allow one to understand better the difficulties that students have, as well as the dysfunctions of our teaching, they more rarely give us inexpensive means of action to make teaching better in an immediate and sensible way.

In spite of these difficulties it is undeniable that research advances, producing at the same time theoretical frameworks for the problems of teaching and learning, methodologies for their study, and results concerning learning in any given domain. In what follows, after having raised the question of the foundations of the research, I shall concentrate on two types of results that seem to me as cutting across the range of approaches: one concerning the discontinuities and breaks in learning, the other concerning questions of cognitive flexibility. I shall illustrate them with precise examples taken from the two mathematical areas that have been favored so far for research at the university level—calculus and linear algebra. In a last section I shall point out a few questions that strike me as insufficiently addressed in research.

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An expanded version of the present article with more references appears with the title "What can we learn from educational research carried out at university level" in a forthcoming ICMI study to be published by Kluwer. For further reading about the levels of teaching that interest us here, the reader may wish to consult [1, 2, 3, 4].

The Foundations of Educational Research

For the past twenty years educational research has been marked by the preeminence of constructivist approaches based on the work of Piaget. Learning is known in these approaches as a process of adaptation in the biological sense of the term, based on processes of assimilation and accommodation: assimilation when newly encountered situations can be managed by a simple adaptation of cognitive schemes already constructed; accommodation when an important imbalance occurs, necessitating a reorganization of previous knowledge. These constructivist approaches have permitted people to have a new look at learning, showing that it cannot be reduced to a simple process of transmission of facts. What can be learned is strongly constrained by the subjects' initial conceptions—by the situations that are proposed to them and the means of action that are given to them for these situations. These things have thus contributed to explaining the recorded limits of teaching strategies that attribute a dominant role to what the teacher says.

As formulated initially, constructivist approaches are increasingly considered as insufficient for modeling in a satisfactory way the processes of learning in mathematics, because the social and cultural dimension of this learning is not sufficiently taken into account. As A. Sierpinska and S. Lerman emphasized in a 1996 article on these questions, awareness of these limitations leads to various constructions that are differentiated notably by the way in which the relationships among the individual, the social, and the cultural are conceived. I am not going to revisit the analysis of this diversity, but I would like to illustrate with two examples how this intervention of the social and the cultural leads one to relativize the classical cognitive analyses. I shall do so by relying on two theoretical frameworks that are particularly familiar to me: the theory of didactic situations [5], whose founding father is G. Brousseau, and the anthropological theory of education developed more recently by Y. Chevallard [6].

In the theory of didactic situations, learning is indeed seen as a process of adaptation, but one recognizes that the processes of adaptation used by the student in a given teaching situation are not all of a mathematical nature. The student adapts by relying on mathematical knowledge, but adapts also by relying on knowledge of the teaching system, its norms and customs, and guesses about the expectations of the teacher—what G. Brousseau isolated and defined as the "didactic contract". A good number of scholastically well-adapted students succeed, including at the university, more by learning to decode the terms of the didactic contract and by conforming to it than by really learning mathematics. Because of the strong effects of the didactic contract, it is not easy to construct learning situations where we can ensure that students' success implies real mathematical engagement. Evaluation in these circumstances is even more difficult, as various works have shown. The theory of didactic situations has developed a set of conceptual tools and techniques to analyze teaching situations from this point of view and to guide the construction of those that optimize the relationships between the mathematical activity of the teacher and activities that can be the students' responsibility. A little later I shall give an example related to integration in calculus.

Anthropological theory is quite distinct from the dominant constructivism. The emphasis in it is on the institutional dimension of learning; our relationships with mathematical objects emerge, according to it, from institutional relationships in force where we encounter them, the term "institution" being understood here in a very broad sense. As different works show, these relationships for an object are not absolute, but vary considerably from one institution to another.

These differences are particularly important to clarify when one studies the problems of institutional transition: for example, the problems of transition between high school and university, which concern us here. A part of the difficulty of this transition can be better understood if one considers that, beyond the common vocabulary and the apparent similarity of tasks and techniques, high school and the university develop profoundly different relationships for common mathematical objects, for example, those of calculus—limits, derivatives, and so on. For this reason university teachers encounter serious difficulties in bringing out the knowledge of the students and are led to the impression that the students know nothing. In addition, the difficulties tied to the culture gap are reinforced by another type of phenomenon brought out by educational research: the fact that much of our knowledge remains strongly contextual, that is, dependent on the situations from which it arises. Some teachers know this phenomenon well and with one word are able to evoke a situation, a moment of their common history with the students, to bring out this type of knowledge in a new context. But changes of class and a fortiori of institution cut the ties of this common memory, thereby limiting the field of knowledge usable to the ones that are already context-independent. Recent research such as in the ongoing thesis of F. Praslon concerning the notion of derivative is devoted to understanding all these phenomena. It constructs mechanisms for university teaching that allow people to work with students to fill the "educational void" of the transition that only a minority of our present-day students seem capable of filling by themselves.

I have discussed in this section the variety of approaches in mathematical education in the light of an increasing recognition of the cultural and social dimension of the learning process. Other evolutions play a role too, some overlapping with the above. For instance, for more than twenty years, research has brought out the fact that the learning of mathematics is not a continuous process, that it necessitates reconstructions, reorganizations, even sometimes veritable breaks with earlier knowledge and modes of thought. This fact has often nourished a vision of a hierarchy in learning, conceived as the progression through a succession of stages, as a progression toward increasing levels of abstraction. More and more, research shows that learning rests, in a quite decisive way, on the flexibility of mathematical functioning via *articulation*[1] of points of view, "registers of representation", and "settings of mathematical functioning". Conceptualization appears also more and more dependent on the concrete and symbolic tools of mathematical work. This dependence, which concerns at the same time what is learned and the methods of learning, is particularly important to take into account today, because of the rapid evolution of tools resulting from technological advances.

Even if certain researchers have been led to develop specific approaches, such as the APOS theory initiated by Ed Dubinsky, it is in this global perspective that research concerning university teaching fits. In effect, even if the concerned audiences of students are cognitively and emotionally more mature, with relationships to mathematics already based on a long history, and if the sought-after knowledge is more complex and more formalized, nothing today says that there exist really specific processes of learning for this level of teaching or that the constructed models are unsuitable for it. This is why we have chosen in what follows to organize the presentation of some results around two questions that simultaneously transcend, it seems to me, the variation in approaches and the levels of teaching: the question of reconstructions and breaks on the one hand, that of flexibility on the other.

[1] *"Articulation" refers to the connections between a part and the whole, or between one part and another part. At the same time it calls to mind both the switching from one part to another and the technical means for making these connections.*

Reconstructions and Breaks in Mathematical Learning

The necessity of reconstructions and breaks in the learning of mathematics might seem banal. However, teaching tends to live on the fiction that learning processes are continuous. It is a fiction that is doubtless convenient for separating the respective responsibilities of the students from those of the teachers. But it is a fiction that generates real difficulties.

The example of calculus illustrates well the variety of discontinuities to take into account. To structure this diversity, we shall distinguish three principal types of reconstructions.

The Reconstruction of Relationships to Familiar Objects

The learning of calculus presupposes first the reconstruction of relationships with mathematical objects that existed for our students even before the official teaching of analysis began. Take an object like the tangent, for example. Research shows, in this case, the difficulties generated by the usual teaching strategies in high school that do not manage this reconstruction, but also the very reasonable cost and efficiency of taking responsibility for it at that level [7]. The real numbers provide another example where the necessary reconstructions prove to be much less easy. Real numbers enter the secondary school curriculum early as algebraic objects with a dense order and a geometrical representation as the real line, and with decimal approximations that can be easily obtained with pocket calculators. Nevertheless, many pieces of research show that even upon entering a university students retain fuzzy conceptions that are barely coherent and poorly adapted to the needs of the calculus world [8]. For instance, real numbers are recognized as having no gaps in their ordering, but, depending on the context, students manage to reconcile this property with the existence of numbers just before or after a given number ($0.999\ldots$ is thus often seen as the predecessor of $1.000\ldots$). More than 40 percent of students entering French universities consider that if two numbers A and B are closer than $1/N$ for every positive N, then they are not necessarily equal, just infinitely close. Relations between irrational numbers and their decimal approximations remain fuzzy. No doubt reconstructions are necessary for understanding "calculus thinking modes". Research proves that these are not easily induced by the kind of intuitive and algebraic analysis that is mainly at play at the high school level and that the constructions of the real number field introduced at the university level remain largely ineffective if students are not confronted with the incoherences of their conceptions and the resulting cognitive conflicts.

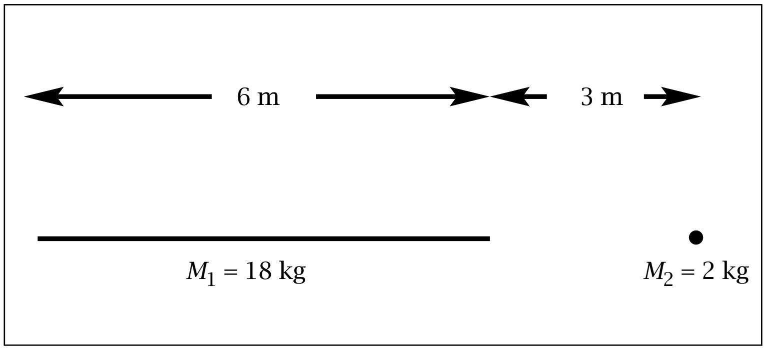

Figure 1.

Integrating New Facets of a Concept

The necessary reconstructions are not limited to those of familiar objects preexisting at the entrance to this area of mathematics. Other reconstructions are going to prove necessary because only certain facets of a concept can be presented on a first exposure. The case of the integral seems to illustrate this situation well. In France, as in numerous other countries, the integral is introduced at the end of secondary school via the notion of indefinite integral, thus as a process inverse to differentiation. It is immediately exploited in simple calculations of areas and volumes that are based on an intuitive approach to these notions and a pragmatic presentation of the Fundamental Theorem of Calculus. It is at the university level that a theory of integration is introduced, via the Riemann integral and then, at more advanced levels, the Lebesgue integral. Necessarily successive reconstructions of relationships to the notion of integral are in play.

During the past twenty years numerous pieces of educational research have been devoted to the concepts of derivative and integral, with a great convergence of results obtained, whatever country is involved. This research shows that students attain a reasonable level of performance in the handling of standard tasks, in particular of tasks of a computational nature, but nothing more. As appears clearly in [9], if, for example, one asks students to decide by themselves whether such-and-such situation in a problem of modeling comes under an integration procedure, they find themselves completely at a loss. They owe their salvation only to the linguistic clues with which the wording of this kind of problem is in general filled and that they have learned to recognize (slices, elements of area or work or force, infinitesimal decomposition, and so on). Worse, a certain number directly questioned, including the best students among them, do not hesitate to declare that in this area the surest thing is not to try to understand but to function mechanically. This circumstance does not need to be seen as a sort of cognitive fatality. What we are observing are the economical ways of adaptation our students develop when exposed to inadequate educational practices.

Fortunately, research is not limited to such negative reports. We come now to a situation constructed by M. Legrand in order to make first-year university students realize by themselves the necessity of the integral concept. The situation is based on the following, apparently simple, problem: a linear bar of mass M_1 and a point mass M_2 are located as in Figure 1, and students are asked to calculate the magnitude of the attracting force between the two masses. This situation has been proved to be effective through various experiments in different contexts. What makes it effective? In order to answer the question, we have to make a brief didactic analysis of it.

When asked this question without any linguistic hint, first-year students do not recognize an integration problem. But they are not blocked, as they can rely on a strategy often used in physics: concentrating the mass of the bar at its center of gravity and applying the familiar attraction law between two point masses. In experiments this strategy was always the predominant one. But in a group of reasonable size, as is easily the case at university level, there are always students who have some doubts: "Is the gravity principle valid in that particular case?" One strength of the situation is that one can test the validity of the gravity principle simply by applying it in another way. What is generally proposed by students is in fact the following: to cut the bar into two halves and apply the gravity principle to each of these parts. Of course, this does not give the same result, and the gravity principle is proved to be invalid in that particular case. But the negative answer is also a positive one, since it brings out one essential fact: the contribution of a piece of the bar to the attraction force depends on its distance to the point mass, and this allows students to propose upper and lower bounds for the required intensity. Moreover, the technique upon which the invalidation process was based can be then engaged in a progressive refinement process, and this leads students to the conviction that the force, whose existence is physically attested, can be approached as accurately as one wants. What is implicitly going on here is the fundamental integration process. Of course, in the educational design elaborated by M. Legrand, this is just the starting point. Students have then to work on situations that, in different contexts, require the same process for being solved. Then they have to look for and discuss the existing analogies between them in order to make the integration process an explicit tool (according to the distinction between the tool and object dimensions of mathematical concepts introduced by Douady [10]). It is only at this point that the teacher connects this mathematical work with the theory of Riemann integral and develops the notion of integration as a mathematical object that will be then reinvested in more complex situations.

Before leaving this example, let me stress the following point: efficiency here is not only linked to the characteristics of the problem that I have just described; it strongly depends on the kind of scenario developed in order to organize the meeting of students with this new facet of the integral concept. This scenario depends in a crucial way on the social character of learning processes: it is through group discussion that the initial strategy is proved to be erroneous; it is the collective game that allows one to find some solution in a reasonable amount of time. Working as a class fosters regularities in the dynamics of the situation that might not be present if students were faced with the same problem individually or in very small groups. No doubt also, the effect would be different if the teacher were simply explaining this particular example during a lecture session.

This example may appear to be idyllic. Unfortunately, educational research does not provide us with effective means to deal with all necessary reconstructions as easily. For instance, differences are evident if one considers the concept of limit, at the core of the field. With the example of the limit concept we come to a third category of reconstructions, reconstructions necessary because, as was already acknowledged by Henri Poincaré at the beginning of this century, concepts cannot always be taught from the start in their definitive form [11].

Changes of Level of Conceptualization

At the high school level in most countries today the impossibility of beginning calculus with a formal development has been acknowledged. Teaching relies both on a dynamic conception of the limit, based on graphical and numerical explorations, and on techniques of an algebraic nature. These things allow students to solve simple but interesting problems of variation and optimization. The transition towards more formal approaches, which takes place at the university level, represents a tremendous jump, both conceptually and technically.

From a conceptual point of view, one crucial point is that the formalization of the limit concept responds to the needs of foundational work, unification, and generalization. Through formalization the concept of limit becomes a "proof-generated concept" in a sense described by Lakatos [12]. It is not easy to make young students sensitive to such concerns; such concerns are not really part of their mathematical culture. And it is not easy to find situations analogous to the bar problem described above to support such concerns. For this reason researchers such as A. Robert (in [3]) suggest for such reconstructions specific educational designs that allow the cultural dimension to be taken into account better.

Nevertheless, one must not underestimate the technical difficulties of this reconstruction. From a technical point of view, one essential point is the following: in the algebraic version of calculus that one meets on first exposure, technical work does not really break with ordinary algebraic work. This is no longer the case when one advances to the more formalized aspects of the subject. For example, students have to reconstruct the meaning of equality and understand that equality does not necessarily result from successive equalities as in algebra, but can instead result from ϵ-proximity for every positive ϵ. We noted at the beginning of this section the difficulties that students have in learning that ϵ-proximity for every positive ϵ implies equality.

Another point is that inequalities become predominant over equalities. This change results in a great increase in technical complexity, all the more so as the associated modes of reasoning often rely on sufficient conditions that are not also necessary. These new modes of reasoning require a carefully controlled loss of information based on a good awareness of the respective orders of magnitude of the different parts of the expressions students have to deal with. In short, students have to identify and learn to master a completely new technical world. Doing so is far from easy; it is necessarily a long-term process.

The mathematical needs for reconstructions discussed above help us to understand, it seems to me, what can separate the capacity to give an intuitive sense to the notion of limit—even to illustrate it by examples and counterexamples—from the capacity to manipulate operationally this notion with its status of constructed object, subject to formal proofs. In the context of increased numbers of secondary school students to be taught, such a reconstruction, without any doubt, is the responsibility of university teaching, to the extent that the university regards this evolution as necessary. But the reconstruction must be carried out within the length of time of the course that is being taught.

In this section we have explained matters in terms of reconstructions of relationships to mathematical objects by distinguishing three different types of reconstructions. We would like, however, to make it clear that, even if researchers recognize the importance of the qualitative changes we have emphasized above, the researchers who work at the university level do not express them forcefully in these terms. Certain researchers express matters more clearly in terms of breaks by referring to the notion of epistemological obstacle borrowed from the philosopher G. Bachelard [13]. This is the case, for example, in various works concerning the notion of limit, such as those cited in the survey article by Cornu [1]. Others, such as E. Dubinsky and A. Sfard, center their study more on the importance and the difficulty of the transition between process and object conceptions. In the first ones, the mathematical notions are conceived as dynamic processes, resulting from the internalizing of effective actions. In the second ones, mathematical notions are perceived as static objects that can be, in their turn, involved in more complex processes. The work they have conducted on the notion of function is in some way prototypical of these approaches [14]. They demonstrate well the type of relationship to functions that a process conception permits and its effectiveness at the high school level. They demonstrate also the limits to such conceptions when in calculus at the university level, one is no longer interested necessarily in particular functions but in classes of functions, defined by properties such as conditions of regularity for which one again envisions processes. These works have shown also the disastrous effects of teaching strategies that aim too soon at set-theoretic static definitions of functional objects without allowing sufficient time for the "process" phase. They have shown finally that programming in specific languages, such as the language ISETL, can help the internalizing of actions into processes and, more delicately, the encapsulizing into objects of the processes.

The results obtained by the research on reconstructions necessary to mathematical learning, on the epistemological obstacles inherent in one kind of learning or another, and on the difficulties on transition between processes and objects certainly help us to understand better the difficulties encountered by students and to take responsibility for these difficulties more effectively in our teaching. However, as was said at the beginning of this article, the results tend to favor a "vertical" and hierarchical vision of mathematical learning and consequently to mask the importance of what one might like to describe in the "horizontal" dimension. It is to restore in some way the equilibrium between these two dimensions that we concentrate in the next section on questions of cognitive flexibility.

Flexibility and Mathematical Learning

Knowledge of the role played by a certain cognitive flexibility in improving mathematical work is not a recent thing. As T. Dreyffus and T. Eisenberg recall in a 1996 article on the multiple facets of mathematical activity, the book *How To Solve It?* by the mathematician H. Polya testifies to this fact. What does educational research add to these initial works? Without any doubt, it adds:

- a better knowledge of the different types of flexibility in play in mathematical activity
- the proof of the limits of teaching strategies that are aimed at developing these flexibilities as skills cutting across different areas but do not take seriously into account the distinctive features of knowledge that support these types of flexibility within the mathematical areas in question [15]
- development and experimentation with teaching designs that aim to make up for the recorded dysfunctions of teaching flexibility in the subject.

To illustrate this, we are going this time to favor the area of linear algebra, more recently investigated in educational research. There these questions are at the heart of various research projects. We are going to rely in particular on the synthesis realized in the work [4] edited by J. L. Dorier. As this author emphasizes, linear algebra finds its source in different mathematical settings that it has allowed us to unify in a certain sense: a geometric setting, a setting of systems of linear equations (finite or infinite dimensional), a setting of matrix calculus, a setting of differential equations, and so on. The development of a flexible articulation among these different settings, as between each of them and abstract linear algebra, appears then as an essential component of learning in this area. This development relies, in turn, on articulation between levels of language and description, between modes of reasoning, between "registers of representation", and between points of view.

Flexibility between Levels of Language and Modes of Reasoning

In [4] J. Hillel analyzes the different languages and associated ways of representing constructs in the subject of linear algebra, as well as their modes of interaction. Principally, he distinguishes three of these: the language of the general theory, the language of $\mathbb{R}^n$, and the geometric language of space in two and three dimensions, which is also used in a metaphorical way in dimensions greater than three. He describes their characteristics and modes of interaction, listing the difficulties that they can induce and to which teaching must be sensitive. Moreover, analyzing videotapes of lectures in courses of five experienced teachers on eigenvalues and eigenvectors, he presents evidence of permanent changes in the language and notation, usually carried out without pause and without any attempt to warn the students that a change is occurring.

A. Sierpinska, A. Defence, T. Khatcherian, and L. Saldanha, also in [4], distinguish three modes of reasoning in linear algebra: a *synthetic-geometric* mode, where the objects are in some way given directly in the spirit that tries to describe them, and two analytic modes where the objects are given indirectly. In the latter two, objects are constructed only by definitions, by properties of their elements. It is the *analytic-arithmetic* mode if the object is defined by a formula that allows one to calculate it, or the *analytic-structural* mode if the object is defined by a set of properties. According to these authors: If one thinks about possible solutions of a system of three linear equations in three unknowns by imagining the respective positioning of three planes in space, one is in the synthetic-geometric mode. If one thinks of this problem in terms of the possible results from reducing a 3-by-3 matrix, one is in the analytic-arithmetic mode. One is in the analytic-structural mode if, for example, one thinks in terms of singular and regular matrices.

Historically the development of linear algebra owes a great deal, as the authors emphasize, to the interaction of these three modes. But, as shown by fine analysis of tutoring situations in the university where this research was carried out, both the tasks proposed to the students and the observed interactions between teachers and students hardly favor the development of flexible coherent management of these different modes. The students on their side develop original systems of intermediate forms among these three modes, and reasons of economy cause mixed forms to appear incorporating the analytical-structural mode. This creativity could, according to the authors, be a source of inspiration for teaching, the educational problem in need of solution being that of finding the means for allowing a consciously controlled management of these different modes and of their flexible articulation. The collected data show clearly that this is not the case if the students are left on their own to do this, despite their manifest creativity.

Flexibility among Registers of Representation

Mathematical work in linear algebra mobilizes several *registers of semiotic representation*,[2] including graphics, pictures, symbolic writing, natural language, and others. As R. Duval [16], among others, emphasizes, semiotic representations are absolutely necessary for mathematical activity because mathematical objects are not directly accessible to perception. Semiotic registers do not have merely a simple function of exteriorization of mental representations, of communication; they are essential to cognitive functioning, to conceptualization. However, according to him, teaching tends to reduce them to this role of exteriorization and communication. Thus it tends to see the capacity to recognize semiotic representations, to form them, to treat them, or to convert them into another register as a simple byproduct of conceptualization. The research of K. Pavlopoulou in his 1994 Strasbourg thesis (cf. [3]) on the coordination of registers of representation in linear algebra brings out well that the relationships between conceptual learning and semiotic learning are much more complex. The module of experimental teaching prepared for beginning students in the context of this research tends again to show that teaching, when it wants to be sensitive to this semiotic dimension of mathematical work, allows one to overcome difficulties, however resistant they appear.

Flexibility between Points of View

Flexibility in articulation between mathematical points of view has been emphasized by several authors, among them M. Alves Diaz in his 1998 thesis at Université Paris VII (cf. [4]). Such a flexibility enters linear algebra in the relationships between "implicit" and "parametric" points of view.[3] In linear algebra one must often pass from one point of view to the other—in a computational way at first, later in a more metaphorical way. The thesis of M. Alves Diaz, conducted at the same time with French and Brazilian students of various levels, shows the great difficulties that students have in developing a flexible articulation between the two points of view. The small percentage of success at all levels in solving the following simple exercise illustrates matters.

> In $\mathbb{R}^3$ let $a=(2,3,-1)$, $b=(1,-1,2)$, $c=(5,0,7)$, and $d=(0,0,1)$. Find an implicit representation of the intersection of the vector spaces E and F generated respectively by $\{a,b\}$ and $\{c,d\}$.

Solving this exercise in the style in which these students have been taught requires passing via Gaussian elimination from a parametric representation to an implicit representation for each of the subspaces E and F, and then the union of the sets of equations for each is the required answer.[4]

The solution of this task leads in particular to numerous formal slips. Students confuse coordinates with parameters and end up with intersections that are in $\mathbb{R}^2$ or $\mathbb{R}^4$ rather than in $\mathbb{R}^3$. They brutally associate equations with vectors, and so on. Visibly the cues from the geometric setting, which ought to be easy to use, have been brought to bear by the questioned students only a little. When geometric intuition has been used, it has not necessarily been used effectively. Finally, contrary to what one might think, the percentage of success is not improved when one speaks to more advanced students.

What this research shows as well, through the analysis of representative teaching manuals in the two countries, is the weak sensitivity to these difficulties that teaching seems to show. Certainly the students are able automatically to use the solution

[2] *R. Duval [16] defines a "register of semiotic representation" as a system of representations by signs that allows the three fundamental activities tied to processes of using signs: the formation of a representation, its treatment in the same register, its conversion into another register.*

[3] *With respect to a vector space, one is thinking about the vector space* implicitly *if one regards it as a set of solutions of a system of linear equations,* parametrically *if one thinks about it in terms of a system of generators.*

[4] *Let us emphasize, however, the perturbation arising here in using the standard technique: the equation of the subspace F is $y=0$, an equation that can moreover be obtained without the least calculation.*

techniques that permit them to manage the articulation technically, but this willingness is insufficient to give the articulation a meaning, to permit them to manage it and control it in an effective way.

Duality, when it is introduced later, ought to allow them to rethink this articulation and to understand better the role that the association of equation with vector plays in it. But the technical world of systems and the theoretical world of duality, as they are classically presented, remain for most present-day students two worlds too weakly connected.

No doubt there would be a need to construct an intermediate discourse permitting the students to put into place, as a reflex action, cues to the technical work of articulation. The history of the development of the notion of rank, such as analyzed in [4], provides interesting insights into this subject, helping us to restore to our understanding of this articulation a complexity that modern presentations, in their apparent simplicity, tend to make us forget. This complexity does not appear in the manuals, where the ability to articulate between points of view is assumed to be automatic once the techniques for it are available.

This weak effect of standard teaching practices on articulation is not an isolated phenomenon. Whether it is a question of articulation of settings, of registers, or of points of view, analysis tends to show that it is difficult for teaching to assume responsibility for the learning of effective independent thinking as part of the articulation. Flexibility seems to be considered as automatically internalized once one has "understood" the notion, as if it were a simple question of homework that one can leave to the private work of the student. Research shows that this is unfortunately not the case. It shows also that flexibility is not out of range of what can be taught if one is attentive to its development. The works cited above tend to show this in the case of linear algebra, and it is also the case in calculus. Much research carried out in the latter area shows in particular that computer technology, if its use is carefully thought out, can play a decisive role in the development of a flexible articulation between the algebraic and graphic registers and can make of this articulation a really efficient instrument of mathematical activity [2]. My own research on the teaching of differential equations goes in the same direction [14], showing notably how the use of computer technology can make approaches via qualitative solutions viable, even with beginning students, and can bring university teaching closer to the present-day development of the field. But the research shows also that the viability of these new teaching strategies requires important changes in the status of the graphical register. Indeed, with beginners viability requires the acceptance of qualitative proofs based on specific graphical arguments. This is hard to negotiate with university teachers, at least in France, where such proofs are not generally accepted at the university level.

Potential and Limits of the Research Enterprise

As this article has tried to show, research carried out at the university level helps us to understand better the difficulties in learning that our students have to face, the surprising resistance to solutions of some of these difficulties, and the limits and dysfunction of our teaching practices. In various cases research has led to the development of teaching designs that have proved to be effective, at least in experimental environments. This article gives only a very partial vision of the diversity of the research enterprise and the results obtained through it. Because of limited space, we have chosen to focus on a few points and to omit others that are certainly also very important. But the presentation here, even though only partial, should be substantial enough to allow me to discuss some limits of the research enterprise in its present state and to suggest some themes for future development.

As to research undertaken up to now, my feeling is that efforts have been concentrated on just a few areas with respect to the diversity of the mathematical topics taught at university level. As was said above, the main effort has been with calculus, a mathematical area that was seen as the main source of failure at the undergraduate level. More recently researchers have investigated the field of linear algebra, and important projects are now going on. But important areas such as probability and statistics remain poorly addressed. Moreover, my feeling is that research in mathematics education, by and large, has more or less consciously taken as a reference point the training of future mathematicians at the expense of the great variety of students taking university-level mathematics courses at university. Without a doubt, research in mathematics education has to be partially reoriented in order to give the place it deserves to issues linked to the mathematical training of elementary and secondary teachers and, more globally, to the mathematical training of all kinds of professionals.

The way the issue of computer technologies has been generally addressed up to now evidences these limitations, in my opinion. It mainly focuses on the ways computer technologies can support conceptualization and the cognitive flexibility recognized as an essential component of it. Research does not investigate with the same attention what a professional mathematical activity assisted by computer technologies really is, as well as what its specific and nonspecific mathematical needs are. These needs might depend on professional specialties if one wants to overcome the mere status of blind user and become an efficient and critical user. As a consequence, research does not pay enough attention to the ways the corresponding knowledge can be constructed in ordinary or service mathematics courses. Nevertheless, this is a real challenge we have to face today, taking into account the fact that at university our main concern is no longer the development of some kind of general mathematical culture.

The limits of the research enterprise are not only those mentioned above, which are linked to the state of development of the field. As university teachers we are faced with more fundamental ones. We would like research to provide us with easy and rather inexpensive means for improving our teaching strategies. But as a researcher I have to concede that only rarely does research give us evidence that through minimal and inexpensive adaptations can we obtain substantial gains. On the contrary, most research designs that have proved to be effective require more engagement and expertise on the part of teachers and significant changes in practices. One essential reason is that the problems are not only with the content of teaching (it is not enough to write or adopt new textbooks); the problems are related to the forms of students' work, the modes of interaction between teachers and students, and the forms and content of students' assessment. Changes are not so easy to achieve; they require time and institutional support, and they are not merely a matter of personal good will.

Another crucial point is the complexity of the systems where teaching and learning processes take place. Because of this complexity the knowledge we can infer from educational research, however useful it is, is necessarily very partial. The models we can elaborate are necessarily simplistic ones. As mathematicians we are well aware that we can learn a great deal even from simplistic models, but we cannot expect them to give us the means to really control educational systems. So we have to be realistic in our expectations, careful with generalizations. This does not mean, in my opinion, that the world of research and the world of practice have to live and develop as separate ones—far from it. But I am convinced that finding the ways of making research-based knowledge useful outside the educational communities and experimental environments where it develops cannot be left solely to the responsibility of researchers. It is our common task.

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