**IMAP^2 WORKSHOP YEAR 3 SUMMER 2009**

**POLYNOMIAL INVESTIGATIONS**

Polynomials (in one and more than one variable) form a rich class of important functions in mathematics, modeling applications, and many other disciplines.They form a cornerstone topic in school mathematics. In these investigations we examine questions about existence and uniqueness of solutions to polynomial equations.

Polynomials in one variable

Let *a0, a1, a2, …, an* be numbers and let *z* be a variable. A polynomial (in *z*) of degree *n* means an expression or function of the form

*p[z]* = *a0+ a1 z + a2\* z2+ a3\* z3 +…+ an\* zn*

*Do you teach any or all of the topics below concerning polynomials? If so, what are the contexts where the topic(s) arise?*

**1.** Factoring polynomials into products of polynomials of lower degree.

EXAMPLES:

*-15 + 2 z + z2 = (z - 3)\*(z+5)*

*-15 + z + z2 = (z - ( -1 + 611/2 ) / 2 )\* (z - ( -1 - 611/2 ) / 2 )*

*15 + 2 z + z2 = (z - [-1+ 141/2 i])\* (z - [-1- 141/2 i])*

 *= (z +1- 141/2 i )\* (z +1+ 141/2 i )*

*NOTE:* ***ALL*** *QUADRATICS FACTOR! RIGHT?!?!?*

*z4 - z3 + 5 z2 -5 z = z (z - 1) (z2 + 5)*

**2.** The division algorithm for polynomials/polynomial division/synthetic division.

Division Algorithm for Polynomials

Let *f(x)* and *g(x)* be polynomials in *x* (real or complex coefficients) with *g(x) ≠ 0*.

Then there exists unique polynomials *q(x), r(x)* such that

*f(x) = g(x)q(x) + r(x)*

and *r(x)=0* or deg *r(x)* is less than deg *g(x).*

In particular, if *f(x)* is a polynomial and *x0* is a root of *f(x) ( f(x0) = 0 ) then*

 *(x-x0)* is a factor of *f(x).*

If *f(x)* is a polynomial then

*f(x0) = 0*

implies

*f(x)= (x-x0)\* q(x)*

where *q(x)* is a polynomial (of degree one less than *f(x)).*

**3.** All polynomials (with real coefficients) factors into products of linear and quadratic factors (where the quadratic factors have no real roots, like *x2 +1*).

All polynomials (with real coefficients) factor into products of linear factors if you also allow complex roots.

All polynomials (with real or complex coefficients) factor into products of linear factors (allowing complex roots).

**4.** Other polynomial topics? Des Cartes rule of signs for determining # of positive (negative) roots?

Wiki reference on polynomial factorization

<http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra>

# Fundamental theorem of algebra

### From Wikipedia, the free encyclopedia

In [mathematics](http://en.wikipedia.org/wiki/Mathematics), the [**fundamental theorem**](http://en.wikipedia.org/wiki/Fundamental_theorem) **of algebra** states that every non-constant single-variable [polynomial](http://en.wikipedia.org/wiki/Polynomial) with [complex](http://en.wikipedia.org/wiki/Complex_number) coefficients has at least one complex [root](http://en.wikipedia.org/wiki/Root_%28mathematics%29). Equivalently, the [field](http://en.wikipedia.org/wiki/Field_%28mathematics%29) of [complex numbers](http://en.wikipedia.org/wiki/Complex_number) is [algebraically closed](http://en.wikipedia.org/wiki/Algebraically_closed_field).

Sometimes, this theorem is stated as: every non-zero single-variable polynomial, with complex coefficients, has exactly as many complex roots as its degree, if each root is counted up to its [multiplicity](http://en.wikipedia.org/wiki/Multiplicity_%28mathematics%29). Although this at first appears to be a stronger statement, it is an easy consequence of the other form of the theorem, through the use of successive [polynomial division](http://en.wikipedia.org/wiki/Polynomial_division) by linear factors.

In spite of its name, there is no known purely algebraic proof of the theorem, and many mathematicians believe that such a proof does not exist.[[1]](http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra#cite_note-0) Besides, it is not fundamental for modern [algebra](http://en.wikipedia.org/wiki/Algebra); its name was given at a time in which algebra was basically about solving polynomial equations with real or complex coefficients.

VERY short proofs are available depending on what background you assume is known to the reader (cf. <http://www.cut-the-knot.org/wiki-math/index.php?n=Calculus.FTAByCauchyTheorem> )

Is there an 'elementary' proof of the FTOA result? None that we know of. And according to Wikipedia, some people think no elementary proof is even possible!

Do you know any other examples of interesting/important/fun mathematical theorems/results/questions, which are easy to state, but for which there are no (known) *simple* proofs? How about mathematical problems for which the answer is not known?

We outline what seems to be a proof of the FTOA which uses the least amount of mathematical machinery from calculus or real or complex analysis. But there is still some hard work related to the use of inequalities.