(1)

F.T.O.A.

Year 3 IMAP^2 Talk -W. Seaman

6/17/09

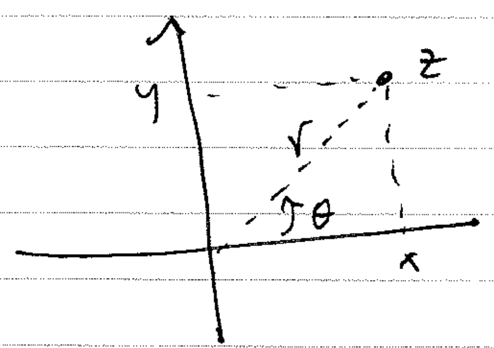
F.T.O.A. Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a complex (or real) polynomial of degree n ($a_0, a_1, \ldots, a_n \in \mathbb{C}$), $z \in \mathbb{C}$.

Then $\exists\, z_0 \in \mathbb{C}$ s.t. $p(z_0) = 0$.

Key Ideas: (1) Complex #s $z = x + iy$, $x, y \in \mathbb{R}$

$i = \sqrt{-1}$

'Polar' representation



$$z = x + iy = r e^{i\theta}$$

$$r = \sqrt{x^2 + y^2} = |z|$$

$\theta = $ "$\arg(z)$" for $z \neq 0$

$\in [0, 2\pi)$

$$z = |z| e^{i \arg(z)}.$$

Triangle-type inequalities: $z, w \in \mathbb{C}$

$$|z + w| \leq |z| + |w|$$

$$|z - w| \geq \big|\, |z| - |w| \,\big|$$

Sps $a, c \in \mathbb{C}$, and $a, c \neq 0$.

Consider the 'simpler' polynomial (Polya!)

$$q(z) = a + cz^k. \qquad (k \in \mathbb{N})$$

$$a = q(0) \neq 0; \quad q(z) = 0 \iff |q(z)| = 0 \iff$$

$$a + cz^k = 0 \iff$$

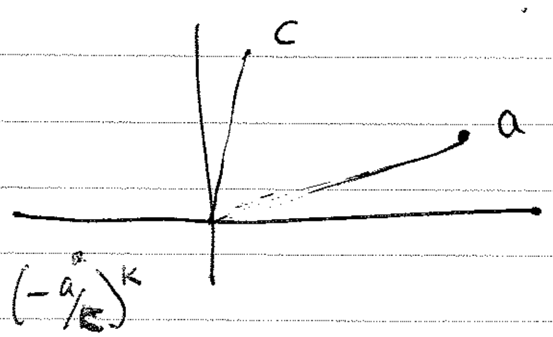
$$z^k = -\frac{a}{c}$$

$$\iff$$

z is a k^{th} root of $-\frac{a}{c}$.

Another view of this result is the following:

If $|a| = |q(0)|$ is not 0, we can ***decrease*** the value of $|q(z)|$ to ***lower*** than $|a| = |q(0)|$ by moving z in the 'direction' of $\left(-\frac{a}{c}\right)^k$.



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$$a = |a| e^{i\arg(a)}$$

$$c = |c| e^{i\arg(c)} \Rightarrow \frac{1}{c} = \frac{1}{|c|} e^{-i\arg(c)}$$

or use $i(2\pi - \arg(c))$

$$-1 = |-1| e^{i\arg(-1)} = e^{i\pi}.$$

$$\therefore \quad -\frac{a}{c} = \frac{|a|}{|c|} e^{i(\pi + \arg(a) - \arg(c))}$$

Direction of $-\frac{a}{c}$ is $\pi + \arg(a) - \arg(c)$

Direction of $\left(-\frac{a}{c}\right)^{\frac{1}{k}}$ is $\dfrac{\pi + \arg(a) - \arg(c)}{k}$

To decrease $|q(z)| = |a + cz^k|$ from $|a| = |q(0)|$, move the input # z from 0 in the direction $\theta = \dfrac{\pi + \arg(a) - \arg(c)}{k}$.

Let $z = r e^{i\left(\frac{\pi + \arg(a) - \arg(c)}{k}\right)}$

Then $z^k = r^k e^{i(\pi + \arg(a) - \arg(c))}$

So

$$q(z) = a + cz^k$$

$$= |a| e^{i\arg(a)} + |c| e^{i\arg(c)} \cdot r^k e^{\pi + \arg(a) - \arg(c)}$$

cancel

$$= |a| e^{i\arg(a)} + |c| r^k e^{i\arg(c)} \cdot \underbrace{e^{\pi}}_{-1} \cdot e^{\arg(a)} \cdot e^{-\arg(c)}$$

$$= e^{i\arg(a)}\left[|a| - r^k |c|\right]$$

$$q(z) = e^{i\arg(a)}\left[|a| - r^k |c|\right].$$

$$|q(z)| = \big||a| - r^k |c|\big|$$

If r is near 0, $|a| - r^k|c|$ is positive and <u>smaller than</u> $|a| = |q(0)|$.

Proof of FTOA. $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ has a 0.

Step 1 $\exists z_0 \in \mathbb{C}$ s.t. $|p(z_0)| \le |p(z)|$ $\forall z \in \mathbb{C}$.

Reason: Triangle inequalities $\Rightarrow$ if R is a big enough #, then $|z| > R$ forces

$$\tfrac{3}{2}|a_n||z|^n > |p(z)| > \tfrac{1}{2}|a_n||z|^n$$

so when $|z|$ is huge, so is $|p(z)|$ ("like" $a_n z^n$).

If $|z| \le R$, $|p(z)|$ has a minimum value, $|p(z_0)|$ and this is smaller than the 'huge' $|p(z)|$.

$\therefore$ $|p(z_0)| \le |p(z)|$ $\forall z \in \mathbb{C}$.

Step 2. Say $z_0 = 0$ (wlog) (why?)

If $p(z_0) \neq 0$, and if $p(z)$ is not constant,

Say $$p(z) = \underbrace{a_0}_{= p(z_0)} + a_k z^k + a_{k+1} z^{k+1} + \dots + a_n z^n$$

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(that is you might have

$a_1 = 0$

$a_2 = 0$

$\vdots$

$a_{k-1} = 0$

but

$a_k \neq 0$.)

$$p(z) = a_0 + a_k z^k + \left(a_{k+1} z^{k+1} + \cdots + a_n z^n\right)$$

Note $a_0 = p(z_0)$.

$$|p(z)| = \left|a_0 + a_k z^k + \left(a_{k+1} z^{k+1} + \cdots + a_n z^n\right)\right|$$

$$\leq \underbrace{|a_0 + a_k z^k|}_{\text{can decrease from } |a_0| \text{ as in previous work pp. 2-4.}} + \underbrace{|a_{k+1} z^{k+1} + \cdots + a_n z^n|}_{\text{For } z \text{ close enough to } 0, \text{ this will be extremely small (like } r^{k+1}) \ r \approx 0.}$$

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<u>Conclusion</u> : if $a_0 \neq 0$, $a_k \neq 0$

we can pick z near 0, in the direction of $\left(-\frac{a_0}{a_k}\right)^{1/k}$, and this z will satisfy

$$|a_0 + a_k z^k| + |a_{k+1} z^{k+1} + \dots + a_n z^n| < |a_0|$$

$\therefore$

$$|p(z)| \le |a_0 + a_k z^k| + |a_{k+1} z^{k+1} + \dots + a_n z^n|$$
$$< |a_0| = |p(z_0)|$$

But $|p(z_0)| \le |p(z)|$ for <u>all</u> $z \in \mathbb{C}$.

and

$|p(z)| < |p(z_0)|$ for this special z. (X)

<u>Conclusion</u> either $a_0 = 0$ or $a_k = 0$

so $p(z_0) = 0$ or $p(z)$ is constant. QED