



Numerical analysis of stochastic elliptic variational inequalities of the first kind

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ABSTRACT

This paper presents a numerical approach to the stochastic obstacle problem using the stochastic Galerkin (SG) method. Due to the low regularity of the solution, linear finite elements are employed in both the physical and random variable spaces. Properties of random fields and variational inequalities of the first kind are employed to establish the well-posedness of the problem. Finite element spaces are introduced to construct suitable approximation subspaces, and a comprehensive SG formulation is proposed to solve the stochastic obstacle problem. Well-posedness of the discrete formulation is shown and an optimal error estimate for the numerical solution in the H^1 -norm is derived. Numerical experiments validate the effectiveness of the SG method, showing that the expectation error converges at a rate of $O(h)$ in the H^1 -norm, consistent with theoretical predictions.

1. Introduction

Variational inequalities (VIs) are a fundamental class of mathematical problems that extend the concepts of optimization and equilibrium. They are extensively applied in fields such as applied mathematics, physics, engineering, and economics to model systems involving constraints, inequalities, or non-smooth behavior ([1–6]). VIs offer a robust framework for addressing challenges in the presence of complex constraints or irregularities. Since closed-form solutions are rarely available for VIs arising in practical applications, numerical methods are required to solve such problems. Various numerical approaches have been developed and studied to this end, including, notably, finite element methods ([7–10]), discontinuous Galerkin methods ([11,12]), and virtual element methods ([13–16]).

In many complex physical and engineering models, significant uncertainties arise from various sources, including parameters, coefficients, forcing terms, and boundary conditions. These models are often described using stochastic partial differential equations (SPDEs). In recent years, significant attention has been given to the stochastic Galerkin method ([17–21]) and stochastic collocation method ([22,23]) as effective tools for uncertainty quantification and propagation in random PDEs.

The stochastic Galerkin method addresses uncertainty by expanding the solution in a polynomial basis with respect to random variables, thereby transforming the problem into a high-dimensional deterministic system. Galerkin finite element techniques, including both h - and p -versions ([17,19,20]), have been effectively used to approximate statistical quantities of interest. In contrast, the stochastic collocation (SC) method ([22,23]) offers a non-intrusive alternative based on evaluating the solution at carefully chosen points in the stochastic space and interpolating the result.

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Building on these methods, several theoretical investigations have been conducted to support their applicability in various contexts. For example, Shen et al. [21] present a detailed analysis of the stochastic Galerkin method for optimal control problems governed by elliptic partial differential equations with random coefficients and state constraints. In more challenging settings, such as stochastic obstacle problems, the convergence behavior may degrade due to the limited regularity of the solution with respect to the random parameters. Consequently, polynomial chaos discretizations fail to achieve exponential convergence rates in the stochastic domain, thereby diminishing the efficiency of SG/SC methods. Nonetheless, if the coincidence set is only mildly sensitive to stochastic variations, satisfactory convergence behavior may still be observed. For instance, Forster and Kornhuber [24] report promising numerical results for elliptic variational inequalities with random diffusion coefficients using a stochastic collocation method; however, a rigorous theoretical justification for the observed performance remains lacking.

One of the main challenges in numerically solving stochastic obstacle problems using stochastic collocation methods arises from the limited global regularity of the solution with respect to the random parameters. This lack of smoothness is a direct consequence of the complementarity condition, which introduces non-smooth dependence on the stochastic variables. Consequently, the efficiency of stochastic collocation approximations may deteriorate, and a rigorous error analysis becomes more challenging in this setting. In contrast, Monte Carlo (MC) methods, which are widely used in statistical simulations, offer a flexible and non-intrusive alternative that remains applicable even in the presence of low regularity. For example, multilevel Monte Carlo finite element methods (MLMC-FEMs) have been applied to stochastic elliptic variational inequalities in [25], and a convergence theory for multilevel sample variance estimators was developed in [26]. The overall error of such MC-based finite element methods is bounded by $O(h) + O(M^{-1/2})$, where M denotes the number of samples.

However, for stochastic obstacle problems, each sample requires the solution of a nonlinear variational inequality, which is significantly more expensive than solving a linear PDE. Consequently, the cost per sample in MC methods becomes substantially higher, making the slow convergence rate with respect to M even more problematic when high accuracy is desired. To overcome this bottleneck, we adopt a stochastic Galerkin finite element discretization, which replaces many independent solves by a single coupled deterministic variational inequality in a tensor-product space ([17]). Once the SG system is solved, statistical quantities of the solution such as the expectation and the second moment are obtained by inexpensive post-processing of the SG coefficients. To the best of our knowledge, a rigorous numerical analysis of SG finite element discretizations for stochastic obstacle problems has been largely absent. This work fills that gap by establishing the well-posedness of both the continuous problem and its SG discretization, and by deriving a priori error estimates.

The main contributions of this paper are summarized as follows. First, we formulate a stochastic Galerkin finite element approximation for the stochastic obstacle problem in a tensor-product space, using linear finite elements in the physical variables and piecewise multilinear finite elements in the stochastic variables to accommodate the limited regularity of the solution. Second, based on a finite-dimensional parametrization of the random data, we establish the well-posedness of the continuous stochastic variational inequality, the corresponding parametric formulation, and the fully discrete SG formulation. Third, we derive an a priori error estimate in the H^1 -norm for the proposed SG finite element approximation. Finally, numerical experiments are presented to confirm the theoretical convergence behavior and to illustrate the accuracy of the method for the expectation and second moment of the solution.

The remainder of this paper is organized as follows. In Section 2, we introduce the stochastic obstacle problem in the form of a variational inequality of the first kind and explore its well-posedness. In Section 3, we construct finite element spaces, employing continuous piecewise linear functions for both the physical and stochastic discretizations. Based on these spaces, in Section 4, we present the full finite element formulation, including the Galerkin approximation and providing a priori error analysis. Finally, in Section 5, we illustrate the performance of the proposed method through numerical experiments that confirm the theoretical convergence behavior across different stochastic configurations.

2. Obstacle problem with random coefficients

2.1. Deterministic obstacle problem

First, we describe a deterministic obstacle problem ([27,28]). Let $D \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain, the positive integer n being the spatial dimension. The weak formulation of the obstacle problem is a variational inequality: Find $u \in K$ such that

$$\int_D a \nabla u \cdot \nabla (v - u) \, dx \geq \int_D f(v - u) \, dx \quad \forall v \in K, \quad (2.1)$$

where $f \in L^2(D)$, $a \in L^\infty(D)$ and K is the set of admissible displacements,

$$K = \{v \in H_0^1(D) \mid v \geq g \text{ a.e. in } D\},$$

in which g denotes the height function of the obstacle. In the space $H_0^1(D)$, we use its canonical inner product

$$(u, v)_{H_0^1(D)} = \int_D \nabla u \cdot \nabla v \, dx$$

and the corresponding norm $\|v\|_{H_0^1(D)} = (v, v)_{H_0^1(D)}^{1/2}$. Assuming the coefficient $a(\cdot)$ is bounded below by a constant $a_0 > 0$, we can conclude that the obstacle problem has a unique solution by applying the following result ([9, page 3] and [28, Theorem 11.3.9]).

Lemma 1. Let V be a real Hilbert space, and let $K \subset V$ be a non-empty, closed and convex set. Assume $b(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is a continuous, V -elliptic bilinear form, $\ell \in V'$. Then the following elliptic variational inequality

$$u \in K, \quad b(u, v - u) \geq \ell(v - u) \quad \forall v \in K$$

has a unique solution.

While the deterministic formulation assumes precise knowledge of the input data, real-world scenarios often involve uncertainties in the coefficient a , the source term f and height function g of the obstacle. These uncertainties will be described by randomness of the data a , f and g .

2.2. Stochastic framework and functional setting

We begin by briefly recalling some standard definitions related to random variables and random fields; cf. e.g., [29–31] for comprehensive introductions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the set of elementary events, $\mathcal{F}$ is a σ -algebra, and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. For any $Y \in L^1(\Omega)$, its expected value is denoted by

$$\mathbb{E}[Y] = \int_{\Omega} Y(\omega) d\mathbb{P}(\omega).$$

Let X be a Banach space of real-valued functions defined on a domain $D \subset \mathbb{R}^n$, equipped with a norm $\|\cdot\|_X$. A measurable real-valued random field $v : D \times \Omega \rightarrow \mathbb{R}$ belongs to the Bochner space $L^p(\Omega; X)$, $1 \leq p < \infty$, if

$$\|v\|_{L^p(\Omega; X)}^p := \int_{\Omega} \|v(\cdot, \omega)\|_X^p d\mathbb{P}(\omega) < \infty.$$

For $p = \infty$, $L^\infty(\Omega; X)$ is equipped with the essential supremum norm

$$\|v\|_{L^\infty(\Omega; X)} := \operatorname{ess\,sup}_{\omega \in \Omega} \|v(\cdot, \omega)\|_X.$$

In particular,

$$\|v\|_{L^2(\Omega; L^2(D))}^2 = \mathbb{E} \left[\|v(\cdot, \omega)\|_{L^2(D)}^2 \right] = \int_{\Omega} \int_D |v(\mathbf{x}, \omega)|^2 d\mathbf{x} d\mathbb{P}(\omega).$$

Given the distinct structural properties of stochastic functions with respect to ω and $\mathbf{x}$, their numerical approximations require the use of tensor spaces. Let H_1 be a Hilbert space of functions defined on the physical domain $D \subset \mathbb{R}^n$, and H_2 a Hilbert space of functions defined on the parameter domain $\Gamma \subset \mathbb{R}^M$. The tensor space $H_1 \otimes H_2$ is the completion of finite sums $u(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^d v_i(\mathbf{x}) w_i(\mathbf{y})$, where $\{v_i\} \subset H_1$ and $\{w_i\} \subset H_2$, with respect to the norm induced by the inner product $(u, \hat{u})_{H_1 \otimes H_2} = \sum_{i,j} (v_i, \hat{v}_j)_{H_1} (w_i, \hat{w}_j)_{H_2}$. For domains D and Γ , the tensor space $H^1(D) \otimes L^2(\Gamma)$ is equipped with the inner product

$$(u, \hat{u})_{H^1(D) \otimes L^2(\Gamma)} = \int_{\Gamma} \int_D (u(\mathbf{x}, \mathbf{y}) \hat{u}(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{x}} u(\mathbf{x}, \mathbf{y}) \cdot \nabla_{\mathbf{x}} \hat{u}(\mathbf{x}, \mathbf{y})) d\mathbf{x} d\mathbf{y}.$$

Thus, if $u \in H^k(D) \otimes L^2(\Gamma)$, it follows that $u(\cdot, \mathbf{y}) \in H^k(D)$ for almost every $\mathbf{y} \in \Gamma$, and for almost every $\mathbf{x} \in D$, $u(\mathbf{x}, \cdot) \in L^2(\Gamma)$. Moreover, there exists an isomorphism $H^k(D) \otimes L^2(\Gamma) \simeq L^2(\Gamma; H^k(D)) \simeq H^k(D; L^2(\Gamma))$ ([19]).

With the above notations, we now consider the stochastic obstacle problem: Given $a \in L^\infty(\Omega; L^\infty(D))$, $f \in L^2(\Omega; L^2(D))$, $g \in L^2(\Omega; H^1(D))$, find a function u such that for almost every $\omega \in \Omega$, $u(\cdot, \omega) \in K_\omega$ and the following inequality holds:

$$\int_D a(\mathbf{x}, \omega) \nabla u(\mathbf{x}, \omega) \cdot \nabla (v(\mathbf{x}) - u(\mathbf{x}, \omega)) d\mathbf{x} \geq \int_D f(\mathbf{x}, \omega) (v(\mathbf{x}) - u(\mathbf{x}, \omega)) d\mathbf{x} \quad \forall v \in K_\omega, \quad (2.2)$$

where $K_\omega = \{v(\cdot, \omega) \in H_0^1(D) \mid v(\cdot, \omega) \geq g(\cdot, \omega) \text{ a.e. in } D\}$. The gradient operator ∇ refers to differentiation with respect to $\mathbf{x} \in D$. The corresponding pointwise formulation is to find a random function $u(\mathbf{x}, \omega)$ such that (cf. [28, Example 11.1.1])

$$\begin{aligned} -\operatorname{div}(a(\mathbf{x}, \omega) \nabla u(\mathbf{x}, \omega)) &\geq f(\mathbf{x}, \omega) && \text{in } D, \\ u(\mathbf{x}, \omega) &\geq g(\mathbf{x}, \omega) && \text{in } D, \\ (\operatorname{div}(a(\mathbf{x}, \omega) \nabla u(\mathbf{x}, \omega)) + f(\mathbf{x}, \omega))(u(\mathbf{x}, \omega) - g(\mathbf{x}, \omega)) &= 0 && \text{in } D, \\ u(\mathbf{x}, \omega) &= 0 && \text{on } \partial D. \end{aligned} \quad (2.3)$$

To ensure the compatibility of the obstacle with the homogeneous Dirichlet boundary condition, we assume that

$$g(\mathbf{x}, \omega) \leq 0 \quad \text{for a.e. } \mathbf{x} \in \partial D \text{ and } \omega \in \Omega.$$

For each admissible realization, the existence and uniqueness of $u(\cdot, \omega)$ follow from Lemma 1 under sample-wise positivity and boundedness of the diffusion coefficient. However, the stochastic variational formulation below is posed in $V = L^2(\Omega; H_0^1(D))$. To obtain the boundedness and coercivity of the stochastic bilinear form with constants independent of ω , we impose the following uniform ellipticity and boundedness assumption ([25]).

Assumption 1. There exist two constants $a_{\min}$ and $a_{\max}$ such that

$$0 < a_{\min} \leq a(\mathbf{x}, \omega) \leq a_{\max} < \infty, \quad \text{for a.e. } (\mathbf{x}, \omega) \in D \times \Omega.$$

2.3. Well-posedness of the stochastic obstacle problem

Let $V = L^2(\Omega; H_0^1(D))$, equipped with the inner product $(v, w)_V = \mathbb{E}[(\nabla v, \nabla w)_{L^2(D)}]$ and the corresponding norm $\|v\|_V = (v, v)_V^{1/2}$ for $v, w \in V$. We consider the following stochastic variational inequality ([24,26]):

$$u \in \mathcal{K}, \quad b(u, v - u) \geq \ell(v - u), \quad \forall v \in \mathcal{K}, \quad (2.4)$$

where $b(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$, $\ell : V \rightarrow \mathbb{R}$ and $\mathcal{K}$ are defined as follows:

$$b(u, v) := \mathbb{E} \left[\int_D a(\mathbf{x}, \omega) \nabla u(\mathbf{x}, \omega) \cdot \nabla v(\mathbf{x}, \omega) d\mathbf{x} \right], \quad (2.5)$$

$$\ell(v) := \mathbb{E} \left[\int_D f(\mathbf{x}, \omega) v(\mathbf{x}, \omega) d\mathbf{x} \right], \quad (2.6)$$

$$\mathcal{K} := \{v \in V \mid v(\mathbf{x}, \omega) \geq g(\mathbf{x}, \omega) \text{ a.e. in } D \times \Omega\}.$$

Theorem 1. Under Assumption 1, for any $f \in L^2(\Omega; L^2(D))$, the problem (2.4) admits a unique solution.

Proof. To prove the existence and uniqueness of a solution to (2.4), we show that the bilinear form is bounded and V -elliptic. Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned} b(u, v) &= \int_{\Omega} \int_D a \nabla u \cdot \nabla v d\mathbf{x} d\mathbb{P}(\omega) \\ &\leq a_{\max} \int_{\Omega} \int_D |\nabla u \cdot \nabla v| d\mathbf{x} d\mathbb{P}(\omega) \\ &\leq a_{\max} \|\nabla u\|_{L^2(\Omega; L^2(D))} \|\nabla v\|_{L^2(\Omega; L^2(D))} \\ &= a_{\max} \|u\|_V \|v\|_V. \end{aligned}$$

By Assumption 1, we have

$$\begin{aligned} b(v, v) &= \int_{\Omega} \int_D a \nabla v \cdot \nabla v d\mathbf{x} d\mathbb{P}(\omega) \\ &\geq a_{\min} \int_{\Omega} \int_D |\nabla v|^2 d\mathbf{x} d\mathbb{P}(\omega) = a_{\min} \|v\|_{L^2(\Omega; H_0^1(D))}^2. \end{aligned}$$

It is readily verified that $f \in L^2(\Omega; L^2(D))$ implies the continuity of the linear form ℓ on V . By applying Lemma 1, we conclude that (2.4) admits a unique solution. $\square$

Since the diffusion coefficient a , the source term f and obstacle function g are typically not known exactly in applications, it is useful to consider approximations for a , f , and g . We now examine a perturbation of the weak formulation (2.4) and analyze the magnitude of the resulting perturbation in the solution. Replacing a , f , and g in (2.4) by their approximations $\tilde{a}, \tilde{f}, \tilde{g} : D \times \Omega \rightarrow \mathbb{R}$ leads to a perturbed stochastic variational inequality:

$$\tilde{u} \in \tilde{\mathcal{K}}, \quad \tilde{b}(\tilde{u}, v - \tilde{u}) \geq \tilde{\ell}(v - \tilde{u}), \quad \forall v \in \tilde{\mathcal{K}}, \quad (2.7)$$

where $\tilde{b} : V \times V \rightarrow \mathbb{R}$ and $\tilde{\ell} : V \rightarrow \mathbb{R}$ are defined by

$$\tilde{b}(u, v) := \mathbb{E} \left[\int_D \tilde{a}(\mathbf{x}, \omega) \nabla u(\mathbf{x}, \omega) \cdot \nabla v(\mathbf{x}, \omega) d\mathbf{x} \right],$$

$$\tilde{\ell}(v) := \mathbb{E} \left[\int_D \tilde{f}(\mathbf{x}, \omega) v(\mathbf{x}, \omega) d\mathbf{x} \right],$$

$$\tilde{\mathcal{K}} := \{v \in V \mid v(\mathbf{x}, \omega) \geq \tilde{g}(\mathbf{x}, \omega) \text{ a.e. in } D \times \Omega\}.$$

As demonstrated in [19], we employ a general framework to establish the well-posedness of problem (2.7).

Proposition 1. Let H be a Hilbert space. Let $\mathcal{K}, \tilde{\mathcal{K}} \subset H$ be non-empty, closed, convex sets. Consider symmetric bilinear forms $B, \tilde{B} : H \times H \rightarrow \mathbb{R}$ that are H -coercive and bounded: there exist $0 < \alpha_{\min} \leq \alpha_{\max}$ such that

$$\alpha_{\min} \|v\|_H^2 \leq \min\{B(v, v), \tilde{B}(v, v)\}, \quad \max\{|B(v, w)|, |\tilde{B}(v, w)|\} \leq \alpha_{\max} \|v\|_H \|w\|_H$$

for all $v, w \in H$. Assume moreover that B and $\tilde{B}$ are comparable: there exists $\gamma \geq 0$ such that

$$|B(v, w) - \tilde{B}(v, w)| \leq \gamma \|v\|_H \|w\|_H \quad \forall v, w \in H. \quad (2.8)$$

Given $\mathcal{L}, \tilde{\mathcal{L}} \in H'$, let $u \in \mathcal{K}, \tilde{u} \in \tilde{\mathcal{K}}$ solve

$$B(u, v - u) \geq \mathcal{L}(v - u) \quad \forall v \in \mathcal{K}, \quad (2.9)$$

$$\tilde{B}(\tilde{u}, v - \tilde{u}) \geq \tilde{\mathcal{L}}(v - \tilde{u}) \quad \forall v \in \tilde{\mathcal{K}}. \quad (2.10)$$

Denote the distances

$$\delta_{\mathcal{K}} := \inf_{v \in \mathcal{K}} \|\tilde{u} - v\|_H, \quad \delta_{\tilde{\mathcal{K}}} := \inf_{v \in \tilde{\mathcal{K}}} \|u - v\|_H,$$

and set $e := u - \tilde{u}$. Then

$$\|e\|_H \leq \frac{1}{\alpha_{\min}} \left(\|\tilde{\mathcal{L}} - \mathcal{L}\|_{H'} + \gamma \|u\|_H \right) + \sqrt{\frac{2}{\alpha_{\min}} \left((\alpha_{\max} \|u\|_H + \|\mathcal{L}\|_{H'}) \delta_{\mathcal{K}} + (\alpha_{\max} \|\tilde{u}\|_H + \|\tilde{\mathcal{L}}\|_{H'}) \delta_{\tilde{\mathcal{K}}} \right)}. \quad (2.11)$$

In particular, if $\mathcal{K} = \tilde{\mathcal{K}}$, then $\delta_{\mathcal{K}} = \delta_{\tilde{\mathcal{K}}} = 0$ and

$$\|u - \tilde{u}\|_H \leq \frac{1}{\alpha_{\min}} \left(\|\tilde{\mathcal{L}} - \mathcal{L}\|_{H'} + \gamma \|u\|_H \right). \quad (2.12)$$

Moreover, if $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are convex cones with vertex at 0, then

$$\|u\|_H \leq \frac{1}{\alpha_{\min}} \|\mathcal{L}\|_{H'}, \quad \|\tilde{u}\|_H \leq \frac{1}{\alpha_{\min}} \|\tilde{\mathcal{L}}\|_{H'},$$

and consequently

$$\|u - \tilde{u}\|_H \leq \frac{1}{\alpha_{\min}} \left(\|\tilde{\mathcal{L}} - \mathcal{L}\|_{H'} + \frac{\gamma}{\alpha_{\min}} \|\mathcal{L}\|_{H'} \right) + \sqrt{\frac{2}{\alpha_{\min}} \left(1 + \frac{\alpha_{\max}}{\alpha_{\min}} \right) (\|\mathcal{L}\|_{H'} \delta_{\mathcal{K}} + \|\tilde{\mathcal{L}}\|_{H'} \delta_{\tilde{\mathcal{K}}})}. \quad (2.13)$$

Proof. Applying Lemma 1, we know that u and $\tilde{u}$ are well defined. Denote $e := u - \tilde{u}$. Choose $\tilde{u}_{\mathcal{K}} \in \mathcal{K}$ and $u_{\tilde{\mathcal{K}}} \in \tilde{\mathcal{K}}$ such that $\|\tilde{u}_{\mathcal{K}} - \tilde{u}\|_H = \delta_{\mathcal{K}}$ and $\|u - u_{\tilde{\mathcal{K}}}\|_H = \delta_{\tilde{\mathcal{K}}}$ (cf. [28, Section 3.4]). Using (2.9) with $v = \tilde{u}_{\mathcal{K}}$ and (2.10) with $v = u_{\tilde{\mathcal{K}}}$ yields

$$\mathcal{B}(u, \tilde{u}_{\mathcal{K}} - u) \geq \mathcal{L}(\tilde{u}_{\mathcal{K}} - u), \quad \tilde{\mathcal{B}}(\tilde{u}, u_{\tilde{\mathcal{K}}} - \tilde{u}) \geq \tilde{\mathcal{L}}(u_{\tilde{\mathcal{K}}} - \tilde{u}).$$

By coercivity of $\tilde{\mathcal{B}}$,

$$\alpha_{\min} \|e\|_H^2 \leq \tilde{\mathcal{B}}(e, e) = \tilde{\mathcal{B}}(u, e) - \tilde{\mathcal{B}}(\tilde{u}, e).$$

Insert and subtract $\mathcal{B}(u, e)$, and rewrite $\tilde{u}_{\mathcal{K}} - u = (\tilde{u}_{\mathcal{K}} - \tilde{u}) - e$ and $u_{\tilde{\mathcal{K}}} - \tilde{u} = e - (u - u_{\tilde{\mathcal{K}}})$ to obtain

$$\begin{aligned} \alpha_{\min} \|e\|_H^2 &\leq (\tilde{\mathcal{B}}(u, e) - \mathcal{B}(u, e)) + (\mathcal{L}(e) - \tilde{\mathcal{L}}(e)) \\ &\quad + (\mathcal{B}(u, \tilde{u}_{\mathcal{K}} - \tilde{u}) - \mathcal{L}(\tilde{u}_{\mathcal{K}} - \tilde{u})) - (\tilde{\mathcal{B}}(\tilde{u}, u - u_{\tilde{\mathcal{K}}}) - \tilde{\mathcal{L}}(u - u_{\tilde{\mathcal{K}}})). \end{aligned}$$

By (2.8) and boundedness,

$$\begin{aligned} |\tilde{\mathcal{B}}(u, e) - \mathcal{B}(u, e)| &\leq \gamma \|u\|_H \|e\|_H, \\ |\mathcal{L}(e) - \tilde{\mathcal{L}}(e)| &\leq \|\tilde{\mathcal{L}} - \mathcal{L}\|_{H'} \|e\|_H, \\ |\mathcal{B}(u, \tilde{u}_{\mathcal{K}} - \tilde{u}) - \mathcal{L}(\tilde{u}_{\mathcal{K}} - \tilde{u})| &\leq (\alpha_{\max} \|u\|_H + \|\mathcal{L}\|_{H'}) \|\tilde{u}_{\mathcal{K}} - \tilde{u}\|_H = (\alpha_{\max} \|u\|_H + \|\mathcal{L}\|_{H'}) \delta_{\mathcal{K}}, \\ |\tilde{\mathcal{B}}(\tilde{u}, u - u_{\tilde{\mathcal{K}}}) - \tilde{\mathcal{L}}(u - u_{\tilde{\mathcal{K}}})| &\leq (\alpha_{\max} \|\tilde{u}\|_H + \|\tilde{\mathcal{L}}\|_{H'}) \|u - u_{\tilde{\mathcal{K}}}\|_H = (\alpha_{\max} \|\tilde{u}\|_H + \|\tilde{\mathcal{L}}\|_{H'}) \delta_{\tilde{\mathcal{K}}}. \end{aligned}$$

Hence

$$\alpha_{\min} \|e\|_H^2 \leq \left(\gamma \|u\|_H + \|\tilde{\mathcal{L}} - \mathcal{L}\|_{H'} \right) \|e\|_H + (\alpha_{\max} \|u\|_H + \|\mathcal{L}\|_{H'}) \delta_{\mathcal{K}} + (\alpha_{\max} \|\tilde{u}\|_H + \|\tilde{\mathcal{L}}\|_{H'}) \delta_{\tilde{\mathcal{K}}}.$$

Applying Young's inequality to the linear term in $\|e\|_H$ with parameter $\alpha_{\min}/2$ gives

$$\frac{\alpha_{\min}}{2} \|e\|_H^2 \leq \frac{1}{2\alpha_{\min}} \left(\gamma \|u\|_H + \|\tilde{\mathcal{L}} - \mathcal{L}\|_{H'} \right)^2 + (\alpha_{\max} \|u\|_H + \|\mathcal{L}\|_{H'}) \delta_{\mathcal{K}} + (\alpha_{\max} \|\tilde{u}\|_H + \|\tilde{\mathcal{L}}\|_{H'}) \delta_{\tilde{\mathcal{K}}},$$

which implies (2.11). If $\mathcal{K} = \tilde{\mathcal{K}}$, the distances vanish and (2.12) follows.

If $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are cones with vertex at 0, testing (2.9) by $v = 0$ and $v = 2u$ yields $\mathcal{B}(u, u) = \mathcal{L}(u)$ and thus $\|u\|_H \leq \alpha_{\min}^{-1} \|\mathcal{L}\|_{H'}$; the same argument for $\tilde{u}$. Substituting these into (2.11) yields (2.13). $\square$

Corollary 1. Assume that $\text{Tr } g = \text{Tr } \tilde{g}$ on ∂D for almost every $\omega \in \Omega$, then the distance $\max(\delta_{\tilde{\mathcal{K}}}, \delta_{\mathcal{K}}) \leq \|\tilde{g} - g\|_H$.

Proof. From definition, $u + \max(\tilde{g} - g, 0) \geq \tilde{g}$. Then,

$$\begin{aligned} \delta_{\tilde{\mathcal{K}}} &= \inf_{v \in \tilde{\mathcal{K}}} \|u - v\|_H \leq \|u - (u + \max(\tilde{g} - g, 0))\|_H \\ &\leq \|g - \tilde{g}\|_H. \end{aligned}$$

Similarly, we have $\delta_{\mathcal{K}} \leq \|g - \tilde{g}\|_H$. $\square$

Corollary 2. Let $H = L^2(\Omega; H_0^1(D))$. Keep the assumptions stated in [Theorem 1](#). Assume the perturbed coefficient $\tilde{a}$ satisfies $0 < a_{\min} \leq \tilde{a} \leq a_{\max} < \infty$, a.e. on $D \times \Omega$ and $f, \tilde{f} \in L^2(\Omega; L^2(D))$. Let $u \in \mathcal{K}$ and $\tilde{u} \in \tilde{\mathcal{K}}$ be the unique solutions to (2.4) and (2.7), respectively.

Then

$$\begin{aligned} \|u - \tilde{u}\|_H &\leq \frac{1}{a_{\min}} \left(\|\tilde{a} - a\|_{L^\infty(\Omega; L^\infty(D))} \|u\|_H + C_D \|\tilde{f} - f\|_{L^2(\Omega; L^2(D))} \right) \\ &\quad + \sqrt{\frac{2}{a_{\min}} \left((a_{\max} \|u\|_H + C_D \|f\|_{L^2(\Omega; L^2(D))}) \delta_{\mathcal{K}} + (a_{\max} \|\tilde{u}\|_H + C_D \|\tilde{f}\|_{L^2(\Omega; L^2(D))}) \delta_{\tilde{\mathcal{K}}} \right)}. \end{aligned} \quad (2.14)$$

In particular, if $g = \tilde{g}$ then (2.14) reduces to

$$\|u - \tilde{u}\|_{L^2(\Omega; H_0^1(D))} \leq a_{\min}^{-1} \left(\|\tilde{a} - a\|_{L^\infty(\Omega; L^\infty(D))} \|u\|_{L^2(\Omega; H_0^1(D))} + C_D \|\tilde{f} - f\|_{L^2(\Omega; L^2(D))} \right), \quad (2.15)$$

where $C_D > 0$ is the Poincaré constant for the domain D , i.e., $\|v\|_{L^2(D)} \leq C_D \|v\|_{H_0^1(D)}$ for any $v \in H_0^1(D)$.

Proof. We apply [Proposition 1](#). From

$$\begin{aligned} |\mathbb{E}[\int_D (a - \tilde{a}) \nabla v \cdot \nabla w \, dx]| &\leq \|\tilde{a} - a\|_{L^\infty(\Omega; L^\infty(D))} \left(\mathbb{E}[\int_D |\nabla v|^2 \, dx] \right)^{1/2} \left(\mathbb{E}[\int_D |\nabla w|^2 \, dx] \right)^{1/2} \\ &= \|\tilde{a} - a\|_{L^\infty(\Omega; L^\infty(D))} \|v\|_H \|w\|_H, \end{aligned}$$

we see that (2.8) holds with $\gamma = \|\tilde{a} - a\|_{L^\infty(\Omega; L^\infty(D))}$. Then, (2.14) follows from (2.11). $\square$

2.4. Karhunen-Loève expansions

Randomness in many systems can often be effectively approximated using a small number of uncorrelated or independent random variables. In this work, we approximate the random fields a , f and g by truncated Karhunen-Loève expansions ([30,32]).

Denote by $\mu_a(\mathbf{x})$ the expectation of $a(\mathbf{x}, \omega)$. Let V_a be the covariance function of a , and let $\mathcal{V}_a : L^2(D) \rightarrow L^2(D)$ be the corresponding covariance operator. We denote by $\{(v_k^a, \phi_k^a)\}_{k \in \mathbb{N}_+}$ the eigenpair sequence of $\mathcal{V}_a$, where $v_1^a \geq v_2^a \geq \dots \geq 0$.

Proposition 2. ([33]) If $a \in L^2(D; L^2(\Omega))$, then there exists a sequence of random variables $\{\xi_k^a\}_{k \geq 1} \subset L^2(\Omega)$ satisfying

$$\int_{\Omega} \xi_i^a(\omega) \, d\mathbb{P}(\omega) = 0, \quad \int_{\Omega} \xi_i^a(\omega) \xi_j^a(\omega) \, d\mathbb{P}(\omega) = \delta_{ij} \quad \forall i, j \geq 1$$

and such that the random field $a(\cdot, \cdot)$ can be expanded in $L^2(D; L^2(\Omega))$ as

$$a(\mathbf{x}, \omega) = \mu_a(\mathbf{x}) + \sum_{k=1}^{\infty} \sqrt{v_k^a} \phi_k^a(\mathbf{x}) \xi_k^a(\omega), \quad (2.16)$$

where $\{(v_k^a, \phi_k^a)\}_{k \in \mathbb{N}_+}$ is the eigenpair sequence of the operator $\mathcal{V}_a$. Moreover, for $v_k^a \neq 0$,

$$\xi_k^a(\omega) := \frac{1}{\sqrt{v_k^a}} \int_D (a(\mathbf{x}, \omega) - \mu_a(\mathbf{x})) \phi_k^a(\mathbf{x}) \, d\mathbf{x} \quad \forall k \geq 1.$$

Based on the expansion (2.16), for any $M_a \in \mathbb{N}$, we define the truncated diffusion coefficient by

$$a_{M_a}(\mathbf{x}, \omega) = \mu_a(\mathbf{x}) + \sum_{k=1}^{M_a} \sqrt{v_k^a} \phi_k^a(\mathbf{x}) \xi_k^a(\omega). \quad (2.17)$$

Similarly, the source term f and the obstacle function g are approximated by

$$f_{M_f}(\mathbf{x}, \omega) = \mu_f(\mathbf{x}) + \sum_{k=1}^{M_f} \sqrt{v_k^f} \phi_k^f(\mathbf{x}) \xi_k^f(\omega), \quad (2.18)$$

and

$$g_{M_g}(\mathbf{x}, \omega) = \mu_g(\mathbf{x}) + \sum_{k=1}^{M_g} \sqrt{v_k^g} \phi_k^g(\mathbf{x}) \xi_k^g(\omega). \quad (2.19)$$

We denote the truncated approximations of the coefficient, source term and obstacle function, by $\tilde{a} = a_{M_a}$, $\tilde{f} = f_{M_f}$ and $\tilde{g} = g_{M_g}$ and use the notation $\tilde{a}, \tilde{f}, \tilde{g}$ throughout to refer to the Karhunen-Loève approximations of a, f, g with truncation levels M_a, M_f, M_g , respectively.

[Proposition 2](#) implies the L^2 -convergence of the truncated Karhunen-Loève expansion. In particular,

$$\|a - a_{M_a}\|_{L^2(\Omega; L^2(D))} \rightarrow 0 \quad \text{as } M_a \rightarrow \infty. \quad (2.20)$$

Similarly, if $f \in L^2(\Omega; L^2(D))$, then

$$\|f - \tilde{f}\|_{L^2(\Omega; L^2(D))} \rightarrow 0 \quad \text{as } M_f \rightarrow \infty.$$

Moreover, if $g \in L^2(\Omega; H^1(D))$ ([34, Theorem 7.2.8] and [33]), then

$$\|g - g_{M_g}\|_{L^2(\Omega; H^1(D))} \rightarrow 0 \quad \text{as } M_g \rightarrow \infty.$$

To apply Corollary 2, a stronger convergence of the truncated diffusion coefficient is needed. Indeed, Proposition 2 only gives L^2 -convergence, while the perturbation estimate requires

$$\|a - a_{M_a}\|_{L^\infty(\Omega; L^\infty(D))} \rightarrow 0 \quad \text{as } M_a \rightarrow \infty.$$

One sufficient condition is given below.

Proposition 3. ([35]) *If the covariance V_a of a is piecewise analytic on $D \times D$ and $\{\xi_k^a\}_{k \in \mathbb{N}_+}$ is bounded uniformly, then*

$$\|a - a_{M_a}\|_{L^\infty(\Omega; L^\infty(D))} \leq c_a \exp\left(-c_{1,a} M_a^{1/n}\right) \quad \forall M_a \in \mathbb{N}, \quad (2.21)$$

where n is the dimension of physical space, c_a is a bound of $\{\xi_k^a\}_{k \in \mathbb{N}_+}$, $c_{1,a}$ is a positive constant.

While Assumption 1 ensures that a is uniformly positive and bounded, this does not necessarily guarantee that the same lower and upper bounds are preserved by its finite-term truncation a_{M_a} , unless M_a is sufficiently large; see Proposition 3. However, if $\{\xi_k^a\}_{k \in \mathbb{N}_+}$ are independent, then under Assumption 1, these bounds hold for $\tilde{a}$ as well for any $M_a \geq 0$. As shown in [35, Remark 3.3], if we denote $\mathcal{F}_{M_a} \subset \mathcal{F}$ as the σ -algebra generated by the random variables $\xi_1, \dots, \xi_{M_a}$, it follows that

$$\mathbb{E}[a | \mathcal{F}_{M_a}] = a_{M_a}.$$

The monotonicity of conditional expectations ensures then that the lower and upper bounds of a hold for a_{M_a} too, i.e.

$$0 < a_{\min} \leq a_{M_a}(\mathbf{x}, \omega) \leq a_{\max} < \infty \quad \text{a.e. in } D \times \Omega.$$

Consequently, problem (2.7) is well-posed. For simplicity, we assume that ξ_k^a , ξ_k^f and ξ_k^g have the same distribution ([30, page 389]).

2.5. The case of a finite-dimensional noise

The weak form of the problem (2.4) is not an ideal starting point for numerical approximation, as it involves integration over the abstract probability space Ω with respect to the measure $\mathbb{P}$. To enable efficient numerical discretization, it is advantageous to express the randomness using a finite number of independent random variables. To this end, it is customary to transform the problem onto a parametric domain $\Gamma \subset \mathbb{R}^M$, where the randomness is encoded by a finite set of independent random variables $\xi = [\xi_1, \dots, \xi_M]^\top$.

Specifically, we focus on the transformation of the bilinear form

$$b(u, v) = \int_{\Omega} \int_D a(\mathbf{x}, \omega) \nabla u(\mathbf{x}, \omega) \cdot \nabla v(\mathbf{x}, \omega) \, d\mathbf{x} \, d\mathbb{P}(\omega).$$

A similar discussion can be made for $\ell(v)$.

To make this formulation more tractable, we assume that the coefficient $\tilde{a}$, source term $\tilde{f}$ and obstacle function $\tilde{g}$ are functions of finitely many random variables $\xi_k: \Omega \rightarrow \Gamma_k \subset \mathbb{R}$, for $1 \leq k \leq M$, as is the case for truncated Karhunen-Loève expansions (see Section 2.4). In this setting, each realization $\omega \in \Omega$ corresponds to a point $\mathbf{y} = [y_1, \dots, y_M]^\top \in \Gamma := \Gamma_1 \times \dots \times \Gamma_M \subset \mathbb{R}^M$, and the stochastic functions $\tilde{a}(\mathbf{x}, \omega)$, $\tilde{f}(\mathbf{x}, \omega)$, $\tilde{g}(\mathbf{x}, \omega)$ and $\tilde{u}(\mathbf{x}, \omega)$ can be equivalently represented as deterministic parametric functions $\tilde{a}(\mathbf{x}, \mathbf{y})$, $\tilde{f}(\mathbf{x}, \mathbf{y})$, $\tilde{g}(\mathbf{x}, \mathbf{y})$ and $\tilde{u}(\mathbf{x}, \mathbf{y})$. This setting is referred to as the finite-dimensional noise case ([30, page 394]).

Definition 1 (Finite-dimensional noise). Let $\xi_k: \Omega \rightarrow \Gamma_k$, $k = 1, \dots, M$, be real-valued random variables, $M < \infty$. A function $v \in L^2(\Omega; L^2(D))$ of the form $v(\mathbf{x}, \xi(\omega))$ for $\mathbf{x} \in D$ and $\omega \in \Omega$, where $\xi = [\xi_1, \dots, \xi_M]^\top: \Omega \rightarrow \Gamma \subset \mathbb{R}^M$ and $\Gamma := \Gamma_1 \times \dots \times \Gamma_M$, is called a finite-dimensional or an M -dimensional noise.

In what follows we use $p(\mathbf{y})$ to denote the joint probability density of $\xi = [\xi_1, \dots, \xi_M]^\top$, and $\Gamma = \prod_{k=1}^M \Gamma_k$. Each ξ_i is assumed to have a density function $p_i(y_i): \Gamma_i \rightarrow \mathbb{R}^+$ for $i = 1, \dots, M$. If $\{\xi_i\}_{i=1}^M$ are independent, then $p(\mathbf{y}) = \prod_{i=1}^M p_i(y_i)$.

For a finite-dimensional noise $v = v(\mathbf{x}, \xi)$, we can compute its expectation using a change of variables. If p is the joint density of ξ , then

$$\|v\|_{L^2(\Omega; L^2(D))}^2 = \mathbb{E}[\|v\|_{L^2(D)}^2] = \int_{\Gamma} p(\mathbf{y}) \|v(\cdot, \mathbf{y})\|_{L^2(D)}^2 \, d\mathbf{y},$$

where $y_k := \xi_k(\omega)$ and $\mathbf{y} = [y_1, \dots, y_M]^\top$. Thus, any random field $v(\mathbf{x}, \xi) \in L^2(\Omega; L^2(D))$ can be associated with a function $v = v(\mathbf{x}, \mathbf{y})$ belonging to the weighted L^2 space

$$L_p^2(\Gamma; L^2(D)) := \left\{ v: D \times \Gamma \rightarrow \mathbb{R} \mid \int_{\Gamma} p(\mathbf{y}) \|v(\cdot, \mathbf{y})\|_{L^2(D)}^2 \, d\mathbf{y} < \infty \right\}.$$

If the diffusion coefficient $a(\mathbf{x}, \omega)$, the source $f(\mathbf{x}, \omega)$ and obstacle function $g(\mathbf{x}, \omega)$ are finite-dimensional noises, we can combine ξ_a , ξ_f and ξ_g to form a finite-dimensional random variable $\xi = [\xi_a, \xi_f, \xi_g]$. In this case, $a(\mathbf{x}, \cdot)$, $f(\mathbf{x}, \cdot)$ and $g(\mathbf{x}, \cdot)$ are $\sigma(\xi)$ -measurable, and the solution $u(\mathbf{x}, \cdot)$ of (2.4) is also $\sigma(\xi)$ -measurable ([25, Theorem 3.3] and [36, Proposition 1.1]). Accordingly, the three groups of components of the parameter vector $\mathbf{y} = [y_1, \dots, y_M]^\top$ corresponding to ξ_a , ξ_f , and ξ_g are, respectively, $\{y_k\}_{k=1}^{M_a}$, $\{y_{M_a+k}\}_{k=1}^{M_f}$, $\{y_{M_a+M_f+k}\}_{k=1}^{M_g}$. Under the assumption of finite-dimensional noise, $u(\mathbf{x}, \omega)$ can be described by the same random variable ξ by Doob-Dynkin lemma (cf. [37, Proposition 3]). Thus, we denote the solution as $u(\mathbf{x}, \mathbf{y})$ to explicitly represent $u(\mathbf{x}, \omega)$.

Returning to the weak form (2.7), we obtain its deterministic equivalent. Let $\tilde{a}$, $\tilde{f}$ and $\tilde{g}$ be the truncated Karhunen-Loève expansions (2.17)–(2.19), define

$$W := L_p^2(\Gamma; H_0^1(D)) = \left\{ v : D \times \Gamma \rightarrow \mathbb{R} \mid \int_{\Gamma} p(\mathbf{y}) \|v(\mathbf{x}, \mathbf{y})\|_{H_0^1(D)}^2 d\mathbf{y} < \infty \right\},$$

$$\tilde{\mathcal{K}} := \{v \in W \mid v(\mathbf{x}, \mathbf{y}) \geq \tilde{g}(\mathbf{x}, \mathbf{y}) \text{ a.e. in } D \times \Gamma\}.$$

The variational inequality (2.7) on $D \times \Gamma$ is

$$\tilde{u} \in \tilde{\mathcal{K}}, \quad \tilde{b}(\tilde{u}, v - \tilde{u}) \geq \tilde{\ell}(v - \tilde{u}), \quad \forall v \in \tilde{\mathcal{K}}, \quad (2.22)$$

where $\tilde{b} : W \times W \rightarrow \mathbb{R}$ and $\tilde{\ell} : W \rightarrow \mathbb{R}$ are defined by

$$\tilde{b}(u, v) := \int_{\Gamma} p(\mathbf{y}) \int_D \tilde{a}(\mathbf{x}, \mathbf{y}) \nabla u(\mathbf{x}, \mathbf{y}) \cdot \nabla v(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}, \quad (2.23)$$

$$\tilde{\ell}(v) := \int_{\Gamma} p(\mathbf{y}) \int_D \tilde{f}(\mathbf{x}, \mathbf{y}) v(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}.$$

The weight $p : \Gamma \rightarrow \mathbb{R}^+$ is the joint density of $\xi = [\xi_1, \dots, \xi_M]^\top$, where $M = M_a + M_f + M_g$ and $\xi : \Omega \rightarrow \Gamma \subset \mathbb{R}^M$. Under these assumptions, $\tilde{u} \in \tilde{\mathcal{K}}$ is uniquely determined by (2.22). Consequently, we have transformed the stochastic problem (2.7) into the deterministic problem (2.22).

Transforming a stochastic variational inequality into its deterministic parametric form is a crucial step that enables the application of finite element methods to approximate solutions. Working over Γ instead of Ω , we have

$$\tilde{a}(\mathbf{x}, \mathbf{y}) = \mu_a(\mathbf{x}) + \sum_{k=1}^{M_a} \sqrt{v_k^a} \phi_k^a(\mathbf{x}) y_k, \quad \mathbf{x} \in D, \mathbf{y} \in \Gamma$$

and a similar expression for $\tilde{f}$. We can ensure the existence of a unique solution of (2.22) under the following conditions.

Assumption 2. There exist two positive constants $\tilde{a}_{\min} \leq \tilde{a}_{\max}$ such that

$$0 < \tilde{a}_{\min} \leq \tilde{a}(\mathbf{x}, \mathbf{y}) \leq \tilde{a}_{\max} < \infty, \quad \text{a.e. in } D \times \Gamma.$$

Recall that if $\xi_1, \dots, \xi_{M_a}$ are independent, then Assumption 1 implies

$$a_{\min} \leq \tilde{a}(\mathbf{x}, \mathbf{y}) \leq a_{\max}.$$

Theorem 2. Under Assumption 2, for any $\tilde{f} \in L_p^2(\Gamma; L^2(D))$, the variational inequality (2.22) admits a unique solution.

Since our primary focus is on developing numerical methods for the stochastic obstacle problem rather than modeling the problem itself, we consider synthetic expansions of the form (2.17). In these expansions, the pairs $(\sqrt{v_k^a}, \phi_k^a)$ are carefully selected so that the solution representation has specific, desirable properties. Notably, in our numerical experiments, the coefficient expansions are synthetically constructed and are not derived from predefined covariance operators. Hence, they do not correspond to classical Karhunen-Loève expansions in the strict sense.

3. Finite element spaces

In this section, we begin by examining standard finite element spaces over the spatial domain $D \subset \mathbb{R}^n$ and the parameter domain $\Gamma \subset \mathbb{R}^M$ independently. Next, we introduce tensor product finite element spaces on $D \times \Gamma$, which will serve as the foundation for approximating solutions to problem (2.22). Finally, we conclude with a discussion of key approximation properties of the tensor product finite element spaces.

3.1. Finite element spaces on the spatial domain

We begin by considering the finite element spaces defined over the spatial domain D . If D is a polyhedral domain, then it can be partitioned into triangles and quadrilaterals. For a general domain, the partition may be more complicated. In this paper, we assume the domain D is a polygon, and consider to partition D into triangles τ . Let $\{\mathcal{T}_h^D\}_{h>0}$ represent a regular family of triangulations of D , such that $\bar{D} = \cup_{\tau \in \mathcal{T}_h^D} \bar{\tau}$. Define $h = \max_{\tau \in \mathcal{T}_h^D} h_\tau$, where h_τ denotes the diameter of the element τ . We assume the triangulations of D are regular, which means there exists a positive constant σ such that $h_\tau/\rho_\tau \leq \sigma$ for all $\tau \in \mathcal{T}_h^D$, where ρ_τ is the diameter of the largest inscribed circle in τ . Moreover, the mesh parameter h approaches zero (cf. [38, Section 4.4] and [39, page 124] and [28, page 411]).

Consider the finite-dimensional space $X^h \subset H_0^1(D)$, consisting of piecewise linear continuous functions defined on $\mathcal{T}_h^D$. We define the interpolation operator Π_h by

$$\Pi_h : H^2(D) \cap H_0^1(D) \rightarrow X^h, \quad \Pi_h v = \sum_{i=1}^I v(\mathbf{x}_i) \varphi_i,$$

where $\{\mathbf{x}_i\}_{i=1}^I \subset \bar{D}$ is the set of interior node points in D and $\{\varphi_i\}_{i=1}^I$ is the corresponding finite element basis functions of the corresponding nodes. Then ([28, Theorem 10.3.9] and [19]) the following approximation properties hold:

There exists a constant $C > 0$ independent of h such that for any $v \in H^2(D) \cap H_0^1(D)$,

$$\|v - \Pi_h v\|_{m,D} \leq Ch^{2-m} |v|_{2,D}, \quad m = 0, 1. \quad (3.1)$$

3.2. Tensor product spaces on $\Gamma \subset \mathbb{R}^M$

Next, we consider the finite-dimensional space defined on $\Gamma \subset \mathbb{R}^M$ [19,21]. Let $\Gamma = \prod_{k=1}^M \Gamma_k$ be as in Section 2.5. Consider a partition of Γ consisting of a finite number of boxes, $B_\gamma = \prod_{k=1}^M [a_k^\gamma, b_k^\gamma] \subset \mathbb{R}^M$, $[a_k^\gamma, b_k^\gamma] \subset \overline{\Gamma_k}$ for $k = 1, \dots, M$, such that, for some finite index set I ,

$$\bar{\Gamma} = \bigcup_{\gamma \in I} B_\gamma = \bigcup_{\gamma \in I} \prod_{k=1}^M [a_k^\gamma, b_k^\gamma],$$

where $B_{\gamma_1}^\circ \cap B_{\gamma_2}^\circ = \emptyset$ for distinct $\gamma_1, \gamma_2 \in I$. The maximum grid size parameter $0 < s_k < 1$ is defined as

$$s_k = \max \{b_k^\gamma - a_k^\gamma \mid \gamma \in I\},$$

for $k = 1, \dots, M$, and we set $\bar{s} = (s_1, \dots, s_M)$. For each nonnegative integer multi-index $q = (q_1, \dots, q_M)$, consider the finite element approximation space of piecewise polynomials of a degree at most q_k in each direction y_k , denoted by $Y_s^q \subset L^2(\Gamma)$. If $\varphi \in Y_s^q$, then its restriction to each of the partition boxes satisfies

$$\varphi|_{B_\gamma} \in \text{span} \left\{ \prod_{k=1}^M y_k^{\alpha_k} \mid \alpha_k \in \mathbb{N}, \alpha_k \leq q_k, k = 1, \dots, M \right\}.$$

The finite element spaces Y_s^q satisfy (cf. [38, Section 4.6] and [19]) the following approximation property: let Π_s be the standard tensor product interpolation operator. Then, there are positive constants $\{C_k\}_{k=1}^M$ depending only on q_k such that for any $v \in H^{q+1}(\Gamma)$,

$$\|v - \Pi_s v\|_{L^2(\Gamma)} \leq \sum_{k=1}^M C_k s_k^{q_k+1} \|\partial_{y_k}^{q_k+1} v\|_{L^2(\Gamma)}.$$

Similarly for $\mathcal{T}_h^D$, we will use the $\mathcal{T}_s^\Gamma$ to denote the division of Γ with respect to $\bar{s}$.

Additionally, if we define $s = \max \{b_k^\gamma - a_k^\gamma \mid \gamma \in I, k = 1, \dots, M\}$, it follows that

$$\|v - \Pi_s v\|_{L^2(\Gamma)} \leq s^\lambda \sum_{k=1}^M C_k \|\partial_{y_k}^{q_k+1} v\|_{L^2(\Gamma)}, \quad (3.2)$$

where $\lambda = \min_{1 \leq j \leq M} \{q_j + 1\}$.

In the subsequent error analysis, we assume that the joint probability density satisfies

$$0 < p_{\min} \leq p(\mathbf{y}) \leq p_{\max} < \infty \quad \text{for a.e. } \mathbf{y} \in \Gamma.$$

Consequently, the weighted and unweighted L^2 norms on Γ are equivalent, and estimate (3.2) also holds in the corresponding weighted Bochner norms. We further assume that the functions to which Π_s is applied are continuous with respect to the parameter variable, so that the nodal tensor-product interpolation operator Π_s is well defined.

In this paper, we select piecewise polynomials of degree 1 in each direction y_n , i.e., $q = (1, \dots, 1)$, and for simplicity, we denote $Y^s = Y_s^q$.

Having defined the finite element spaces on D and Γ , we now construct the tensor product space over the product domain $D \times \Gamma$.

3.3. Tensor product finite element spaces on $D \times \Gamma$

Combining the finite element spaces X^h and Y^s , we construct the following tensor product space:

$$X^h \otimes Y^s = \text{span} \{ \varphi_i(\mathbf{x}) \psi_j(\mathbf{y}) \mid \varphi_i(\mathbf{x}) \in X^h, \psi_j(\mathbf{y}) \in Y^s \}.$$

Let $\{\mathbf{x}_i\}_{i=1}^I \subset \bar{D}$ be the set of interior spatial nodes from the triangulation $\mathcal{T}_h^D$ and $\{\mathbf{y}_j\}_{j=1}^J \subset \Gamma$ be the set of parameter nodes from the partition $\mathcal{T}_s^\Gamma$, which are the vertices of the boxes B_γ . The finite element nodes of $X^h \otimes Y^s$ are obtained as the Cartesian product of the spatial and parameter nodes:

$$\{(\mathbf{x}_i, \mathbf{y}_j) \in D \times \Gamma \mid i = 1, \dots, I, j = 1, \dots, J\}.$$

The corresponding basis functions of $X^h \otimes Y^s$ are given by the tensor products

$$\Phi_{ij}(\mathbf{x}, \mathbf{y}) = \varphi_i(\mathbf{x})\psi_j(\mathbf{y}), \quad i = 1, \dots, I, \quad j = 1, \dots, J,$$

where φ_i are the piecewise linear finite element basis functions of X^h and ψ_j are the piecewise multilinear finite element basis functions of Y^s . This tensor product framework generates a total of $I \times J$ basis functions, facilitating the approximation of solutions over the product domain $D \times \Gamma$.

4. Stochastic finite element methods

In this section, we consider a fully discrete method. In the fully discrete method, we discretize both the spatial and parameter domains.

4.1. Fully discrete method

Let $W^{hs} = X^h \otimes Y^s$ and define

$$\tilde{\mathcal{K}}^{hs} = \{v^{hs} \in W^{hs} \mid v^{hs}(\mathbf{x}, \mathbf{y}) \geq \tilde{g}(\mathbf{x}, \mathbf{y}) \text{ for any node } (\mathbf{x}, \mathbf{y})\}.$$

Then, the fully discrete method to solve (2.22) is

$$\tilde{u}^{hs} \in \tilde{\mathcal{K}}^{hs}, \quad \tilde{b}(\tilde{u}^{hs}, v^{hs} - \tilde{u}^{hs}) \geq \tilde{\ell}(v^{hs} - \tilde{u}^{hs}), \quad \forall v^{hs} \in \tilde{\mathcal{K}}^{hs}, \quad (4.1)$$

where $\tilde{b}: W \times W \rightarrow \mathbb{R}$ and $\tilde{\ell}: W \rightarrow \mathbb{R}$ are defined as in (2.23).

Clearly, $\tilde{\mathcal{K}}^{hs}$ is a closed convex subset of W^{hs} .

Theorem 3. *If the conditions of Theorem 2 hold, then (4.1) admits a unique solution.*

Now we provide an error analysis for the fully discrete method. For this purpose, we recall the following generalized Céa lemma ([28, page 445]).

Lemma 2 ([28]). *Let V be a real Hilbert space and $K \subset V$ be non-empty, closed and convex. Let $V^h \subset V$ be a finite element space and $K^h \subset V^h$ be non-empty, closed and convex. Assume the bilinear form $b(\cdot, \cdot)$ is bounded and V -elliptic, and $l \in V'$.*

If u is the solution of

$$u \in K, \quad b(u, v - u) \geq (f, v - u) \quad \forall v \in K,$$

and $u^h \in K^h$ is the finite element solution defined by

$$u^h \in K^h, \quad b(u^h, v^h - u^h) \geq (f, v^h - u^h) \quad \forall v^h \in K^h,$$

then

$$\|u - u^h\|^2 \leq c \inf_{v \in K} R(v - u^h) + c \inf_{v^h \in K^h} \left[R(v^h - u) + c_1 \|u - v^h\|^2 \right], \quad (4.2)$$

where

$$R(v) = b(u, v) - \ell(v).$$

Observe that if $K^h \subset K$, then the inequality (4.2) is reduced to

$$\|u - u^h\|^2 \leq c \inf_{v^h \in K^h} \left[R(v^h - u) + c_1 \|u - v^h\|^2 \right].$$

Theorem 4. *Let Assumption 2 hold, and let $\tilde{u}$ and $\tilde{u}^{hs}$ be solutions of the problem (2.22) and (4.1). Assume $\tilde{f} \in L_p^2(\Gamma; L^2(D))$, $\text{div}(\tilde{a}\nabla\tilde{u}) \in L_p^2(\Gamma; L^2(D))$ and $\tilde{u}, \tilde{g} \in H_p^2(\Gamma; H^2(D)) \cap C(\bar{\Gamma}; H^2(D))$. Then*

$$\|\tilde{u} - \tilde{u}^{hs}\|_W \leq c(h + s),$$

where c depending only on $\|\tilde{u}\|_{H_p^2(\Gamma; H^2(D))}$, $\|\tilde{f}\|_{L_p^2(\Gamma; L^2(D))}$ and $\|\tilde{g}\|_{H_p^2(\Gamma; H^2(D))}$.

Proof. Recall that $W = L_p^2(\Gamma; H_0^1(D))$. Assumption 2 implies

$$\tilde{a}_{\min} \|\tilde{u} - \tilde{u}^{hs}\|_W^2 \leq \tilde{b}(\tilde{u} - \tilde{u}^{hs}, \tilde{u} - \tilde{u}^{hs}) \leq \tilde{a}_{\max} \|\tilde{u} - \tilde{u}^{hs}\|_W^2.$$

Denote

$$\tilde{R}(v) = \tilde{b}(\tilde{u}, v) - \tilde{\ell}(v).$$

First, we bound the term $\tilde{R}(v)$:

$$\tilde{R}(v) = \tilde{b}(\tilde{u}, v) - \tilde{\ell}(v) = \int_{\Gamma} p(\mathbf{y}) \int_D \left[\tilde{a}(\mathbf{x}, \mathbf{y}) \nabla \tilde{u} \cdot \nabla v - \tilde{f}(v) \right] d\mathbf{x} d\mathbf{y}.$$

Applying integration by parts, we have

$$\tilde{R}(v) = \int_{\Gamma} p(y) \int_D \left[-\operatorname{div}(\tilde{a}(\mathbf{x}, \mathbf{y}) \nabla \tilde{u}) - \tilde{f} \right] v \, d\mathbf{x} \, dy \leq \left\| -\operatorname{div}(\tilde{a} \nabla \tilde{u}) - \tilde{f} \right\|_{L_p^2(\Gamma; L^2(D))} \|v\|_{L_p^2(\Gamma; L^2(D))}.$$

Lemma 2 implies

$$\begin{aligned} \|\tilde{u} - \tilde{u}^{hs}\|_W^2 &\leq c \inf_{v^{hs} \in \mathcal{K}^{hs}} \left[\tilde{R}(v^{hs} - \tilde{u}) + \|\tilde{u} - v^{hs}\|_W^2 \right] + c \inf_{v \in \mathcal{K}} \tilde{R}(v - \tilde{u}^{hs}) \\ &\leq c \left\{ \inf_{v^{hs} \in \mathcal{K}^{hs}} \left[\|\tilde{u} - v^{hs}\|_W^2 + \left\| -\operatorname{div}(\tilde{a} \nabla \tilde{u}) - \tilde{f} \right\|_{L_p^2(\Gamma; L^2(D))} \|v^{hs} - \tilde{u}\|_{L_p^2(\Gamma; L^2(D))} \right] \right. \\ &\quad \left. + \left\| -\operatorname{div}(\tilde{a} \nabla \tilde{u}) - \tilde{f} \right\|_{L_p^2(\Gamma; L^2(D))} \inf_{v \in \mathcal{K}} \|v - \tilde{u}^{hs}\|_{L_p^2(\Gamma; L^2(D))} \right\}. \end{aligned}$$

Apparently, $\Pi_s \Pi_h \tilde{u} \in \mathcal{K}^{hs}$, where Π_h is the continuous piecewise linear interpolation operator (3.1), and Π_s is the tensor product interpolation operator (3.2).

According to finite element interpolation error estimates (3.1), we have

$$\|v - \Pi_h v\|_{m,D} \leq Ch^{2-m} |v|_{2,D}, \quad m = 0, 1.$$

It follows that

$$\|v - \Pi_h v\|_{L_p^2(\Gamma; H^m(D))} \leq Ch^{2-m} |v|_{L_p^2(\Gamma; H^2(D))}, \quad m = 0, 1.$$

From the triangle inequality of the norm, we have

$$\|\Pi_h \tilde{u}\|_{L_p^2(\Gamma; L^2(D))} \leq c \|\tilde{u}\|_{L_p^2(\Gamma; H^2(D))} \quad \text{and} \quad \|\Pi_h \tilde{u}\|_{L_p^2(\Gamma; H_0^1(D))} \leq c \|\tilde{u}\|_{L_p^2(\Gamma; H^2(D))}.$$

Due to $\partial_y^\alpha (\Pi_h \tilde{u}(\cdot, y)) = \Pi_h (\partial_y^\alpha \tilde{u}(\cdot, y))$ for all multi-indices α , we have

$$\|\Pi_h \tilde{u}\|_{H_p^2(\Gamma; L^2(D))}^2 = \sum_{|\alpha| \leq 2} \int_{\Gamma} p(y) \left\| \Pi_h (\partial_y^\alpha \tilde{u}) \right\|_{L^2(D)}^2 dy \leq c \sum_{|\alpha| \leq 2} \|\partial_y^\alpha \tilde{u}\|_{L_p^2(\Gamma; H^2(D))}^2 = c \|\tilde{u}\|_{H_p^2(\Gamma; H^2(D))}^2.$$

Then

$$\begin{aligned} \|\Pi_s \Pi_h \tilde{u} - \tilde{u}\|_{L_p^2(\Gamma; L^2(D))} &\leq \|\Pi_s \Pi_h \tilde{u} - \Pi_h \tilde{u}\|_{L_p^2(\Gamma; L^2(D))} + \|\Pi_h \tilde{u} - \tilde{u}\|_{L_p^2(\Gamma; L^2(D))} \\ &\leq cs^2 \|\Pi_h \tilde{u}\|_{H_p^2(\Gamma; L^2(D))} + ch^2 \|\tilde{u}\|_{L_p^2(\Gamma; H^2(D))} \\ &\leq cs^2 \|\tilde{u}\|_{H_p^2(\Gamma; H^2(D))} + ch^2 \|\tilde{u}\|_{L_p^2(\Gamma; H^2(D))} \\ &\leq c(s^2 + h^2) \|\tilde{u}\|_{H_p^2(\Gamma; H^2(D))}. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} \|\Pi_s \Pi_h \tilde{g} - \tilde{g}\|_{L_p^2(\Gamma; L^2(D))} &\leq c(s^2 + h^2) \|\tilde{g}\|_{H_p^2(\Gamma; H^2(D))}, \\ \|\Pi_s \Pi_h \tilde{u} - \tilde{u}\|_{L_p^2(\Gamma; H_0^1(D))} &\leq c(s^2 + h) \|\tilde{u}\|_{H_p^2(\Gamma; H^2(D))}. \end{aligned}$$

Hence,

$$\begin{aligned} \inf_{v^{hs} \in \mathcal{K}^{hs}} \left[\|\tilde{u} - v^{hs}\|_W^2 + \left\| -\operatorname{div}(\tilde{a} \nabla \tilde{u}) - \tilde{f} \right\|_{L_p^2(\Gamma; L^2(D))} \|v^{hs} - \tilde{u}\|_{L_p^2(\Gamma; L^2(D))} \right] \\ \leq \|\tilde{u} - \Pi_s \Pi_h \tilde{u}\|_W^2 + \left\| -\operatorname{div}(\tilde{a} \nabla \tilde{u}) - \tilde{f} \right\|_{L_p^2(\Gamma; L^2(D))} \|\Pi_s \Pi_h \tilde{u} - \tilde{u}\|_{L_p^2(\Gamma; L^2(D))} \\ \leq c(h^2 + s^2). \end{aligned}$$

Following the idea of Falk [7], we bound the $\inf_{v \in \mathcal{K}} \|v - \tilde{u}^{hs}\|_{L_p^2(\Gamma; L^2(D))}$.

Denote

$$\tilde{u}^{hs,*}(\mathbf{x}, \mathbf{y}) = \max\{\tilde{u}^{hs}(\mathbf{x}, \mathbf{y}), \tilde{g}(\mathbf{x}, \mathbf{y})\}, \quad (D \times \Gamma)^* = \{\mathbf{x} \in D, \mathbf{y} \in \Gamma \mid \tilde{u}^{hs}(\mathbf{x}, \mathbf{y}) < \tilde{g}(\mathbf{x}, \mathbf{y})\}.$$

Then, over $D \times \Gamma \setminus (D \times \Gamma)^*$, $\tilde{u}^{hs,*} = \tilde{u}^{hs}$. Since $\tilde{u}^{hs} \in L_p^2(\Gamma; H_0^1(D))$ and $\tilde{g} \in L_p^2(\Gamma; H^2(D))$ with $\tilde{g} \leq 0$ on $\partial D \times \Gamma$, clearly, $\tilde{u}^{hs,*} \in \mathcal{K}$,

$$\inf_{v \in \mathcal{K}} \|v - \tilde{u}^{hs}\|_{L_p^2(\Gamma; L^2(D))}^2 \leq \|\tilde{u}^{hs,*} - \tilde{u}^{hs}\|_{L_p^2(\Gamma; L^2(D))}^2 = \int_{(D \times \Gamma)^*} |\tilde{u}^{hs,*}(\mathbf{x}, \mathbf{y}) - \tilde{u}^{hs}(\mathbf{x}, \mathbf{y})|^2 p(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y}.$$

Let $\Pi_s \Pi_h \tilde{g}$ be the continuous piecewise linear interpolant of $\tilde{g}$. Since the discrete solution satisfies the nodal constraint, at any node $(\mathbf{x}_i, \mathbf{y}_j)$, $\tilde{u}^{hs}(\mathbf{x}_i, \mathbf{y}_j) \geq \tilde{g}(\mathbf{x}_i, \mathbf{y}_j)$, we have $\tilde{u}^{hs} \geq \Pi_s \Pi_h \tilde{g}$ in $D \times \Gamma$. Hence, over $(D \times \Gamma)^*$, we have

$$0 < |\tilde{u}^{hs}(\mathbf{x}, \mathbf{y}) - \tilde{g}(\mathbf{x}, \mathbf{y})| = \tilde{g}(\mathbf{x}, \mathbf{y}) - \tilde{u}^{hs}(\mathbf{x}, \mathbf{y}) \leq \tilde{g}(\mathbf{x}, \mathbf{y}) - \Pi_s \Pi_h \tilde{g}(\mathbf{x}, \mathbf{y}).$$

Thus, we get

$$\begin{aligned} \int_{(D \times \Gamma)^*} |\tilde{u}^{hs,*}(\mathbf{x}, \mathbf{y}) - \tilde{u}^{hs}(\mathbf{x}, \mathbf{y})|^2 p(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} &\leq \int_{(D \times \Gamma)^*} |\tilde{g}(\mathbf{x}, \mathbf{y}) - \Pi_s \Pi_h \tilde{g}(\mathbf{x}, \mathbf{y})|^2 p(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} \\ &\leq \int_{D \times \Gamma} |\tilde{g}(\mathbf{x}, \mathbf{y}) - \Pi_s \Pi_h \tilde{g}(\mathbf{x}, \mathbf{y})|^2 p(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} \\ &= \|\tilde{g} - \Pi_s \Pi_h \tilde{g}\|_{L_p^2(\Gamma; L^2(D))}^2. \end{aligned}$$

Then

$$\inf_{v \in \mathcal{K}} \|v - \tilde{u}^{hs}\|_{L_p^2(\Gamma; L^2(D))} \leq c(s^2 + h^2) \|\tilde{g}\|_{H_p^2(\Gamma; H^2(D))}.$$

Finally, we get the order estimate

$$\|\tilde{u} - \tilde{u}^{hs}\|_W \leq c(h + s),$$

where the constant $c > 0$ depending only on $\|\tilde{u}\|_{L_p^2(\Gamma; H^2(D))}$, $\|\tilde{f}\|_{H_p^2(\Gamma; L^2(D))}$ and $\|\tilde{g}\|_{H_p^2(\Gamma; H^2(D))}$. $\square$

Then, by Jensen's inequality [29, Theorem 1.6.2] we obtain

$$\|\mathbb{E}[\tilde{u}] - \mathbb{E}[\tilde{u}^{hs}]\|_{H_0^1(D)}^2 \leq \mathbb{E} \left[\|\tilde{u} - \tilde{u}^{hs}\|_{H_0^1(D)}^2 \right].$$

Using the definition $\|v\|_W^2 = \mathbb{E} \left[\|v\|_{H_0^1(D)}^2 \right]$, we conclude that

$$\|\mathbb{E}[\tilde{u}] - \mathbb{E}[\tilde{u}^{hs}]\|_{H_0^1(D)}^2 \leq \|\tilde{u} - \tilde{u}^{hs}\|_W^2.$$

Taking the square root on both sides,

$$\|\mathbb{E}[\tilde{u}] - \mathbb{E}[\tilde{u}^{hs}]\|_{H_0^1(D)} \leq \|\tilde{u} - \tilde{u}^{hs}\|_W \leq c(h + s).$$

Similarly, following [22], we derive the following estimate for the second-moment error:

$$\begin{aligned} \|\mathbb{E}[(\tilde{u})^2 - (\tilde{u}^{hs})^2]\|_{L^1(D)} &= \|\mathbb{E}[(\tilde{u} - \tilde{u}^{hs})(\tilde{u} + \tilde{u}^{hs})]\|_{L^1(D)} \\ &\leq \|\tilde{u} - \tilde{u}^{hs}\|_{L_p^2(\Gamma; L^2(D))} \|\tilde{u} + \tilde{u}^{hs}\|_{L_p^2(\Gamma; L^2(D))} \\ &\leq c \|\tilde{u} - \tilde{u}^{hs}\|_W (\|\tilde{u}\|_W + \|\tilde{u}^{hs}\|_W) \\ &\leq c \|\tilde{u} - \tilde{u}^{hs}\|_W \\ &\leq c(h + s). \end{aligned}$$

4.2. Stochastic Galerkin system

We now discuss the nonlinear algebraic system that must be solved to compute mean and variance following the notation of [30, page 405]. Our approximate space is the tensor-product space $W^{hs} = X^h \otimes Y^s$, where X^h is the finite element space of dimension I and Y^s is the piecewise polynomial space of dimension J . Consequently, the total number of degrees of freedom is IJ .

Starting from the fully discrete formulation (4.1),

$$\begin{aligned} \tilde{b}(u, v) &= \int_{\Gamma} p(\mathbf{y}) \int_D \tilde{a}(\mathbf{x}, \mathbf{y}) \nabla u(\mathbf{x}, \mathbf{y}) \cdot \nabla v(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y}, \\ \tilde{\ell}(v) &= \int_{\Gamma} p(\mathbf{y}) \int_D \tilde{f}(\mathbf{x}, \mathbf{y}) v(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y}. \end{aligned}$$

The parameter fields are represented by the truncated Karhunen-Loève expansion (2.17) and (2.18)

$$\tilde{a}(\mathbf{x}, \mathbf{y}) = \mu_a(\mathbf{x}) + \sum_{k=1}^{M_a} \sqrt{v_k^a} \phi_k^a(\mathbf{x}) y_k, \quad \tilde{f}(\mathbf{x}, \mathbf{y}) = \mu_f(\mathbf{x}) + \sum_{k=1}^{M_f} \sqrt{v_k^f} \phi_k^f(\mathbf{x}) y_{M_a+k}.$$

We seek $\tilde{u}^{hs} \in \tilde{\mathcal{K}}^{hs}$ such that

$$\tilde{u}^{hs} \in \tilde{\mathcal{K}}^{hs}, \quad \tilde{b}(\tilde{u}^{hs}, v - \tilde{u}^{hs}) \geq \tilde{\ell}(v - \tilde{u}^{hs}), \quad \forall v \in \tilde{\mathcal{K}}^{hs}.$$

By expanding $\tilde{u}^{hs}$ as in (4.1), $\tilde{u}^{hs}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^I \sum_{j=1}^J u_{ij} \phi_i(\mathbf{x}) \psi_j(\mathbf{y})$ and testing against all basis functions $v^{hs} = \phi_r \psi_t$ for $r = 1, \dots, I$ and $t = 1, \dots, J$. This process yields a matrix $A \in \mathbb{R}^{IJ \times IJ}$ representing the discretized bilinear form and a vector $\mathbf{b} \in \mathbb{R}^{IJ}$ representing the discretized linear form:

$$\begin{aligned} [A]_{(r,t),(i,j)} &= \tilde{b}(\phi_i \psi_j, \phi_r \psi_t), \\ [\mathbf{b}]_{(r,t)} &= \tilde{\ell}(\phi_r \psi_t), \end{aligned} \quad \text{for } r, i = 1, \dots, I \text{ and } t, j = 1, \dots, J.$$

This results in a block-structured matrix and corresponding vectors:

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1J} \\ A_{21} & A_{22} & \dots & A_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ A_{J1} & A_{J2} & \dots & A_{JJ} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_J \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_J \end{pmatrix}.$$

The j^{th} block of the solution vector is given by

$$\mathbf{u}_j = [u_{1j}, u_{2j}, \dots, u_{Ij}]^T, \quad j = 1, \dots, J.$$

The constituent blocks are given by

$$A_{tj} = \langle \psi_j, \psi_t \rangle_p K_0 + \sum_{k=1}^{M_a} \langle y_k \psi_j, \psi_t \rangle_p K_k, \\ \mathbf{b}_t = \langle \psi_t, 1 \rangle_p \mathbf{f}_0 + \sum_{k=1}^{M_f} \langle y_{M_a+k}, \psi_t \rangle_p \mathbf{f}_k, \quad \text{for } t, j = 1, \dots, J.$$

The finite element matrices $K_0, K_k \in \mathbb{R}^{I \times I}$ and vector $\mathbf{f}_0, \mathbf{f}_k \in \mathbb{R}^I$ are defined as

$$[K_0]_{ri} = \int_D \mu_a(\mathbf{x}) \nabla \phi_i(\mathbf{x}) \cdot \nabla \phi_r(\mathbf{x}) \, d\mathbf{x}, \quad [K_k]_{ri} = \int_D \left(\sqrt{v_k^a} \phi_k^a(\mathbf{x}) \right) \nabla \phi_i(\mathbf{x}) \cdot \nabla \phi_r(\mathbf{x}) \, d\mathbf{x}, \\ [\mathbf{f}_0]_r = \int_D \mu_f(\mathbf{x}) \phi_r(\mathbf{x}) \, d\mathbf{x}, \quad [\mathbf{f}_k]_r = \int_D \left(\sqrt{v_k^f} \phi_k^f(\mathbf{x}) \right) \phi_r(\mathbf{x}) \, d\mathbf{x}.$$

This block structure admits a compact Kronecker product representation [40,41]:

$$A = G_0 \otimes K_0 + \sum_{k=1}^{M_a} G_k \otimes K_k, \\ \mathbf{b} = \mathbf{g}_0 \otimes \mathbf{f}_0 + \sum_{k=1}^{M_f} \mathbf{g}_k \otimes \mathbf{f}_k,$$

where the matrices $G_0, G_k \in \mathbb{R}^{J \times J}$ and vectors $\mathbf{g}_0, \mathbf{g}_k \in \mathbb{R}^J$ are defined by

$$[G_0]_{jt} = \langle \psi_j, \psi_t \rangle_p, \quad [G_k]_{jt} = \langle y_k \psi_j, \psi_t \rangle_p, \\ [\mathbf{g}_0]_t = \langle \psi_t, 1 \rangle_p, \quad [\mathbf{g}_k]_t = \langle y_{M_a+k}, \psi_t \rangle_p.$$

Crucially, the discretization of the variational inequality (4.1) is not the linear system $\mathbf{A}\mathbf{u} = \mathbf{b}$, but rather the nonlinear complementarity problem:

$$\mathbf{u} \geq \mathbf{g}, \quad \mathbf{A}\mathbf{u} - \mathbf{b} \geq \mathbf{0}, \quad (\mathbf{u} - \mathbf{g})^\top (\mathbf{A}\mathbf{u} - \mathbf{b}) = 0,$$

where $\mathbf{g} \in \mathbb{R}^{IJ}$ is the vector of nodal values of the obstacle, $g_{ij} = \tilde{g}(\mathbf{x}_i, \mathbf{y}_j)$ and the last condition implies component-wise complementarity:

$$(u_{ij} - g_{ij})[A\mathbf{u} - \mathbf{b}]_{ij} = 0 \quad i = 1, \dots, I \text{ and } j = 1, \dots, J.$$

When adapting the active set method [42,43] to solve the variational inequality (4.1), at any node $(\mathbf{x}_{i'}, \mathbf{y}_{j'})$, we need only satisfy the condition

$$\tilde{u}^{hs}(\mathbf{x}_{i'}, \mathbf{y}_{j'}) = \sum_{i=1}^I \sum_{j=1}^J u_{ij} \phi_i(\mathbf{x}_{i'}) \psi_j(\mathbf{y}_{j'}) \geq g(\mathbf{x}_{i'}, \mathbf{y}_{j'}),$$

which is equivalent to the condition $u_{ij} \geq g_{ij}$ for $i = 1, \dots, I$ and $j = 1, \dots, J$.

5. Numerical experiments

In this section, we present numerical experiments to illustrate the performance of the proposed stochastic Galerkin method. The main quantities of interest are the expected value and the second moment of the solution. We use the finite element spaces X^h and Y^s introduced in Section 3, with bases $\{\phi_i\}_{i=1}^I$ and $\{\psi_j\}_{j=1}^J$, respectively. The fully discrete stochastic Galerkin approximation is written as

$$\tilde{u}^{hs}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^I \sum_{j=1}^J u_{ij} \phi_i(\mathbf{x}) \psi_j(\mathbf{y}) = \sum_{j=1}^J \tilde{u}_j(\mathbf{x}) \psi_j(\mathbf{y}), \quad \tilde{u}_j(\mathbf{x}) = \sum_{i=1}^I u_{ij} \phi_i(\mathbf{x}).$$

Table 1Relative errors of expected values with $s = (e - e^{-1})/2^4$.

h	$e_{L^2}^{rel,1}(u, u^{hs})$	order	$e_{H^1}^{rel,1}(u, u^{hs})$	order
$3/2^2$	3.6004e-01		4.7088e-01	
$3/2^3$	7.6845e-02	2.2281	2.3794e-01	0.9847
$3/2^4$	2.1563e-02	1.8333	1.2199e-01	0.9638
$3/2^5$	5.9221e-03	1.8644	6.1089e-02	0.9979

Table 2Relative errors of second moments with $s = (e - e^{-1})/2^4$.

h	$e_{L^2}^{rel,2}(u, u^{hs})$	order	$e_{H^1}^{rel,2}(u, u^{hs})$	order
$3/2^2$	8.0635e-01		7.6605e-01	
$3/2^3$	2.2570e-01	1.8370	4.2420e-01	0.8527
$3/2^4$	5.9037e-02	1.9347	2.1764e-01	0.9628
$3/2^5$	1.6042e-02	1.8797	1.0952e-01	0.9907

Table 3Relative errors of expected values with $h = \frac{3}{2(e-e^{-1})}s$.

s	$e_{L^2}^{rel,1}(u, u^{hs})$	order	$e_{H^1}^{rel,1}(u, u^{hs})$	order
$(e - e^{-1})/2^0$	4.1212e-01		4.7822e-01	
$(e - e^{-1})/2^1$	8.9613e-02	2.2013	2.3942e-01	0.9981
$(e - e^{-1})/2^2$	2.4148e-02	1.8918	1.2207e-01	0.9717
$(e - e^{-1})/2^3$	5.9221e-03	2.0277	6.1089e-02	0.9988

After the coefficients u_{ij} are obtained, the expected value and the second moment are computed by

$$\mathbb{E}[\tilde{u}^{hs}](\mathbf{x}) = \sum_{j=1}^J \tilde{u}_j(\mathbf{x}) \int_{\Gamma} p(\mathbf{y}) \psi_j(\mathbf{y}) d\mathbf{y},$$

and

$$\mathbb{E}[(\tilde{u}^{hs})^2](\mathbf{x}) = \sum_{k=1}^J \sum_{j=1}^J \tilde{u}_k(\mathbf{x}) \tilde{u}_j(\mathbf{x}) \int_{\Gamma} p(\mathbf{y}) \psi_k(\mathbf{y}) \psi_j(\mathbf{y}) d\mathbf{y}.$$

The numerical errors are evaluated for the expected value and the second moment of the solution. For the expected value, we define

$$e_{L^2}^{rel,1}(u, u^{hs}) = \frac{\|\mathbb{E}[u] - \mathbb{E}[u^{hs}]\|_{L^2(D)}}{\|\mathbb{E}[u]\|_{L^2(D)}}, \quad e_{H^1}^{rel,1}(u, u^{hs}) = \frac{\|\nabla \mathbb{E}[u] - \nabla \mathbb{E}[u^{hs}]\|_{L^2(D)}}{\|\nabla \mathbb{E}[u]\|_{L^2(D)}}.$$

For the second moment, we define

$$e_{L^2}^{rel,2}(u, u^{hs}) = \frac{\|\mathbb{E}[u^2] - \mathbb{E}[(u^{hs})^2]\|_{L^2(D)}}{\|\mathbb{E}[u^2]\|_{L^2(D)}}, \quad e_{H^1}^{rel,2}(u, u^{hs}) = \frac{\|\nabla \mathbb{E}[u^2] - \nabla \mathbb{E}[(u^{hs})^2]\|_{L^2(D)}}{\|\nabla \mathbb{E}[u^2]\|_{L^2(D)}}.$$

We consider two numerical examples. [Example 1](#) involves a stochastic diffusion coefficient, while [Example 2](#) involves a stochastic source term. These examples are designed to test the convergence behavior of the method under different types of stochasticity.

Example 1. Let $D = (-1.5, 1.5)^2$ and

$$a(x_1, x_2, \omega) = 1 + \exp(\xi_1) + 2 \exp(\xi_2),$$

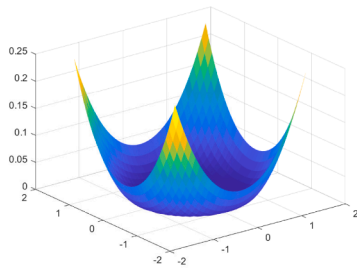
$$f(x_1, x_2, \omega) = -2,$$

where $\xi_1, \xi_2 \sim \text{i.i.d. } U(-1, 1)$, and the obstacle function $g = 0$. The exact solution is

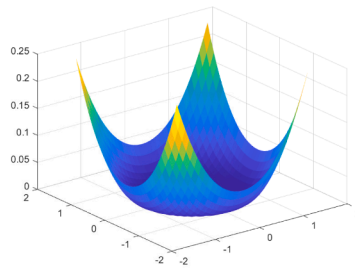
$$u(x_1, x_2, \xi_1, \xi_2) = \begin{cases} 0, & x_1^2 + x_2^2 \leq 1, \\ \left(\frac{x_1^2 + x_2^2}{2} - \frac{\log(x_1^2 + x_2^2)}{2} - \frac{1}{2} \right) * \frac{1}{1 + \exp(\xi_1) + 2 \exp(\xi_2)}, & x_1^2 + x_2^2 > 1. \end{cases}$$

From [Tables 1](#) and [2](#), we observe approximately first-order convergence in the H^1 -norm for both the expected value and the second moment. The result for the expected value is consistent with the theoretical estimate, while the observed H^1 -convergence of the second-moment error is stronger than the L^1 -estimate established above.

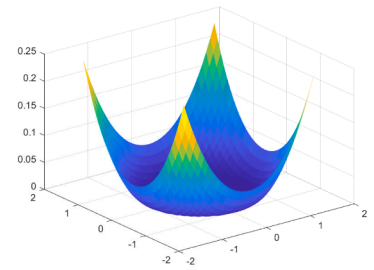
Since the error in the probability space is not significant, it is difficult to observe the convergence order with respect to s when h is fixed, even at $h = 2^{-5}$. To mitigate this issue, in [Table 3](#), we adopt a compromise by setting $h = \frac{3}{2(e-e^{-1})}s$, aiming to better capture the convergence behavior with respect to s .



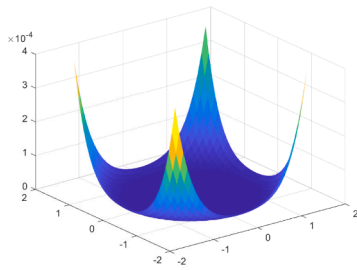
(a) Expectation of the Exact solution



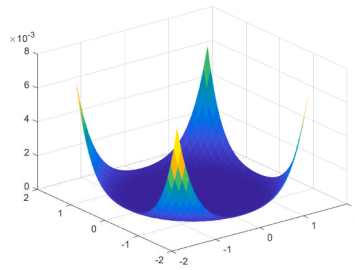
(b) Expectation of the SG solution



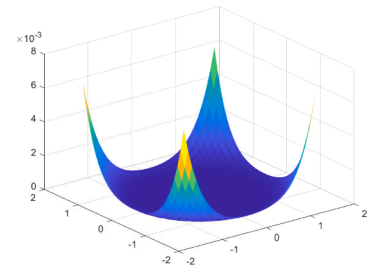
(c) Mean of the MC solution



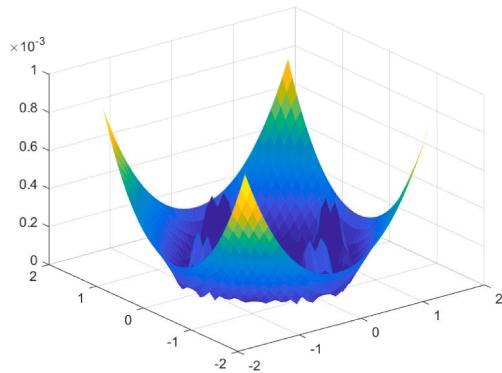
(d) Variance of the Exact solution



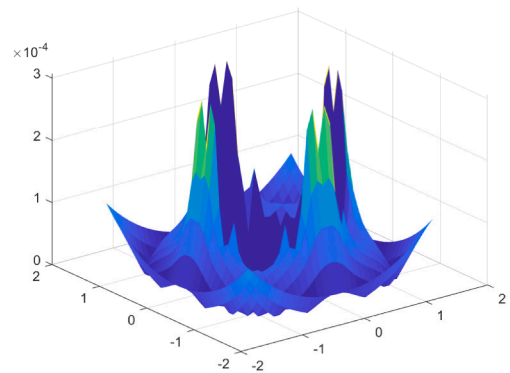
(e) Variance of the SG solution



(f) Variance of the MC solution

Fig. 1. Comparison of statistical properties: expectation and variance of the exact solution, SG solution, and MC solution.

(a) Error between expectation of exact solution and SG solution



(b) Error between expectation of exact solution and MC solution

Fig. 2. Absolute error in the expectation: comparison between the SG and MC solutions with respect to the exact solution.

Fig. 1 demonstrates the high accuracy of the SG method, which also exhibits superior computational efficiency, completing in just 10.49 seconds compared to the MC method's 344.56 seconds for processing 2^{15} samples. However, as shown in **Fig. 2**, while the MC solution outperforms the SG solution, it does so at the cost of increased computational resource consumption.

Example 2. Let $D = (-1, 1)^2$ and

$$a(x_1, x_2, \omega) = 1,$$

$$f(x_1, x_2, \omega) = \begin{cases} 8r^2(x_1^2 + x_2^2 - 1 - r^2) * (\exp(\xi_1) + 2 \exp(\xi_2)), & x_1^2 + x_2^2 \leq r^2, \\ -8(2x_1^2 + 2x_2^2 - r^2) * (\exp(\xi_1) + 2 \exp(\xi_2)), & x_1^2 + x_2^2 > r^2, \end{cases}$$

Table 4Relative errors of expected values with $s = (e - e^{-1})/2^3$.

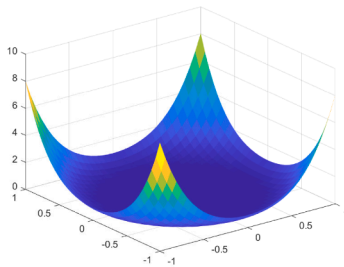
h	$e_{L^2}^{rel,1}(u, u^{hs})$	order	$e_{H^1}^{rel,1}(u, u^{hs})$	order
$1/2^2$	1.3864e-01		3.2220e-01	
$1/2^3$	3.5350e-02	1.9716	1.6350e-01	0.9786
$1/2^4$	8.9860e-03	1.9760	8.2092e-02	0.9940
$1/2^5$	2.2315e-03	2.0096	4.1091e-02	0.9984

Table 5Relative errors of second moments with $s = (e - e^{-1})/2^3$.

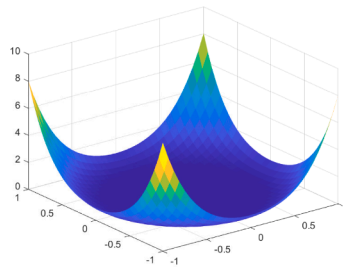
h	$e_{L^2}^{rel,2}(u, u^{hs})$	order	$e_{H^1}^{rel,2}(u, u^{hs})$	order
$1/2^2$	4.1182e-01		6.0197e-01	
$1/2^3$	1.0932e-01	1.9134	3.1815e-01	0.9200
$1/2^4$	2.7766e-02	1.9772	1.6141e-01	0.9789
$1/2^5$	6.9689e-03	1.9944	8.1007e-02	0.9947

Table 6Relative errors of expected values with $h = s/(e - e^{-1})$.

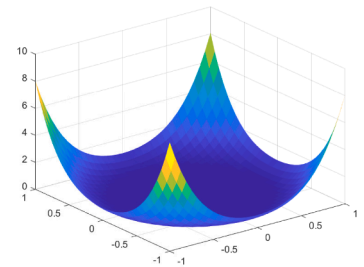
s	$e_{L^2}^{rel,1}(u, u^{hs})$	order	$e_{H^1}^{rel,1}(u, u^{hs})$	order
$(e - e^{-1})/2^1$	5.4709e-01		6.0731e-01	
$(e - e^{-1})/2^2$	1.3864e-01	1.9804	3.2220e-01	0.9145
$(e - e^{-1})/2^3$	3.5350e-02	1.9716	1.6350e-01	0.9786
$(e - e^{-1})/2^4$	8.9860e-03	1.9760	8.2092e-02	0.9940



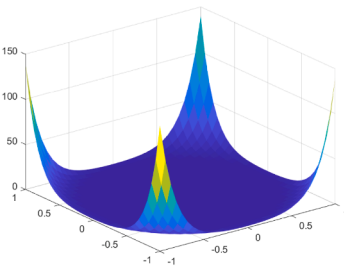
(a) Expectation of the Exact solution



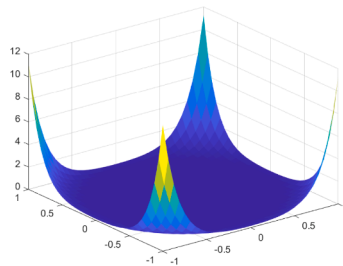
(b) Expectation of the SG solution



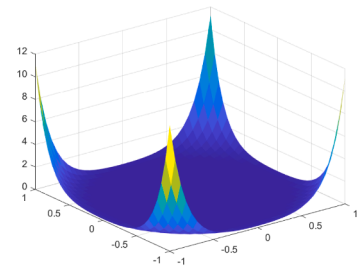
(c) Mean of the MC solution



(d) Variance of the Exact solution



(e) Variance of the SG solution

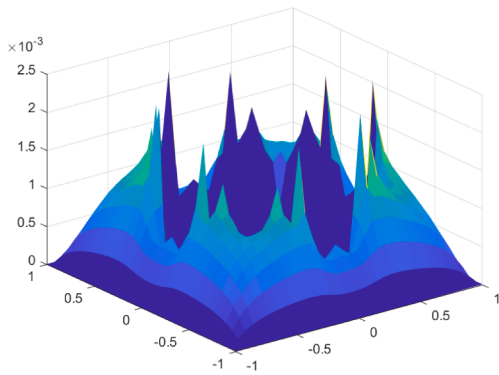


(f) Variance of the MC solution

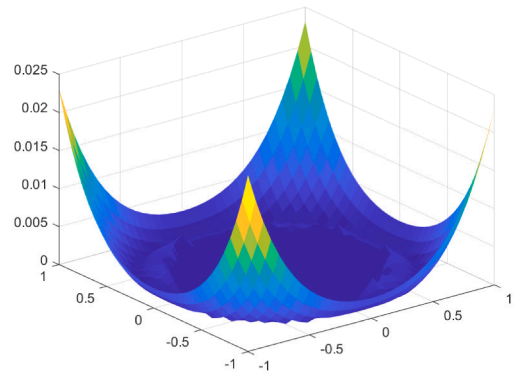
Fig. 3. Comparison of statistical properties: expectation and variance of the exact solution, SG solution, and MC solution.

for $\xi_1, \xi_2 \sim \text{i.i.d. } U(-1, 1)$, $g = 0$, and $r = 0.7$. The exact solution is

$$u(x, y, \omega) = \begin{cases} 0, & x_1^2 + x_2^2 \leq r^2, \\ (x_1^2 + x_2^2 - r^2) * (\exp(\xi_1) + 2 \exp(\xi_2)), & x_1^2 + x_2^2 > r^2. \end{cases}$$



(a) Error between expectation of exact solution and SG solution



(b) Error between expectation of exact solution and MC solution

Fig. 4. Absolute error in the expectation: comparison between the SG and MC solutions with respect to the exact solution.

From Tables 4 and 5, we again observe approximately first-order convergence in the H^1 -norm for both the expected value and the second moment. To investigate the convergence behavior as s decreases while balancing the spatial and stochastic discretization errors, we set $h = s/(e - e^{-1})$; the corresponding results are presented in Table 6.

The SG method not only achieves high accuracy, as demonstrated in Figs. 3 and 4, but also exhibits superior computational efficiency, completing in just 9.82 s. In contrast, the MC method requires 349.29 s to process 2^{15} samples, yet it still falls short of the true solution. Consequently, the MC method would need significantly more samples to approximate the true solution, thereby increasing its computational demands.

Our numerical experiments demonstrate that both the expectation error and the second moment error converge at an $O(h)$ rate in the H^1 -norm.

6. Conclusion

In this study, we developed stochastic Galerkin formulations to solve stochastic obstacle problems and established the well-posedness of the discrete formulation. Our numerical analysis provided an optimal-order error estimate for linear finite element in the H^1 -norm with respect to the physical variable. Numerical experiments supported these theoretical findings, showing that the expectation error and the second-moment error of the solution converge at a rate of $O(h)$ in the H^1 -norm. This agrees with the derived estimate for the expectation, while the H^1 -convergence of the second-moment is stronger than the theoretical $L^1(D)$ -estimate.

The main limitation of the proposed SG method is the possible curse of dimensionality. When the random inputs have strong spatial variability or complex stochastic structures, more random variables may be needed for an accurate representation, leading to high-dimensional tensor-product discretizations and increased computational cost. Moreover, the SG method is intrusive and leads to a coupled deterministic variational inequality, whose assembly and solution are more involved than those of non-intrusive sampling-based methods. The theoretical error estimate also relies on sufficient regularity of the parametric solution, which may be weakened when the coincidence set is highly sensitive to the random variables.

This approach can be extended to more complex stochastic problems, including time-dependent variational inequalities, while adaptive strategies will be designed to refine the stochastic discretization dynamically based on error indicators, focusing computational effort on regions of high variability. We will also explore the comparative advantages of stochastic collocation methods against the SG approach in handling diverse randomness and solution structures. To address the challenges of high-dimensional random spaces, sparse grids, low-rank tensor approximations, and reduced-order modeling will be employed to mitigate the curse of dimensionality and enhance efficiency. Finally, to improve accuracy in problems with limited regularity, we aim to develop specialized finite element spaces or enriched basis functions that better capture solution features.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

- [1] G. Duvaut, J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin; New York, 1976.
- [2] N. Kikuchi, J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, 1988.
- [3] W. Han, B.D. Reddy, *Plasticity: Mathematical Theory and Numerical Analysis*, Springer, New York, second edition, 2013.
- [4] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, SIAM, Philadelphia, 2000.
- [5] W. Han, M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, vol. 30 of *Studies in Advanced Mathematics*, American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2002.
- [6] C. Eck, J. Jarušek, M. Krbeč, *Unilateral Contact Problems: Variational Methods and Existence Theorems*, vol. 270 of *Monographs and Textbooks in Pure and Applied Mathematics*, CRC Press, Boca Raton, 2005.
- [7] R.S. Falk, Error estimates for the approximation of a class of variational inequalities, *Math. Comput.* 28 (1974) 963–971.
- [8] R. Glowinski, J.-L. Lions, R. Trémoilères, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.
- [9] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.
- [10] I. Hlaváček, J. Haslinger, J. Nečas, J. Lovíšek, *Solution of Variational Inequalities in Mechanics*, Springer-Verlag, New York, 1988.
- [11] F. Wang, W. Han, X. Cheng, Discontinuous Galerkin methods for solving elliptic variational inequalities, *SIAM J. Numer. Anal.* 48 (2010) 708–733.
- [12] F. Wang, W. Han, X. Cheng, Discontinuous Galerkin methods for solving a quasistatic contact problem, *Numer. Math.* 126 (2014) 771–800.
- [13] F. Wang, H. Wei, Virtual element method for simplified friction problem, *Appl. Math. Lett.* 85 (2018) 125–131.
- [14] F. Wang, H. Wei, Virtual element methods for the obstacle problem, *IMA J. Numer. Anal.* 40 (2020) 708–728.
- [15] F. Feng, W. Han, X. Cheng, Virtual element methods for elliptic variational inequalities of the second kind, *J. Sci. Comput.* 80 (2019) 60–80.
- [16] X. Huang, Q. Ran, F. Wang, Virtual element method for an elastic-rigid obstacle problem, *Commun. Nonlinear Sci. Numer. Simul.* 194 (2026) 110256.
- [17] M.K. Deb, I. Babuška, J.T. Oden, Solution of stochastic partial differential equations using galerkin finite element techniques, *Comput. Methods Appl. Mech. Eng.* 190 (2001) 6359–6372.
- [18] I. Babuška, P. Chatzipantelidis, On solving elliptic stochastic partial differential equations, *Comput. Methods Appl. Mech. Eng.* 191 (2002) 4093–4122.
- [19] I. Babuška, R. Tempone, G.E. Zouraris, Galerkin finite element approximations of stochastic elliptic partial differential equations, *SIAM J. Numer. Anal.* 42 (2004) 800–825.
- [20] I. Babuška, R. Tempone, G.E. Zouraris, Solving elliptic boundary value problems with uncertain coefficients by the finite element method: the stochastic formulation, *Comput. Methods Appl. Mech. Eng.* 194 (2005) 1251–1294.
- [21] W. Shen, L. Ge, W. Liu, Stochastic Galerkin method for optimal control problem governed by random elliptic PDE with state constraints, *J. Sci. Comput.* 78 (3) (2019) 1571–1600.
- [22] I. Babuška, F. Nobile, R. Tempone, A stochastic collocation method for elliptic partial differential equations with random input data, *SIAM J. Numer. Anal.* 45 (2007) 1005–1034.
- [23] D. Xiu, Efficient collocational approach for parametric uncertainty analysis, *Commun. Comput. Phys.* 2 (2007) 293–309.
- [24] R. Forster, R. Kornhuber, A polynomial chaos approach to stochastic variational inequalities, *J. Numer. Math.* 18 (4) (2010) 235–255.
- [25] R. Kornhuber, C. Schwab, M.W. Wolf, Multilevel monte carlo finite element methods for stochastic elliptic variational inequalities, *SIAM J. Numer. Anal.* 52 (2014) 1243–1268.
- [26] C. Bierig, A. Chernov, Convergence analysis of multilevel Monte Carlo variance estimators and application for random obstacle problems, *Numer. Math.* 130 (2015) 579–613.
- [27] J.F. Rodrigues, *Obstacle Problems in Mathematical Physics*, North-Holland, Amsterdam, 1987.
- [28] K. Atkinson, W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, Springer, New York, Third edition, 2009.
- [29] R. Durrett, *Probability: Theory and Examples*, Cambridge University Press, Cambridge; New York, Fifth edition, 2019.
- [30] G.J. Lord, C.E. Powell, T. Shardlow, *An Introduction to Computational Stochastic PDEs*, Cambridge University Press, New York, 2014.
- [31] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, Springer, Berlin; Heidelberg, Sixth edition, 2013.
- [32] O. Le Maître, O.M. Knio, *Spectral Methods for Uncertainty Quantification: With Applications to Computational Fluid Dynamics*, Springer, Dordrecht, 2010.
- [33] C. Schwab, R.A. Todor, Karhunen-Loève approximation of random fields by generalized fast multipole methods, *J. Comput. Phys.* 217 (2006) 100–122.
- [34] T. Hsing, R.L. Eubank, *Theoretical Foundations of Functional Data Analysis, with an Introduction to Linear Operators*, John Wiley & Sons, Chichester, 2015.
- [35] R.A. Todor, C. Schwab, Convergence rates for sparse chaos approximations of elliptic problems with stochastic coefficients, *IMA J. Numer. Anal.* 27 (2007) 232–261.
- [36] J. Gwinner, A class of random variational inequalities and simple random unilateral boundary value problems-existence, discretization, finite element approximation, *Stoch. Anal. Appl.* 18 (2000) 967–993.
- [37] M.M. Rao, R.J. Swift, *Probability Theory with Applications*, Springer, New York, Second edition, 2006.
- [38] S.C. Brenner, L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, New York, Third edition, 2008.
- [39] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
- [40] C.E. Powell, H.C. Elman, Block-diagonal preconditioning for spectral stochastic finite-element systems, *IMA J. Numer. Anal.* 29 (2) (2009) 350–375.
- [41] O.G. Ernst, C.E. Powell, D.J. Silvester, E. Ullmann, Efficient solvers for a linear stochastic Galerkin mixed formulation of diffusion problems with random data, *SIAM J. Sci. Comput.* 31 (2) (2009) 1424–1447.
- [42] S. Hübner, B.I. Wohlmuth, A primal-dual active set strategy for non-linear multibody contact problems, *Comput. Methods Appl. Mech. Eng.* 194 (2005) 3147–3166.
- [43] T. Kärkkäinen, K. Kunisch, P. Tarvainen, Augmented Lagrangian active set methods for obstacle problems, *J. Optim. Theory Appl.* 119 (2003) 499–533.