Journal of Computational and Applied Mathematics 459 (2025) 116366

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Well-posedness of a class of evolutionary variational–hemivariational inequalities in contact mechanics

Wei Xu [a], Weimin Han [b], Ting Li [c,*], Ziping Huang [c]

[a] Tongji Zhejiang College, Jiaxing 314051, China
[b] Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA
[c] School of Mathematical Sciences, Tongji University, Shanghai 200092, China

ARTICLE INFO

Keywords:
Variational–hemivariational inequality
Convex constraint
Rothe method
Existence and uniqueness
Lipschitz continuous dependence
Viscoelastic material
Frictional contact

ABSTRACT

A class of evolutionary variational–hemivariational inequalities with a convex constraint is studied in this paper. An inequality in this class involves a first-order derivative and a history-dependent operator. Existence and uniqueness of a solution to the inequality is established by the Rothe method, in which the first-order temporal derivative is approximated by backward Euler's formula, and the history-dependent operator is approximated by a modified left endpoint rule. The proof of the result relies on basic results in functional analysis only, and it does not require the notion of pseudomonotone operators and abstract surjectivity results for such operators, used in other papers on the Rothe method for other evolutionary variational–hemivariational inequalities. Moreover, a Lipschitz continuous dependence conclusion of the solution on the right-hand side is proved. Finally, a new frictional contact problem for viscoelastic material is discussed, which illustrates an application of the theoretical results.

1. Introduction

In the study of nonlinear nonsmooth problems arising in science and engineering, hemivariational inequalities have been shown to be a powerful mathematical tool. Since the early 1980s [1], the mathematical theory of hemivariational inequalities has developed rapidly; see [2–5]. Meanwhile, considerable progress has been made on numerical methods for solving hemivariational inequalities; see [6–12] and the lengthy survey paper [13].

In [14], a class of evolutionary hemivariational inequalities without constraint is studied. Through applications of the theory of pseudomonotone operators, a well-posedness result of the inequalities is proved, and a Céa-type inequality is derived for fully discrete approximation. In [15,16], minimization principles serve as a starting point to prove existence and uniqueness results of stationary hemivariational inequalities. Under suitable assumptions, the solution of certain hemivariational inequality is also the minimizer of a corresponding energy functional. Then for general variational–hemivariational inequalities, existence and uniqueness of solutions are proved by an additional fixed-point argument.

Motivated by [15,16], in this paper, we explore well-posedness results for a class of evolutionary variational–hemivariational inequalities without the need of the notion of the pseudomonotone operator and an abstract surjectivity result for such an operator. In contrast to [14], we study variational–hemivariational inequalities in this paper, and the inequalities are posed over a convex set. The Rothe method has been applied to the study of evolutionary hemivariational inequalities, starting with [17], followed by [8,18–21], etc. Using the arguments on stationary variational–hemivariational inequalities in [15,16], we show the existence and uniqueness of temporally semi-discrete solutions to the evolutionary inequality. Furthermore, piecewise affine functions and

* Corresponding author.
E-mail address: liting2020@tongji.edu.cn (T. Li).

https://doi.org/10.1016/j.cam.2024.116366
Received 25 July 2024; Received in revised form 24 October 2024
Available online 10 November 2024

piecewise constant functions are constructed based on the semi-discrete solutions and are proved to converge to the solution of the evolutionary inequality.

The rest of the paper is as follows. In Section 2, some preliminaries in nonlinear functional analysis are recalled. In Section 3, an evolutionary variational–hemivariational inequality and assumptions on the problem data are introduced. In Section 4, the Rothe method is considered to solve the inequality. Well-posedness results of the inequality are presented in Section 5. Finally, a new frictional contact problem for viscoelastic material is analyzed in Section 6, which illustrates an application of the theoretical results.

2. Preliminaries

We review some basic notions and results in this section. Let X be a normed space with a dual space X^* and $\langle \cdot, \cdot \rangle_{X^* \times X}$ be the duality pairing between X and X^*. Let Y be a normed space. The norms in X and Y are written by $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively. The symbol $\to$ denotes strong convergence, and the symbol $\rightharpoonup$ means weak convergence.

Definition 2.1. Let $\Psi : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of Ψ at a point $x \in X$ in the direction $z \in X$ is defined by

$$\Psi^0(x; z) = \limsup_{y \to x, \lambda \downarrow 0} \frac{\Psi(y + \lambda z) - \Psi(y)}{\lambda}.$$

Definition 2.2. The generalized gradient of $\Psi : X \to \mathbb{R}$ at a point $x \in X$ is defined by

$$\partial \Psi(x) = \{ \xi \in X^* \mid \Psi^0(x; z) \geq \langle \xi, z \rangle_{X^* \times X} \text{ for all } z \in X \}.$$

Let $(0, T)$ be the time interval, where $T > 0$ is a fixed.

Definition 2.3 (*[22]*). An operator $S : L^2(0, T; X) \to L^2(0, T; X^*)$ is called history-dependent, if there exists a constant $C_T > 0$ such that

$$\|(S x_1)(t) - (S x_2)(t)\|_{X^*} \leq C_T \int_0^t \|x_1(s) - x_2(s)\|_X ds \text{ for all } x_1, x_2 \in L^2(0, T; X), \text{ a.e. } t \in (0, T).$$

Let $[0, T]$ be divided into a finite number of disjoint subintervals $\triangle_i = [l_i, r_i]$ such that $[0, T] = \cup_{i=1}^n \triangle_i$. Such a partition is denoted by Π, and the family of all such partitions is denoted by $\mathcal{P}$. Let $1 \leq p, q < \infty$. We introduce the space

$$BV^q(0, T; X) := \{ x : (0, T) \to X \mid \|x\|_{BV^q(0,T;X)} < \infty \},$$

where $\|x\|_{BV^q(0,T;X)}$ stands for the seminorm of $x \in BV^q(0, T; X)$ given by

$$\|x\|_{BV^q(0,T;X)}^q := \sup_{\Pi \in \mathcal{P}} \sum_{\triangle_i \in \Pi} \|x(r_i) - x(l_i)\|_X^q.$$

If X and Y are Banach spaces and the embedding $X \subset Y$ is continuous, then the space

$$M^{p,q}(0, T; X, Y) := L^p(0, T; X) \cap BV^q(0, T; Y)$$

is a Banach space equipped with the norm $\| \cdot \|_{M^{p,q}(0,T;X,Y)} := \| \cdot \|_{L^p(0,T;X)} + \| \cdot \|_{BV^q(0,T;Y)}$.

We recall a compactness result next (cf. [17, Proposition 2.8]).

Lemma 2.4. *Let $1 \leq p, q < \infty$ and $X_1 \subset X_2 \subset X_3$ be Banach spaces such that X_1 is reflexive, the embedding $X_1 \subset X_2$ is compact and the embedding $X_2 \subset X_3$ is continuous. Then, any bounded subset of $M^{p,q}(0, T; X_1, X_3)$ is relatively compact in $L^p(0, T; X_2)$.*

3. An evolutionary variational–hemivariational inequality

In this section, we introduce an evolutionary variational–hemivariational inequality which involves a first-order temporal derivative and a history-dependent operator. Let $V \subset H \subset V^*$ be an evolution triple, where V is a separable Hilbert space with the dual V^*, H is a separable Hilbert space, the dual of H is identified with H itself, the embedding $V \subset H$ is compact, and the embedding of $H \subset V^*$ is continuous. Let K be a nonempty convex and closed subset of V. The duality pairing between V and V^* is denoted by $\langle \cdot, \cdot \rangle$, and the norms in V, H, V^* are written by $\| \cdot \|$, $\| \cdot \|_H$, $\| \cdot \|_{V^*}$, respectively. Let X be a Banach space with a dual space X^* and $\langle \cdot, \cdot \rangle_{X^* \times X}$ be the duality pairing.

For a fixed $T > 0$, denote $I = (0, T)$ and $\overline{I} = [0, T]$. Let $\mathcal{V} = L^2(I; V)$, $\mathcal{H} = L^2(I; H)$, $\mathcal{V}^* = L^2(I; V^*)$, $\mathcal{X} = L^2(I; X)$ and $\mathcal{X}^* = L^2(I; X^*)$. Identifying $\mathcal{H}$ with its dual, we have the continuous embeddings $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$. Define

$$\langle v^*, v \rangle_{\mathcal{V}^* \times \mathcal{V}} = \int_0^T \langle v^*(t), v(t) \rangle dt \quad \forall v^* \in \mathcal{V}^*, v \in \mathcal{V}. \tag{3.1}$$

Denote $\dot{v} = \partial v/\partial t$ the time derivative of v in the sense of distributions. Define

$$\langle x^*, x\rangle_{\mathcal{X}^*\times\mathcal{X}} = \int_0^T \langle x^*(t), x(t)\rangle_{X^*\times X} dt \quad \forall\, x^* \in \mathcal{X}^*, x \in \mathcal{X}. \tag{3.2}$$

Let $A : V \to V^*$, $B : V \to V^*$, $R : V \to V^*$, $\gamma : V \to X$ be given operators with norms $\|A\|$, $\|B\|$, $\|R\|$ and $\|\gamma\|$, respectively. Let $q : \overline{I}\times\overline{I} \to \mathcal{L}(V;V)$ be an operator-valued function. Define an operator $\mathcal{S} : \mathcal{V} \to \mathcal{V}^*$ by

$$(\mathcal{S}v)(t) = R\Big(\int_0^t q(t,s)v(s)ds + h_0\Big) \quad \text{for all } v \in \mathcal{V},\ \text{a.e. } t \in I, \tag{3.3}$$

where $h_0 \in V$. Let $\Phi : V \to \mathbb{R}$, $\Psi : X \to \mathbb{R}$ be given functionals.

3.1. An evolutionary variational–hemivariational inequality

The evolutionary variational–hemivariational inequality to be studied is as follows.

Problem 3.1. Find a function $u \in \mathcal{V}$ such that $\dot{u} \in \mathcal{V}$, for a.e. $t \in I$, $u(t) \in K$,

$$\langle A\dot{u}(t) + Bu(t) + (\mathcal{S}u)(t), v - u(t)\rangle + \Phi(v) - \Phi(u(t)) + \langle \xi(t), \gamma(v - u(t))\rangle_{X^*\times X} \geq \langle f(t), v - u(t)\rangle \quad \forall\, v \in K, \tag{3.4}$$

where

$$\xi \in \mathcal{X}^*, \quad \xi(t) \in \partial\Psi(\gamma u(t)) \quad \text{a.e. } t \in I, \tag{3.5}$$

and

$$u(0) = u_0. \tag{3.6}$$

On the problem data, we introduce the following hypotheses.

$H(A)$: The operator $A : V \to V^*$ satisfies
(a) $A \in \mathcal{L}(V;V^*)$;
(b) $\langle Av, v\rangle \geq \alpha_A\|v\|^2$ for all $v \in V$ with $\alpha_A > 0$;
(c) $\langle Av, w\rangle = \langle Aw, v\rangle$ for all $v, w \in V$.

$H(B)$: The operator $B : V \to V^*$ satisfies
(a) $B \in \mathcal{L}(V;V^*)$;
(b) $\langle Bv, v\rangle \geq \alpha_B\|v\|^2$ for all $v \in V$ with $\alpha_B > 0$.

$H(\mathcal{S})$: The operator $\mathcal{S} : \mathcal{V} \to \mathcal{V}^*$ is defined by (3.3), where $h_0 \in V$, $R \in \mathcal{L}(V;V^*)$, and $q \in C(\overline{I}\times\overline{I};\mathcal{L}(V;V))$ is uniformly Lipschitz continuous with respect to the first variable, i.e., there exists a constant $L_q > 0$ such that

$$\|q(t_1,s) - q(t_2,s)\| \leq L_q|t_1 - t_2| \text{ for a.e. } t_1, t_2, s \in I.$$

$H(\Phi)$: The functional $\Phi : V \to \mathbb{R}$ is convex and bounded above on a non-empty open set in V.

$H(\Psi)$: The functional $\Psi : X \to \mathbb{R}$ satisfies
(a) Ψ is locally Lipschitz;
(b) $\|\partial\Psi(z)\|_{X^*} \leq c_0 + c_1\|z\|_X$ for all $z \in X$ with $c_0, c_1 > 0$;
(c) there exists a constant $\alpha_\Psi \geq 0$ such that

$$\Psi^0(z_1; z_2 - z_1) + \Psi^0(z_2; z_1 - z_2) \leq \alpha_\Psi\|z_1 - z_2\|_X^2 \quad \text{for all } z_1, z_2 \in X.$$

$H(\gamma)$: The operator $\gamma : V \to X$ satisfies
(a) $\gamma \in \mathcal{L}(V;X)$;
(b) its Nemytskii operator $\widetilde{\gamma} : M^{2,2}(I;V,V^*) \to \mathcal{X}$ is compact, where $\widetilde{\gamma}$ is defined by

$$(\widetilde{\gamma}v)(t) = \gamma v(t) \quad \text{for all } v \in \mathcal{V},\ \text{a.e. } t \in I.$$

$H(f)$: $f \in H^1(I;V^*)$.

$H(P)$: $\alpha_B > \alpha_\Psi\|\gamma\|^2$.

The novelty in the problem we consider is that the inequality (3.4) is posed over a convex set $K \subset V$. The operator $\mathcal{S}$ is a history-dependent operator according to Definition 2.3, due to $H(\mathcal{S})$. Denote $\|q\| = \|q\|_{C(\overline{I}\times\overline{I};\mathcal{L}(V;V))}$ for simplicity. From $H(\Phi)$, Φ is locally Lipschitz continuous on V (cf. [16, Lemma 2.2]). $H(\gamma)$ is used in analysis of evolutionary hemivariational inequalities (cf. [21]), and specific examples of Nemytskii operators are described in [17,18].

3.2. A stationary variational–hemivariational inequality

We introduce a stationary variational–hemivariational inequality and recall an existence and uniqueness result which will be used to show the well-posedness of the temporally semi-discrete solution to Problem 3.1.

Problem 3.2. Find a function $\overline{u} \in K$ such that

$$\langle G\overline{u}, v - \overline{u} \rangle + \Phi(v) - \Phi(\overline{u}) + \langle \overline{\xi}, \gamma v - \gamma \overline{u} \rangle_{X^* \times X} \geq \langle \overline{f}, v - \overline{u} \rangle \quad \forall\, v \in K, \tag{3.7}$$

where $G : V \to V^*$ is an operator, $\overline{\xi} \in \partial \Psi(\gamma \overline{u}) \subset X^*$.

Similar to [16, Theorem 4.3], we have the following result.

Lemma 3.3. *Assume $H(\Phi)$, $H(\gamma)(a)$, $H(\Psi)(a),(c)$, and $\overline{f} \in V^*$. Assume G is Lipschitz continuous and strongly monotone with a constant α_G such that $\alpha_G > \alpha_\Psi \|\gamma\|^2$. Then, Problem 3.2 has a unique solution.*

Remark 3.4. Assume the hypotheses of Lemma 3.3 are satisfied. If, in addition, G is a potential operator with the potential F_G (cf. [23, Section 41.3]), then there exists a unique function $\overline{u} \in K$ such that

$$\overline{u} \in \operatorname*{argmin}_{v \in K} E(v),$$

where $E(v) = F_G(v) + \Phi(v) + \Psi(\gamma v) - \langle \overline{f}, v \rangle$, $v \in V$. Moreover, $\overline{u}$ is also the solution of Problem 3.2.

4. Rothe method

In this section, the Rothe method is used to prove the existence and uniqueness result for Problem 3.1. Let N be a positive integer and $k = T/N$ be the temporal step-size. Denote $t_n = kn$ and $u_n = u(t_n)$ for $n = 0, 1, \ldots, N$. Furthermore, the following approximations are adopted. Let $\dot{u}_n \approx (u_n - u_{n-1})/k$ for $n = 1, 2, \ldots, N$. Denote $v^k := \{v_n\}_{n=0}^N$ for $v_0, \ldots, v_N \in V$. To approximate $\mathcal{S}$, a modified left endpoint rule $\mathcal{S}_n^k$ is defined by

$$\mathcal{S}_n^k v^k = \begin{cases} R(h_0), \ n = 0, \\ R\Big(\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} q(t_n, s) v_i \, ds + h_0\Big), \ n = 1, 2, \ldots, N. \end{cases} \tag{4.1}$$

Moreover, we define

$$f_n^k = \begin{cases} f(0), \ n = 0, \\ \frac{1}{k} \int_{t_{n-1}}^{t_n} f(s) ds, \ n = 1, 2, \ldots, N. \end{cases} \tag{4.2}$$

On the basis of (4.2), we construct a piecewise constant by

$$f_c^k(t) = \begin{cases} f(0) & \text{for } t = t_0, \\ f_n^k & \text{for } t \in (t_{n-1}, t_n], \ n = 1, 2, \ldots, N. \end{cases} \tag{4.3}$$

Then, a temporally semi-discrete scheme for Problem 3.1 is as follows.

Problem 4.1. Find a discrete solution $u^k := \{u_n^k\}_{n=0}^N \subset K$ such that

$$\langle A u_n^k + k B u_n^k + k \mathcal{S}_n^k u^k, v - u_n^k \rangle + k\Phi(v) - k\Phi(u_n^k) + k \langle \xi_n^k, \gamma(v - u_n^k) \rangle_{X^* \times X} \geq \langle k f_n^k + A u_{n-1}^k, v - u_n^k \rangle \quad \forall\, v \in K, \tag{4.4}$$

for $n = 1, 2, \ldots, N$, where

$$\xi_n^k \in \partial \Psi(\gamma u_n^k) \subset X^*, \tag{4.5}$$

and

$$u_0^k = u_0. \tag{4.6}$$

Next, we consider the existence and uniqueness of a solution to Problem 4.1.

Lemma 4.2. *Assume $H(A)$, $H(B)$, $H(\mathcal{S})$, $H(\Phi)$, $H(\Psi)$, $H(\gamma)$, $H(f)$, and $H(P)$. Then, Problem 4.1 has a unique solution u^k.*

Proof. We prove the result by an induction argument. Note that $u_0^k = u_0$ is given. For $1 \leq n \leq N$, assume $\{u_i^k\}_{i=0}^{n-1}$ and $\{\xi_i^k\}_{i=0}^{n-1}$ are known. We rewrite (4.4) as follows: find $u_n^k \in K$ such that

$$\langle (A + kB) u_n^k, v - u_n^k \rangle + k\Phi(v) - k\Phi(u_n^k) + k \langle \xi_n^k, \gamma(v - u_n^k) \rangle_{X^* \times X}$$

$$\geq \langle kf_n^k + Au_{n-1}^k - kR\Big(\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} q(t_n,s)u_i^k ds + h_0\Big), v - u_n^k\rangle \quad \forall v \in K, \tag{4.7}$$

where $\xi_n^k \in \partial\Psi(\gamma u_n^k)$. The operator $(A + kB)$ is Lipschitz continuous and strongly monotone with a constant $\alpha_A + k\alpha_B$. Moreover, $\alpha_A + k\alpha_B > k\alpha_\Psi \|\gamma\|^2$ is satisfied for any $k > 0$, due to $H(A)(b)$, $H(B)(b)$, and $H(P)$. Then, we utilize Lemma 3.3 to deduce that the inequality (4.7) has a unique solution u_n^k. By induction, Problem 4.1 has a unique solution u^k. ■

Recall the modified Cauchy–Schwarz inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2 \quad \forall a, b \in \mathbb{R},\ \forall \epsilon > 0. \tag{4.8}$$

This inequality is usually applied with ϵ sufficiently small. Denote v_0 a fixed element in K. Below, we will use c for a generic constant that depends only on $\|A\|$, $\|B\|$, $\|R\|$, $\|q\|$, $\|\gamma\|$, $\|h_0\|$, $\|u_0\|$, $\|v_0\|$, T, and ϵ. Define

$$\delta u_n^k := (u_n^k - u_{n-1}^k)/k, \quad n = 1, 2, \ldots, N.$$

Now, we show the boundedness of u_n^k, ξ_n^k and δu_n^k, respectively.

Lemma 4.3. *Assume $H(A)$, $H(B)$, $H(S)$, $H(\Phi)$, $H(\Psi)$, $H(\gamma)$, $H(f)$, and $H(P)$. Then, there exists a constant c such that*

$$\max_{1\leq n\leq N} \|u_n^k\| \leq c, \tag{4.9}$$

$$\max_{1\leq n\leq N} \|\xi_n^k\|_{X^*} \leq c, \tag{4.10}$$

$$k\sum_{n=1}^{N} \|\delta u_n^k\|^2 \leq c. \tag{4.11}$$

Proof. Choose any element $v_0 \in K$. In (4.4), we take $v = v_0$ and get

$$\begin{aligned}&\langle A(u_n^k - u_{n-1}^k), v_0\rangle + k\langle Bu_n^k + S_n^k u^k, v_0 - u_n^k\rangle + k\Phi(v_0) - k\Phi(u_n^k)\\ &\quad + k\langle \xi_n^k, \gamma(v_0 - u_n^k)\rangle_{X^*\times X} + k\langle f_n^k, u_n^k - v_0\rangle \geq \langle A(u_n^k - u_{n-1}^k), u_n^k\rangle.\end{aligned} \tag{4.12}$$

Next, we bound each term in (4.12). Using $H(A)$, we obtain

$$\langle Au_n^k - Au_{n-1}^k, u_n^k\rangle = \frac{1}{2}\big(\langle Au_n^k, u_n^k\rangle - \langle Au_{n-1}^k, u_{n-1}^k\rangle + \langle A(u_n^k - u_{n-1}^k), u_n^k - u_{n-1}^k\rangle\big) \geq \frac{1}{2}\langle Au_n^k, u_n^k\rangle - \frac{1}{2}\langle Au_{n-1}^k, u_{n-1}^k\rangle. \tag{4.13}$$

We apply Cauchy–Schwarz inequality and (4.8) to derive that for any small $\epsilon > 0$, there exists a constant c such that

$$\langle f_n^k, u_n^k - v_0\rangle \leq \|f_n^k\|_{V^*}(\|u_n^k\| + \|v_0\|) \leq \epsilon\, \|u_n^k\|^2 + c\, \|f_n^k\|_{V^*}^2 + c. \tag{4.14}$$

Utilizing $H(B)$, we get

$$\langle Bu_n^k, v_0 - u_n^k\rangle = \langle Bu_n^k, v_0\rangle - \langle Bu_n^k, u_n^k\rangle \leq (\epsilon - \alpha_B)\, \|u_n^k\|^2 + c. \tag{4.15}$$

Denote $a_{n-1} = \sum_{i=0}^{n-1} \|u_i^k\|$ for convenience. Then,

$$\langle S_n^k u^k, v_0 - u_n^k\rangle \leq \|S_n^k u^k\|_{V^*}\, \|v_0 - u_n^k\| \leq \|R\|\,\big(k\|q\|a_{n-1} + \|h_0\|\big)\,\big(\|v_0\| + \|u_n^k\|\big) \leq c\,\big(1 + k\,a_{n-1}\big)\,\|u_n^k\| + ck\,a_{n-1} + c. \tag{4.16}$$

Apply (4.8) on the first term on the right side of (4.16):

$$c\,\big(1 + k\,a_{n-1}\big)\,\|u_n^k\| \leq \epsilon\,\|u_n^k\|^2 + c\,\big(k\,a_{n-1}\big)^2 + c, \tag{4.17}$$

moreover,

$$k\,a_{n-1} \leq \big(k\,a_{n-1}\big)^2 + 1/4. \tag{4.18}$$

Note that

$$\big(k\,a_{n-1}\big)^2 \leq k^2 n \sum_{i=0}^{n-1} \|u_i^k\|^2 \leq T\,k \sum_{i=0}^{n-1} \|u_i^k\|^2. \tag{4.19}$$

Together with (4.16)–(4.19), we have

$$\langle S_n^k u^k, v_0 - u_n^k\rangle \leq \epsilon\,\|u_n^k\|^2 + ck\sum_{i=0}^{n-1} \|u_i^k\|^2 + c. \tag{4.20}$$

Thanks to $H(\Phi)$, Φ is bounded below by an affine functional ([24, Lemma 11.3.5]). Thus, there exist two constants c_{Φ_0}, c_{Φ_1} such that

$$\Phi(u_n^k) \geq c_{\Phi_0} + c_{\Phi_1}\|u_n^k\|.$$

Hence,

$$\Phi(v_0) - \Phi(u_n^k) \le -c_{\Phi_1}\|u_n^k\| - c_{\Phi_0} + \Phi(v_0) \le \epsilon\,\|u_n^k\|^2 + c. \tag{4.21}$$

With a fixed $\xi_0 \in \partial\Psi(\gamma(v_0))$, we use $H(\Psi)$(b) to derive that

$$|\langle \xi_0, \gamma(u_n^k - v_0)\rangle_{X^*\times X}| \le \|\xi_0\|_{X^*}\|\gamma(u_n^k - v_0)\|_X \le \frac{\epsilon}{2}\,\|u_n^k\|^2 + c. \tag{4.22}$$

By $H(\Psi)$(c) and $H(\gamma)$(a), the following inequality holds:

$$\begin{aligned}\langle \xi_n^k, \gamma(v_0 - u_n^k)\rangle_{X^*\times X} + \langle \xi_0, \gamma(u_n^k - v_0)\rangle_{X^*\times X} &\le \Psi^0(\gamma u_n^k; \gamma v_0 - \gamma u_n^k) + \Psi^0(\gamma v_0; \gamma u_n^k - \gamma v_0)\\ &\le \alpha_\Psi\|\gamma\|^2\|u_n^k\|^2 + \frac{\epsilon}{2}\,\|u_n^k\|^2 + c.\end{aligned} \tag{4.23}$$

Together with (4.22) and (4.23), we derive

$$\langle \xi_n^k, \gamma(v_0 - u_n^k)\rangle_{X^*\times X} \le (\alpha_\Psi\,\|\gamma\|^2 + \epsilon)\,\|u_n^k\|^2 + c. \tag{4.24}$$

Combining (4.13)–(4.15), (4.20), (4.21), and (4.24), we derive from (4.12) that

$$\frac{1}{2}\langle Au_n^k, u_n^k\rangle - \frac{1}{2}\langle Au_{n-1}^k, u_{n-1}^k\rangle \le (\alpha_\Psi\|\gamma\|^2 - \alpha_B + 5\epsilon)k\,\|u_n^k\|^2 + c\,k^2\sum_{j=0}^{n-1}\|u_j^k\|^2 + c\,k\|f_n^k\|_{V^*}^2 + c\,k + \langle A(u_n^k - u_{n-1}^k), v_0\rangle. \tag{4.25}$$

Replacing n by i and adding (4.25) from $i = 1$ to $i = n$, we have

$$\frac{1}{2}\langle Au_n^k, u_n^k\rangle - \frac{1}{2}\langle Au_0^k, u_0^k\rangle \le (\alpha_\Psi\|\gamma\|^2 - \alpha_B + 5\epsilon)k\sum_{i=1}^{n}\|u_i^k\|^2 + c\,k^2\sum_{i=1}^{n}\sum_{j=0}^{i-1}\|u_j^k\|^2 + c\,k\sum_{i=1}^{n}\|f_i^k\|_{V^*}^2 + \langle A(u_n^k - u_0^k), v_0\rangle + c. \tag{4.26}$$

For the second term on the right side of (4.26),

$$k\sum_{i=1}^{n}\sum_{j=0}^{i-1}\|u_j^k\|^2 \le nk\sum_{i=0}^{n-1}\|u_i^k\|^2 \le T\sum_{i=0}^{n-1}\|u_i^k\|^2. \tag{4.27}$$

Note that $\|f_c^k\|_{\mathcal{V}^*}^2 = k\sum_{i=1}^{N}\|f_i^k\|_{V^*}^2$, $f_c^k \to f$ in $\mathcal{V}^*$ ([20, Lemma 3]), and we have

$$k\sum_{i=1}^{n}\|f_i^k\|_{V^*}^2 \le k\sum_{i=1}^{N}\|f_i^k\|_{V^*}^2 \le c. \tag{4.28}$$

Moreover,

$$\langle A(u_n^k - u_0^k), v_0\rangle \le \|A\|\|v_0\|(\|u_n^k\| + \|u_0\|) \le \epsilon\,\|u_n^k\|^2 + c. \tag{4.29}$$

Combining (4.26)–(4.29), we have

$$\frac{1}{2}\langle Au_n^k, u_n^k\rangle - \frac{1}{2}\langle Au_0^k, u_0^k\rangle \le (\alpha_\Psi\|\gamma\|^2 - \alpha_B + 5\epsilon)k\sum_{i=1}^{n}\|u_i^k\|^2 + c\,k\sum_{i=0}^{n-1}\|u_i^k\|^2 + \epsilon\,\|u_n^k\|^2 + c. \tag{4.30}$$

Because of $H(P)$, we may choose a positive number $\epsilon < \min\{\frac{1}{5}(\alpha_B - \alpha_\Psi\|\gamma\|^2), \frac{1}{2}\alpha_A\}$. Using $H(A)$(b), we find from (4.30) that

$$\Big(\frac{1}{2}\alpha_A - \epsilon\Big)\|u_n^k\|^2 \le c\,k\sum_{i=0}^{n-1}\|u_i^k\|^2 + c. \tag{4.31}$$

We then apply a discrete Gronwall's inequality (cf. [25, Lemma 7.25]) to get (4.9). In addition, (4.10) follows from (4.9) and $H(\Psi)$(b).

Next, we prove the relation (4.11). Let

$$u_{-1}^k = u_0 + kA^{-1}(Bu_0 + R(h_0) + \eta_0 + \gamma^*\overline{\xi}_0 - f(0)),$$

where $A^{-1} : V^* \to V$ is the inverse operator of A, $\eta_0 \in \partial\Phi(u_0)$, $\overline{\xi}_0 \in \partial\Psi(\gamma u_0)$, and $\gamma^* : X^* \to V^*$ is the adjoint operator of γ. Then,

$$\delta u_0^k = A^{-1}(f(0) - Bu_0 - R(h_0) - \eta_0 - \gamma^*\overline{\xi}_0).$$

Similar to (4.13),

$$\langle A(\delta u_n^k - \delta u_{n-1}^k), \delta u_n^k\rangle \ge \frac{1}{2}\langle A(\delta u_n^k), \delta u_n^k\rangle - \frac{1}{2}\langle A(\delta u_{n-1}^k), \delta u_{n-1}^k\rangle. \tag{4.32}$$

Take $v = u_{n-1}^k$ in (4.4) and divide the inequality by k to get

$$k\langle A(\delta u_n^k), \delta u_n^k\rangle + k\langle Bu_n^k + \mathcal{S}_n^k u^k, \delta u_n^k\rangle + \Phi(u_n^k) - \Phi(u_{n-1}^k) + k\langle \xi_n^k, \gamma(\delta u_n^k)\rangle_{X^*\times X} \le k\langle f_n^k, \delta u_n^k\rangle. \tag{4.33}$$

Then we take $v = u_n^k$ in (4.4) with n replaced by $n-1$,

$$-k\langle A(\delta u_{n-1}^k), \delta u_n^k\rangle - k\langle Bu_{n-1}^k + \mathcal{S}_{n-1}^k u^k, \delta u_n^k\rangle + \Phi(u_{n-1}^k) - \Phi(u_n^k) - k\langle \xi_{n-1}^k, \gamma(\delta u_n^k)\rangle_{X^*\times X} \le -k\langle f_{n-1}^k, \delta u_n^k\rangle. \tag{4.34}$$

We add (4.33) and (4.34), and divide k on its both sides,

$$\langle A(\delta u_n^k - \delta u_{n-1}^k), \delta u_n^k\rangle + \langle Bu_n^k - Bu_{n-1}^k + S_n^k u^k - S_{n-1}^k u^k, \delta u_n^k\rangle + \langle \xi_n^k - \xi_{n-1}^k, \gamma(\delta u_n^k)\rangle_{X^*\times X} \le \langle f_n^k - f_{n-1}^k, \delta u_n^k\rangle. \tag{4.35}$$

Combine (4.32) and (4.35) to get

$$\begin{aligned}&\frac{1}{2}\langle A(\delta u_n^k), \delta u_n^k\rangle - \frac{1}{2}\langle A(\delta u_{n-1}^k), \delta u_{n-1}^k\rangle + \langle Bu_n^k - Bu_{n-1}^k, \delta u_n^k\rangle \\ &\quad\le \langle \xi_{n-1}^k - \xi_n^k, \gamma(\delta u_n^k)\rangle_{X^*\times X} + \langle S_{n-1}^k u^k - S_n^k u^k, \delta u_n^k\rangle + \langle f_n^k - f_{n-1}^k, \delta u_n^k\rangle.\end{aligned} \tag{4.36}$$

Adding (4.36) from $n = 1$ to $n = N$, we have

$$\begin{aligned}&\frac{1}{2}\langle A(\delta u_N^k), \delta u_N^k\rangle - \frac{1}{2}\langle A(\delta u_0^k), \delta u_0^k\rangle + \sum_{n=1}^N \langle Bu_n^k - Bu_{n-1}^k, \delta u_n^k\rangle \\ &\quad\le \sum_{n=1}^N \langle \xi_{n-1}^k - \xi_n^k, \gamma(\delta u_n^k)\rangle_{X^*\times X} + \sum_{n=1}^N \langle S_{n-1}^k u^k - S_n^k u^k, \delta u_n^k\rangle + \sum_{n=1}^N \langle f_n^k - f_{n-1}^k, \delta u_n^k\rangle.\end{aligned} \tag{4.37}$$

To proceed further, we bound each term in (4.37) in turn. Using $H(B)$(b), we have

$$\sum_{n=1}^N \langle Bu_n^k - Bu_{n-1}^k, \delta u_n^k\rangle \ge \alpha_B k \sum_{n=1}^N \|\delta u_n^k\|^2. \tag{4.38}$$

Moreover, we utilize $H(\Psi)$(b) and $H(\gamma)$(a) to derive

$$\begin{aligned}\sum_{n=1}^N \langle \xi_{n-1}^k - \xi_n^k, \gamma(\delta u_n^k)\rangle_{X^*\times X} &\le \frac{1}{k}\sum_{n=1}^N \Big(\Psi^0(\gamma u_{n-1}^k; \gamma u_n^k - \gamma u_{n-1}^k) + \Psi^0(\gamma u_n^k; \gamma u_{n-1}^k - \gamma u_n^k)\Big) \\ &\le \alpha_\Psi \|\gamma\|^2 k \sum_{n=1}^N \|\delta u_n^k\|^2.\end{aligned} \tag{4.39}$$

By (4.1) and $H(S)$, we obtain

$$\begin{aligned}\|S_{n-1}^k u^k - S_n^k u^k\|_{V^*} &= \|R\Big(\sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}} q(t_{n-1}, s) u_i^k ds - \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} q(t_n, s) u_i^k ds\Big)\|_{V^*} \\ &\le \|R\|\Big(\sum_{i=0}^{n-2}\|\int_{t_i}^{t_{i+1}} (q(t_n, s) - q(t_{n-1}, s)) u_i^k ds\| + \|\int_{t_{n-1}}^{t_n} q(t_n, s) u_{n-1}^k ds\|\Big) \\ &\le \|R\|\Big(k^2 L_q \sum_{i=0}^{n-2}\|u_i^k\| + k\|q\|\|u_{n-1}^k\|\Big).\end{aligned}$$

Apply the bound (4.9),

$$\|S_{n-1}^k u^k - S_n^k u^k\|_{V^*} \le c\,k. \tag{4.40}$$

Then, we apply (4.40) on the second term on the right side of (4.37):

$$\sum_{n=1}^N \langle S_{n-1}^k u^k - S_n^k u^k, \delta u_n^k\rangle \le \sum_{n=1}^N \|S_{n-1}^k u^k - S_n^k u^k\|_{V^*}\|\delta u_n^k\| \le \epsilon k \sum_{n=1}^N \|\delta u_n^k\|^2 + c. \tag{4.41}$$

Define $f(t) = f(0)$ for $t \in (-k, 0)$ and $t_{-1} = -k$. By the definition (4.2),

$$f_n^k - f_{n-1}^k = \frac{1}{k}\int_{t_{n-1}}^{t_n} \big(f(s) - f(s-k)\big)\, ds = \frac{1}{k}\int_{t_{n-1}}^{t_n}\int_{s-k}^{s} \dot f(r)\, dr\, ds. \tag{4.42}$$

Then,

$$\|f_n^k - f_{n-1}^k\|_{V^*} \le \int_{t_{n-2}}^{t_n} \|\dot f(s)\|_{V^*} ds \le \sqrt{2k}\Big(\int_{t_{n-2}}^{t_n} \|\dot f(s)\|_{V^*}^2 ds\Big)^{\frac{1}{2}}. \tag{4.43}$$

Since $\dot f(t) = 0$ for a.e. $t \in (-k, 0)$,

$$\frac{1}{k}\sum_{n=1}^N \|f_n^k - f_{n-1}^k\|_{V^*}^2 \le 4\int_{-k}^T \|\dot f(s)\|_{V^*}^2 ds = 4\,\|\dot f\|_{\mathcal{V}^*}^2. \tag{4.44}$$

Utilizing (4.44), we obtain

$$\sum_{n=1}^N \langle f_n^k - f_{n-1}^k, \delta u_n^k\rangle \le \sum_{n=1}^N \|f_n^k - f_{n-1}^k\|_{V^*}\|\delta u_n^k\| \le \epsilon k \sum_{n=1}^N \|\delta u_n^k\|^2 + c\,\|\dot f\|_{\mathcal{V}^*}^2. \tag{4.45}$$

Using (4.38)–(4.39), (4.41), and (4.45) in (4.37),

$$(\alpha_B - \alpha_\Psi \|\gamma\|^2 - 2\epsilon) k \sum_{n=1}^N \|\delta u_n^k\|^2 + \frac{1}{2}\alpha_A \|\delta u_N^k\|^2 \le c\,\|\dot f\|_{\mathcal{V}^*}^2 + \frac{1}{2}\|A\|\|\delta u_0^k\|^2 + c. \tag{4.46}$$

Then, (4.11) follows from (4.46) with a sufficiently small $\epsilon > 0$. ■

Based on u^k, we construct a piecewise constant u^k_c and a piecewise affine function u^k_a by

$$u^k_c(t) = \begin{cases} u^k_{n-1} & \text{for } t \in [t_{n-1}, t_n),\ 1 \le n \le N, \\ u^k_N & \text{for } t = t_N, \end{cases} \tag{4.47}$$

and

$$u^k_a(t) = \begin{cases} u^k_{n-1} + \frac{t-t_{n-1}}{k}(u^k_n - u^k_{n-1}) & \text{for } t \in [t_{n-1}, t_n),\ 1 \le n \le N, \\ u^k_N & \text{for } t = t_N. \end{cases} \tag{4.48}$$

Moreover, we define ξ^k_c by

$$\xi^k_c(t) = \begin{cases} \xi^k_{n-1} & \text{for } t \in [t_{n-1}, t_n),\ 1 \le n \le N, \\ \xi^k_N & \text{for } t = t_N. \end{cases} \tag{4.49}$$

Regarding the sequences u^k_c, u^k_a and ξ^k_c, we have the following property.

Lemma 4.4. *Assume $H(A)$, $H(B)$, $H(S)$, $H(\Phi)$, $H(\Psi)$, $H(\gamma)$, $H(f)$, and $H(P)$. Then, there exists a constant c such that*

$$\|u^k_c\|_{\mathcal{V}} + \|u^k_c\|_{BV^2(I;V^*)} \le c, \tag{4.50}$$

$$\|u^k_a\|_{\mathcal{V}} + \|\dot{u}^k_a\|_{\mathcal{V}} \le c, \tag{4.51}$$

$$\|\xi^k_c\|_{\mathcal{X}} \le c. \tag{4.52}$$

Proof. Using (4.9), we have

$$\|u^k_c\|^2_{\mathcal{V}} = \int_0^T \|u^k_c(t)\|^2 dt = k \sum_{n=0}^{N-1} \|u^k_n\|^2 \le c. \tag{4.53}$$

Without loss of generality, we assume the semi-norm of u^k_c in $BV^2(I;V^*)$ is achieved on a division $0 = s_0 < s_1 < \cdots < s_j = T$ and each s_i in different interval $[m_i k, (m_i+1)k)$ such that $u^k_c(s_i) = u^k_{m_i}$ with $m_0 = 0$, $m_j = N$ and $m_{i+1} > m_i$ for $i = 1, \ldots, j-1$. Then,

$$\begin{aligned} \|u^k_c\|^2_{BV^2(I;V^*)} &= \sum_{i=0}^{j-1} \|u^k_{m_{i+1}} - u^k_{m_i}\|^2_{V^*} \le \sum_{i=0}^{j-1} \Big((m_{i+1} - m_i) \sum_{n=m_i}^{m_{i+1}-1} \|u^k_{n+1} - u^k_n\|^2_{V^*} \Big) \\ &\le \Big(\sum_{i=0}^{j-1} (m_{i+1} - m_i) \Big) \Big(\sum_{i=0}^{j-1} \sum_{n=m_i}^{m_{i+1}-1} \|u^k_{n+1} - u^k_n\|^2_{V^*} \Big) = N \sum_{n=0}^{N-1} \|u^k_{n+1} - u^k_n\|^2_{V^*}. \end{aligned} \tag{4.54}$$

Note that the embedding $V \subset V^*$ is continuous, then $\|u^k_{n+1} - u^k_n\|_{V^*} \le c\,\|u^k_{n+1} - u^k_n\|_V$. Together with (4.54) and (4.11), we derive

$$\|u^k_c\|^2_{BV^2(I;V^*)} \le c\,N \sum_{n=0}^{N-1} \|u^k_{n+1} - u^k_n\|^2_V \le c\,Tk \sum_{n=1}^{N} \|\delta u^k_n\|^2 \le c. \tag{4.55}$$

Combining (4.53) and (4.55), we get (4.50). Similarly, we have

$$\begin{aligned} \|u^k_a\|^2_{\mathcal{V}} + \|\dot{u}^k_a\|^2_{\mathcal{V}} &= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|u^k_{n-1} + \frac{t - t_{n-1}}{k}(u^k_n - u^k_{n-1})\|^2 dt + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\frac{u^k_n - u^k_{n-1}}{k}\|^2 dt \\ &\le 2 \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \Big(\|u^k_{n-1}\|^2 + \frac{(t-t_{n-1})^2}{k^2} \|u^k_n - u^k_{n-1}\|^2 \Big) dt + k \sum_{n=1}^{N} \|\delta u^k_n\|^2 \\ &\le 2k \sum_{n=1}^{N} \Big(\|u^k_{n-1}\|^2 + \frac{1}{3} \|u^k_n - u^k_{n-1}\|^2 \Big) + k \sum_{n=1}^{N} \|\delta u^k_n\|^2. \end{aligned} \tag{4.56}$$

Then, we obtain (4.51). In addition, (4.52) follows from (4.10). ■

On the basis of Lemma 4.4, we investigate weak convergence of these sequences.

Lemma 4.5. *Assume $H(A)$, $H(B)$, $H(S)$, $H(\Phi)$, $H(\Psi)$, $H(\gamma)$, $H(f)$, and $H(P)$. Let $\{k\}$ be a sequence converging to zero. Then, there exists a subsequence, still denoted by $\{k\}$, such that*

$$u^k_c \rightharpoonup u \text{ in } \mathcal{V},\ u^k_a \rightharpoonup u \text{ in } \mathcal{V},\ \dot{u}^k_a \rightharpoonup \dot{u} \text{ in } \mathcal{V}, \text{ and } \xi^k_c \rightharpoonup \xi \text{ in } \mathcal{X}^*. \tag{4.57}$$

where u and ξ satisfy Problem 3.1.

Proof. Recall that $\mathcal{V}$ is a reflexive Banach space, and the sequence $\{u^k_c\}$ is bounded by (4.50). Hence, there exists an element $u \in \mathcal{V}$ and a subsequence, still denoted by $\{u^k_c\}$, such that $u^k_c \rightharpoonup u$ in $\mathcal{V}$. Similarly, there exists an element $\xi \in \mathcal{X}^*$ and a subsequence, still denoted by $\{\xi^k_c\}$, such that $\xi^k_c \rightharpoonup \xi$ in $\mathcal{X}^*$.

The sequence $\{u_a^k\}$ is bounded in $\mathcal{V}$ by (4.51). There exists a subsequence, still denoted by u_a^k, and an element $u_1 \in \mathcal{V}$, such that $u_a^k \rightharpoonup u_1$ in $\mathcal{V}$. Hence $u_a^k - u_c^k \rightharpoonup u_1 - u$ in $\mathcal{V}$. Moreover,

$$\|u_a^k - u_c^k\|_{\mathcal{V}}^2 \le \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \frac{(t-t_{n-1})^2}{k^2} \|u_n^k - u_{n-1}^k\|^2 dt = \frac{k^3}{3} \sum_{n=1}^{N} \|\delta u_n^k\|^2.$$

Hence, we derive $\|u_a^k - u_c^k\|_{\mathcal{V}} \to 0$ as $k \to 0$ from (4.11). Thus, $u = u_1$ is valid. Furthermore, we use [26, Proposition 23.19] to get $\dot{u}_a^k \rightharpoonup \dot{u}$ in $\mathcal{V}$.

Next, let us show u and ξ satisfy Problem 3.1. Firstly, we establish an inequality that corresponds to (3.4). Rewrite (4.4) to get the following relation

$$\langle A(\frac{u_n^k - u_{n-1}^k}{k}) + Bu_n^k + S_n^k u^k, v - u_n^k\rangle + \Phi(v) - \Phi(u_n^k) + \langle \xi_n^k, \gamma(v - u_n^k)\rangle_{X^*\times X} \ge \langle f_n^k, v - u_n^k\rangle \quad \forall\, v \in K, \tag{4.58}$$

for $n = 1, 2, \dots, N$. Denote

$$\eta_c^k(t) = \begin{cases} \displaystyle\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} q(t_n, s) u_i^k ds + h_0 & \text{for } t \in [t_{n-1}, t_n),\ n = 1, 2, \dots, N, \\ \displaystyle\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} q(t_N, s) u_i^k ds + h_0 & \text{for } t = t_N. \end{cases} \tag{4.59}$$

Combining (4.58), (4.59), (4.47)–(4.49), we derive a pointwise inequality

$$\begin{aligned} &\langle A(\dot{u}_a^k(t)) + Bu_c^k(t) + R(\eta_c^k(t)), v(t) - u_c^k(t)\rangle + \Phi(v(t)) - \Phi(u_c^k(t)) \\ &\quad + \langle \xi_c^k(t), \gamma(v(t) - u_c^k(t))\rangle_{X^*\times X} \ge \langle f_c^k, v(t) - u_c^k(t)\rangle \quad \forall\, v \in L^2(I; K),\ \text{a.e. } t \in I. \end{aligned} \tag{4.60}$$

Define Nemytskii operators $\mathscr{A}, \mathscr{B}, \mathscr{R}, \mathscr{S}$: $\mathcal{V} \to \mathcal{V}^*$ by $(\mathscr{A}v)(t) = Av(t)$, $(\mathscr{B}v)(t) = Bv(t)$, $(\mathscr{R}v)(t) = R(v(t))$, and $(\mathscr{S}v)(t) = R\Big(\int_0^t q(t,s)v(s)ds\Big)$ for any $v \in \mathcal{V}$, a.e. $t \in I$. Moreover, define $\widetilde{\Phi}(v) = \int_0^T \Phi(v(t))dt$ for $v \in \mathcal{V}$. Integrate (4.60) over I,

$$\begin{aligned} &\langle \mathscr{A}\dot{u}_a^k, v - u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mathscr{B}u_c^k, v - u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mathscr{R}(\eta_c^k), v - u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} + \widetilde{\Phi}(v) - \widetilde{\Phi}(u_c^k) \\ &\quad + \langle \xi_c^k, \widetilde{\gamma}(v - u_c^k)\rangle_{\mathcal{X}^*\times\mathcal{X}} \ge \langle f_c^k, v - u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} \quad \forall\, v \in L^2(I; K). \end{aligned} \tag{4.61}$$

Secondly, we use Lemma 4.5 to illustrate the convergence result of (4.61) as $k \to 0$. By $H(A)$(a), the operator $\mathscr{A} : \mathcal{V} \to \mathcal{V}^*$ is bounded and linear, hence it is weakly continuous. For $\dot{u}_a^k \rightharpoonup \dot{u}$ in $\mathcal{V}$, we have $\mathscr{A}\dot{u}_a^k \rightharpoonup \mathscr{A}\dot{u}$ in $\mathcal{V}^*$ as $k \to 0$. Similarly, the operator $\mathscr{B}$ is weakly continuous on $\mathcal{V}$. For $u_c^k \rightharpoonup u$ in $\mathcal{V}$, the relation $\mathscr{B}u_c^k \rightharpoonup \mathscr{B}u$ is valid. In addition, the functional $V \ni u \mapsto \langle Au, u\rangle$ is weakly l.s.c. on V, due to the convexity and continuity of A (cf. [25, Corollary 1.50]). Analogously, $\mathcal{V} \ni v \mapsto \langle \mathscr{B}v, v\rangle_{\mathcal{V}^*\times\mathcal{V}}$ is weakly l.s.c. on $\mathcal{V}$. Note that

$$\limsup_{k\to 0} \langle \mathscr{A}\dot{u}_a^k, v - u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} \le \limsup_{k\to 0} \langle \mathscr{A}\dot{u}_a^k, v\rangle_{\mathcal{V}^*\times\mathcal{V}} - \liminf_{k\to 0} \langle \mathscr{A}\dot{u}_a^k, u_a^k\rangle_{\mathcal{V}^*\times\mathcal{V}} + \limsup_{k\to 0} \langle \mathscr{A}\dot{u}_a^k, u_a^k - u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{4.62}$$

Moreover, the following equation holds:

$$\langle \mathscr{A}\dot{u}_a^k, u_a^k\rangle_{\mathcal{V}^*\times\mathcal{V}} = \int_0^T \langle A\dot{u}_a^k(t), u_a^k(t)\rangle dt = \frac{1}{2}\langle Au_a^k(T), u_a^k(T)\rangle - \frac{1}{2}\langle Au(0), u(0)\rangle. \tag{4.63}$$

By (4.57), we get $u_a^k \rightharpoonup u$ in $W^{1,2}(I; V)$. Furthermore $u_a^k \rightharpoonup u$ in $C(\overline{I}; V)$ is valid, due to the continuous embedding $W^{1,2}(I; V) \subset C(\overline{I}; V)$. Then, for all $t \in \overline{I}$, we have $u_a^k(t) \rightharpoonup u(t)$ in V. In particular, $u_a^k(T) \rightharpoonup u(T)$. Hence,

$$\liminf_{k\to 0} \langle Au_a^k(T), u_a^k(T)\rangle \ge \langle Au(T), u(T)\rangle.$$

We take the lower limit of (4.63) to get

$$\liminf_{k\to 0} \langle \mathscr{A}\dot{u}_a^k, u_a^k\rangle_{\mathcal{V}^*\times\mathcal{V}} \ge \frac{1}{2}\langle Au(T), u(T)\rangle - \frac{1}{2}\langle Au(0), u(0)\rangle = \langle \mathscr{A}\dot{u}, u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{4.64}$$

Together with (4.62) and (4.64), we have

$$\limsup_{k\to 0} \langle \mathscr{A}\dot{u}_a^k, v - u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} \le \langle \mathscr{A}\dot{u}, v - u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{4.65}$$

Similar to the derivation of (4.65), we get

$$\limsup_{k\to 0} \langle \mathscr{B}u_c^k, v - u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} \le \limsup_{k\to 0} \langle \mathscr{B}u_c^k, v\rangle_{\mathcal{V}^*\times\mathcal{V}} - \liminf_{k\to 0} \langle \mathscr{B}u_c^k, u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} \le \langle \mathscr{B}u, v - u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{4.66}$$

To proceed further, we have

$$\begin{aligned} \limsup_{k\to 0} \langle \mathscr{R}\eta_c^k, v - u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} &\le \limsup_{k\to 0} \langle \mathscr{R}(\eta_c^k - h_0) - \mathscr{S}u_c^k, v - u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} + \limsup_{k\to 0} \langle \mathscr{S}u_c^k, v\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ &\quad - \liminf_{k\to 0} \langle \mathscr{S}u_c^k, u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} + \limsup_{k\to 0} \langle \mathscr{R}h_0, v - u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}}. \end{aligned} \tag{4.67}$$

For $t \in [t_{n-1}, t_n)$, we use (4.59) and $H(S)$ to derive that

$$\|\eta_c^k(t)-h_0-\int_0^t q(t,s)u_c^k(s)ds\| \le \|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} q(t_n,s)u_i^k ds-\int_0^t q(t,s)u_c^k(s)ds\|$$
$$\le \|\int_0^{t_n}(q(t_n,s)-q(t,s))u_c^k ds\|+\|\int_t^{t_n} q(t,s)u_c^k(s)ds\| \le ck\,\|u_c^k\|_{L^\infty(I;V)}.$$

Therefore, $\|\mathscr{R}(\eta_c^k-h_0)-\mathscr{S}u_c^k\|_{\mathcal{V}^*} \le ck\,\|u_c^k\|_{L^\infty(I;V)}$. Thus, $\mathscr{R}(\eta_c^k-h_0)-\mathscr{S}u_c^k \to 0$ in $\mathcal{V}^*$ as $k\to 0$. Note that $\mathscr{S}$ is weakly continuous and the functional $v\mapsto \langle \mathscr{S}v,v\rangle_{\mathcal{V}^*\times\mathcal{V}}$ is weakly l.s.c. on $\mathcal{V}$. Applying these properties on (4.67), we obtain

$$\limsup_{k\to 0}\langle \mathscr{R}\eta_c^k, v-u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}} \le \langle \mathscr{S}u, v-u\rangle_{\mathcal{V}^*\times\mathcal{V}}+\langle \mathscr{R}h_0, v-u\rangle_{\mathcal{V}^*\times\mathcal{V}}=\langle Su, v-u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{4.68}$$

Let us show that $\widetilde{\Phi}$ is weakly l.s.c. on $\mathcal{V}$. Assume $w_n\subset\mathcal{V}$ and $w_n\to w$ in $\mathcal{V}$. Utilizing [3, Theorem 2.39], passing to a subsequence if necessary, we have $w_n(t)\to w(t)$ in V for a.e. $t\in I$. In addition, we have that

$$\Phi(w_n(t)) \ge c_{\Phi_0}+c_{\Phi_1}\|w_n(t)\| \quad \text{for a.e. } t\in I.$$

Using Fauto's lemma (cf. [3, Theorem 1.64]) and the lower semicontinuity of Φ, we obtain

$$\liminf_{n\to\infty}\widetilde{\Phi}(w_n) \ge \int_0^T \liminf_{n\to\infty}\Phi(w_n(t))dt \ge \widetilde{\Phi}(w),$$

i.e., $\widetilde{\Phi}$ is l.s.c. on $\mathcal{V}$. The convexity of $\widetilde{\Phi}$ is obvious. Applying [25, Corollary 1.50] again, we know that $\widetilde{\Phi}$ is weakly l.s.c. on $\mathcal{V}$. Therefore,

$$\limsup_{k\to 0}(\widetilde{\Phi}(v)-\widetilde{\Phi}(u_c^k))=\widetilde{\Phi}(v)-\liminf_{k\to 0}\widetilde{\Phi}(u_c^k)\le \widetilde{\Phi}(v)-\widetilde{\Phi}(u). \tag{4.69}$$

The relation $u_c^k\to u$ in $M^{2,2}(I;V,V^*)$ is valid, due to (4.50). Therefore, we use $H(\gamma)$(b) to get $\widetilde{\gamma}u_c^k\to\widetilde{\gamma}u$ in $\mathcal{X}$. Together with $\xi_c^k\rightharpoonup\xi$ in $\mathcal{X}^*$, we have

$$\limsup_{k\to 0}\langle \xi_c^k,\widetilde{\gamma}(v-u_c^k)\rangle_{\mathcal{X}^*\times\mathcal{X}}=\langle \xi,\widetilde{\gamma}(v-u)\rangle_{\mathcal{X}^*\times\mathcal{X}}. \tag{4.70}$$

Observing $f_c^k\to f$ in $\mathcal{V}^*$, we get

$$\limsup_{k\to 0}\langle f_c^k, v-u_c^k\rangle_{\mathcal{V}^*\times\mathcal{V}}=\langle f, v-u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{4.71}$$

Combining (4.61), (4.65), (4.66), (4.68)–(4.71), we obtain that for all $v\in L^2(I;K)$,

$$\langle \mathscr{A}\dot{u}+\mathscr{B}u+Su, v-u\rangle_{\mathcal{V}^*\times\mathcal{V}}+\widetilde{\Phi}(v)-\widetilde{\Phi}(u)+\langle \xi,\widetilde{\gamma}(v-u)\rangle_{\mathcal{X}^*\times\mathcal{X}} \ge \langle f, v-u\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{4.72}$$

Thirdly, we illustrate that ξ satisfies (3.5). Note that $\xi_c^k(t)\in\partial\Psi(\gamma u_c^k(t))$ for a.e. $t\in I$, and $\widetilde{\gamma}u_c^k\to\widetilde{\gamma}u$ in $\mathcal{X}$. By the converse Lebesgue's dominated theorem [3, Theorem 2.39], by passing to a subsequence if necessary, $\gamma u_c^k(t)\to\gamma u(t)$ in X for a.e. $t\in I$. It follows from [3, Theorem 3.13] that $\xi(t)\in\partial\Psi(\gamma u(t))$ for a.e. $t\in I$. Thus, we draw the conclusion that u and ξ satisfy Problem 3.1.

The uniqueness of u can be proved through a standard procedure (cf. [14, Theorem 11]) and the argument is omitted here. ■

5. Main results

We summarize the lemmas proved in the previous section in the form of a theorem.

Theorem 5.1. *Assume $H(A)$, $H(B)$, $H(S)$, $H(\Phi)$, $H(\Psi)$, $H(\gamma)$, $H(f)$, and $H(P)$. Then, there exists a unique solution to Problem 3.1.*

Moreover, an equivalent form for Problem 3.1 is illustrated as follows.

Problem 5.2. Find a function $u\in\mathcal{V}$ such that $\dot{u}\in\mathcal{V}$, for a.e. $t\in I$, $u(t)\in K$,

$$\langle A\dot{u}(t)+Bu(t)+(Su)(t), v-u(t)\rangle+\Phi(v)-\Phi(u(t))+\Psi^0(\gamma u(t);\gamma v-\gamma u(t))\ge\langle f(t), v-u(t)\rangle \quad \forall\, v\in K, \tag{5.1}$$

and

$$u(0)=u_0. \tag{5.2}$$

Theorem 5.3. *Assume $H(A)$, $H(B)$, $H(S)$, $H(\Phi)$, $H(\Psi)$, $H(\gamma)$, $H(f)$, and $H(P)$. Then, there exists a unique solution of Problem 5.2.*

Proof. Let u be the solution of Problem 3.1. For $\xi(t)\in\partial\Psi(\gamma u(t))$, we have

$$\langle \xi(t),\gamma(v-u(t))\rangle_{X^*\times X}\le\Psi^0(\gamma u(t);\gamma v-\gamma u(t)). \tag{5.3}$$

Thus, we derive (5.1) from (3.4), i.e., u is the solution of Problem 5.2. The argument of the uniqueness of u is standard and is hence omitted. ■

Next, we provide the Lipschitz continuous dependence of the solution on the right-hand side.

Theorem 5.4. *Assume $H(A)$, $H(B)$, $H(\mathcal{S})$, $H(\Phi)$, $H(\Psi)$, $H(\gamma)$, $H(f)$, and $H(P)$. Then, the solution u of Problem 3.1 depends Lipschitz continuously on f.*

Proof. Let u_1, u_2 be the solutions of Problem 3.1 corresponding to f_1, f_2 respectively, where $f_1, f_2 \in H^1(I; V^*)$. Thus, we have

$$\langle A\dot{u}_1(t) + Bu_1(t) + (\mathcal{S}u_1)(t), v - u_1(t)\rangle + \Phi(v) - \Phi(u_1(t)) + \langle \xi_1(t), \gamma(v - u_1(t))\rangle_{X^*\times X} \geq \langle f_1(t), v - u_1(t)\rangle \quad \forall\, v \in K, \tag{5.4}$$

where $\xi_1 \in \mathcal{X}^*$, $\xi_1(t) \in \partial\Psi(\gamma u_1(t))$, and

$$\langle A\dot{u}_2(t) + Bu_2(t) + (\mathcal{S}u_2)(t), v - u_2(t)\rangle + \Phi(v) - \Phi(u_2(t)) + \langle \xi_2(t), \gamma(v - u_2(t))\rangle_{X^*\times X} \geq \langle f_2(t), v - u_2(t)\rangle \quad \forall\, v \in K, \tag{5.5}$$

where $\xi_2 \in \mathcal{X}^*$, $\xi_2(t) \in \partial\Psi(\gamma u_2(t))$. Taking $v = u_2(t)$ in (5.4) and $v = u_1(t)$ in (5.5), then we add the resulting inequalities to get

$$\begin{aligned}
&\langle A(\dot{u}_1(t) - \dot{u}_2(t)), u_1(t) - u_2(t)\rangle + \langle B(u_1(t) - u_2(t)), u_1(t) - u_2(t)\rangle \\
&\quad \leq \langle (\mathcal{S}u_1)(t) - (\mathcal{S}u_2)(t), u_2(t) - u_1(t)\rangle + \langle f_1(t) - f_2(t), u_1(t) - u_2(t)\rangle \\
&\qquad + \Psi^0(\gamma u_1(t); \gamma u_2(t) - \gamma u_1(t)) + \Psi^0(\gamma u_2(t); \gamma u_1(t) - \gamma u_2(t)).
\end{aligned} \tag{5.6}$$

Applying $H(B)$(b), $H(\mathcal{S})$, $H(\Psi)$(c) and Cauchy–Schwarz inequality on (5.6), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\langle A(u_1(t) - u_2(t)), u_1(t) - u_2(t)\rangle + (\alpha_B - \alpha_\Psi\|\gamma\|^2 - \epsilon)\|u_1(t) - u_2(t)\|^2 \\
&\quad \leq \|R\|\|q\| \int_0^t \|u_1(s) - u_2(s)\| ds \|u_1(t) - u_2(t)\| + \frac{1}{4\epsilon}\|f_1(t) - f_2(t)\|^2_{V^*}.
\end{aligned} \tag{5.7}$$

Since $u_1(0) - u_2(0) = 0$, then we integrate (5.7) over $[0, t]$ to get

$$\frac{1}{2}\langle A(u_1(t) - u_2(t)), u_1(t) - u_2(t)\rangle \leq \|R\|\|q\|\Big(\int_0^t \|u_1(s) - u_2(s)\| ds\Big)^2 + \frac{1}{4\epsilon}\int_0^t \|f_1(s) - f_2(s)\|^2_{V^*} ds. \tag{5.8}$$

By $H(A)$(a), we derive

$$\frac{1}{2}\alpha_A\|u_1(t) - u_2(t)\|^2 \leq \|R\|\|q\|\Big(\int_0^t \|u_1(s) - u_2(s)\| ds\Big)^2 + \frac{1}{4\epsilon}\int_0^T \|f_1(s) - f_2(s)\|^2_{V^*} ds. \tag{5.9}$$

Thus, we get

$$\|u_1(t) - u_2(t)\| \leq \sqrt{\frac{2\|R\|\|q\|}{\alpha_A}}\int_0^t \|u_1(s) - u_2(s)\| ds + \sqrt{\frac{1}{2\epsilon\,\alpha_A}}\|f_1 - f_2\|_{\mathcal{V}^*}. \tag{5.10}$$

Now, we use Gronwall's inequality (cf. [25, Lemma 7.24]) to obtain

$$\|u_1(t) - u_2(t)\| \leq e^{\sqrt{\frac{2\|R\|\|q\|}{\alpha_A}}T}\sqrt{\frac{1}{2\epsilon\,\alpha_A}}\|f_1 - f_2\|_{\mathcal{V}^*}. \tag{5.11}$$

Finally, we utilize (5.11) to derive the inequality that

$$\|u_1 - u_2\|_{\mathcal{V}} \leq e^{\sqrt{\frac{2\|R\|\|q\|}{\alpha_A}}T}\sqrt{\frac{T}{2\epsilon\,\alpha_A}}\|f_1 - f_2\|_{\mathcal{V}^*}, \tag{5.12}$$

which shows the Lipschitz continuity of u respect to f. ■

6. Application in contact mechanics

In this section, we study a frictional contact problem between a viscoelastic body and a rigid foundation. The viscoelastic body occupies a domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz boundary Γ. The boundary is divided into three measurable parts Γ_1, Γ_2 and Γ_3, and $(d-1)$-dimensional measure $|\Gamma_1| > 0$. The body is fixed on Γ_1, thus the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in Ω, and a surface contraction of density $\boldsymbol{f}_2$ acts on Γ_2. On Γ_3, the body is in frictional contact with the foundation, which is made of a rigid obstacle covered with a layer of elastic material. A unilateral constraint condition combined with Tresca friction law [25] is used to describe the frictional contact behavior.

We recall the canonical inner product and norm in $\mathbb{R}^d$ that

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad |\boldsymbol{u}| = (\boldsymbol{u}, \boldsymbol{u})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d.$$

Let $\mathbb{S}^d$ be the space of $d \times d$ symmetric matrices. The corresponding inner product and norm are defined by

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau}, \boldsymbol{\tau})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Since Γ is Lipschitz, the outward unit normal $\boldsymbol{\nu}$ exists a.e. on Γ. The quantities $v_\nu := \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau := \boldsymbol{v} - v_\nu\boldsymbol{\nu}$ are the normal and tangential components of $\boldsymbol{v}$, respectively. For a stress tensor $\boldsymbol{\sigma}$, $\sigma_\nu := (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$ represent the normal and tangential components of σ, respectively. For a displacement field u, $\varepsilon(u) := (\nabla u + (\nabla u)^T)/2$ is the linearized strain tensor. Denote by $\dot{u}$ the time derivative of u. The classical formulation of the frictional contact problem is as follows.

Problem 6.1. Find a displacement $u : \Omega \times I \to \mathbb{R}^d$ and a stress field $\sigma : \Omega \times I \to \mathbb{S}^d$ such that for a.e. $t \in I$,

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{B}\varepsilon(u(t)) + \int_0^t \mathcal{R}(t-s)\varepsilon(u(s))ds \text{ in } \Omega, \tag{6.1}$$

$$\mathrm{Div}\sigma(t) + f_0(t) = \mathbf{0} \text{ in } \Omega, \tag{6.2}$$

$$u(t) = \mathbf{0} \text{ on } \Gamma_1, \tag{6.3}$$

$$\sigma(t)\nu = f_2(t) \text{ on } \Gamma_2, \tag{6.4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g,\ \sigma_\nu(t) + \xi_\nu(t) \le 0, \\ (\sigma_\nu(t) + \xi_\nu(t))(u_\nu(t) - g) = 0, \\ \xi_\nu(t) \in \partial\psi_\nu(u_\nu(t)) \end{array}\right\} \text{ on } \Gamma_3, \tag{6.5}$$

$$|\sigma_\tau(t)| \le F_b,\ -\sigma_\tau(t) = F_b u_\tau(t)/|u_\tau(t)| \text{ if } u_\tau(t) \ne \mathbf{0} \text{ on } \Gamma_3, \tag{6.6}$$

and

$$u(0) = u_0 \text{ in } \Omega. \tag{6.7}$$

We give a brief description on Problem 6.1. Eq. (6.1) is the constitutive law of the viscoelastic material, where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$ describe the viscous, elastic and relaxation properties, respectively, and the integration term characterizes the long memory of the material. Relation (6.2) is the equilibrium equation, where Div is the divergence operator defined by $\mathrm{Div}\sigma = (\frac{\partial \sigma_{ij}}{\partial x_j})$, f_0 is the volume force density acting on Ω. The body is clamped on Γ_1. A surface traction of density f_2 is applied on Γ_2. The relation (6.5) is a unilateral constraint condition on Γ_3. The relation (6.6) is the Tresca law for friction. In (6.5), the function g denotes the thickness of the elastic layer, and $u_\nu \le g$ sets a restriction on the normal displacement. If the normal penetration does not reach the bound g, i.e., $u_\nu < g$, the relation $-\sigma_\nu = \xi_\nu \in \partial\psi_\nu(u_\nu)$ holds, which is the usual normal compliance condition. In (6.6), F_b represents the friction bound. When $|\sigma_\tau| < F_b$, the material point is in the stick zone; when $|\sigma_\tau| = F_b$, i.e., the friction traction reaches the bound, the material point is in the slip zone. Eq. (6.7) is the initial condition. For details of this kind of contact models, we refer the reader to [11,27].

To introduce a weak formulation of Problem 6.1, we need an evolution triple of spaces. Define

$$V = \{v = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid v = \mathbf{0} \text{ a.e. on } \Gamma_1\}$$

which is a subspace of $H^1(\Omega; \mathbb{R}^d)$. Let V^* be the dual space of V. Define

$$H = L^2(\Omega; \mathbb{R}^d)$$

and

$$\mathcal{H} = \{\tau = (\tau_{ij}) \in L^2(\Omega; \mathbb{S}^d) \mid \tau_{ij} = \tau_{ji}, 1 \le i, j \le d\}.$$

Moreover, the space $\mathcal{H}$ is equipped with the inner product

$$(\tau, \sigma)_{\mathcal{H}} = \int_\Omega \tau_{ij}(x)\sigma_{ij}(x)dx \text{ for all } \tau, \sigma \in \mathcal{H}.$$

On V, an inner product is defined by

$$(u, v)_V = \big(\varepsilon(u), \varepsilon(u)\big)_{\mathcal{H}} \quad \text{for all } u, v \in V.$$

Since $|\Gamma_1| > 0$, it follows from Korn's inequality that V is a Hilbert space. Then, $V \subset H \subset V^*$ forms an evolution triple. Besides, we define

$$U = \{v \in V \mid v_\nu \le g \text{ a.e. on } \Gamma_3\}$$

which is a nonempty, closed and convex subset of V. Denote $X = L^2(\Gamma_3)$. Furthermore, we define a space of fourth-order tensor fields

$$\mathcal{Q}_\infty = \{Q = (Q_{ijkl}) \mid Q_{ijkl} = Q_{jikl} = Q_{klij} \in L^\infty(\Omega), 1 \le i, j, k, l \le d\}.$$

This is a Banach space with the norm defined by

$$\|Q\|_{\mathcal{Q}_\infty} = \sum_{1 \le i,j,k,l \le d} \|Q_{ijkl}\|_{L^\infty(\Omega)}.$$

Define $\gamma_\nu : V \to X$ by $\gamma_\nu v = v_\nu$ and $\gamma_\tau : V \to L^2(\Gamma_3; \mathbb{R}^d)$ by $\gamma_\tau v = v_\tau$, i.e., γ_ν and γ_τ denote the trace operators for the normal and tangential components on Γ_3, respectively.

Assumptions on the data of Problem 6.1 are listed as follows. On the viscosity tensor $\mathcal{A}$, assume

$$\begin{cases} \mathcal{A} : \Omega \to \mathcal{Q}_\infty \text{ satisfies that} \\ \text{(a) } \mathcal{A} = (a_{ijkl}) \in \mathcal{Q}_\infty; \\ \text{(b) there exists a constant } \alpha_1 > 0 \text{ such that} \\ \quad \mathcal{A}\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \geq \alpha_1 |\boldsymbol{\varepsilon}|^2 \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ a.e. in } \Omega. \end{cases} \tag{6.8}$$

On the elasticity tensor $\mathcal{B}$, assume

$$\begin{cases} \mathcal{B} : \Omega \to \mathcal{Q}_\infty \text{ satisfies that} \\ \text{(a) } \mathcal{B} = (b_{ijkl}) \in \mathcal{Q}_\infty; \\ \text{(b) there exists a constant } \alpha_2 > 0 \text{ such that} \\ \quad \mathcal{B}\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \geq \alpha_2 |\boldsymbol{\varepsilon}|^2 \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ a.e. in } \Omega. \end{cases} \tag{6.9}$$

For the relaxation tensor $\mathcal{R}$, assume

$$\mathcal{R} : \overline{I} \to \mathrm{Q}_\infty \text{ is Lipschitz continuous with a constant } L_{\mathcal{R}} > 0. \tag{6.10}$$

On the function ψ_ν, assume

$$\begin{cases} \psi_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ satisfies that} \\ \text{(a) } \psi_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(b) } \psi_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for any } r \in \mathbb{R}, \text{ and there exists } \overline{e} \in L^2(\Gamma_3) \\ \quad \text{such that } \psi_\nu(\cdot, \overline{e}(\cdot)) \in L^1(\Gamma_3); \\ \text{(c) there exist } \overline{c}_0, \overline{c}_1 > 0 \text{ such that } |\partial\psi_\nu(r)| \leq \overline{c}_0 + \overline{c}_1 |r| \quad \forall\, r \in \mathbb{R}; \\ \text{(d) there exists } \alpha_3 > 0 \text{ such that} \\ \quad \psi_\nu^0(r_1; r_2 - r_1) + \psi_\nu^0(r_2; r_1 - r_2) \leq \alpha_3 |r_1 - r_2|^2 \quad \forall\, r_1, r_2 \in \mathbb{R}. \end{cases} \tag{6.11}$$

On the densities of volume force $\boldsymbol{f}_0$ and surface contraction $\boldsymbol{f}_2$, assume

$$\boldsymbol{f}_0 \in H^1(I; L^2(\Omega; \mathbb{R}^d)), \ \boldsymbol{f}_2 \in H^1(I; L^2(\Gamma_2; \mathbb{R}^d)). \tag{6.12}$$

For the functions F_b and g, assume

$$F_b \text{ and } g \text{ are measurable on } \Gamma_3, \ F_b(\boldsymbol{x}) \text{ and } g(\boldsymbol{x}) \text{ are nonnegative for a.e. } \boldsymbol{x} \in \Gamma_3. \tag{6.13}$$

On the parameters α_2 and α_3, assume

$$\alpha_2 \geq \alpha_3 \|\gamma_\nu\|^2. \tag{6.14}$$

On the initial displacement $\boldsymbol{u}_0$, assume

$$\boldsymbol{u}_0 \in U. \tag{6.15}$$

By the Riesz representation theorem, we define a function $\boldsymbol{f} : I \to V^*$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v} \rangle = (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \text{for all } \boldsymbol{v} \in V, \ \text{a.e. } t \in I.$$

Following a standard approach (cf. [3,25]), we can derive the following weak formulation for Problem 6.1.

Problem 6.2. Find a displacement $\mathbf{u} : \Omega \times I \to \mathbb{R}^d$ such that for a.e. $t \in I$,

$$\Big(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\Big)_{\mathcal{H}} + \int_{\Gamma_3} \psi_\nu^0(u_\nu(t); v_\nu - u_\nu(t))d\Gamma$$

$$+ \int_{\Gamma_3} F_b(|\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau(t)|)d\Gamma \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t) \rangle \quad \forall\, \boldsymbol{v} \in U, \tag{6.16}$$

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{6.17}$$

Now we apply Theorem 5.3 to study Problem 6.2. Define an operator $A : V \to V^*$ by

$$\langle A\boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_1), \boldsymbol{\varepsilon}(\boldsymbol{v}_2)\big)_{\mathcal{H}} \quad \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V. \tag{6.18}$$

Define an operator $B : V \to V^*$ by

$$\langle B\boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \big(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{v}_1), \boldsymbol{\varepsilon}(\boldsymbol{v}_2)\big)_{\mathcal{H}} \quad \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V. \tag{6.19}$$

For all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathcal{V}$, a.e. $t \in I$, define $\mathcal{S} : \mathcal{V} \to \mathcal{V}^*$ by

$$\langle (\mathcal{S}\boldsymbol{w}_1)(t), \boldsymbol{w}_2(t) \rangle = \Big(\int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{w}_1(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{w}_2(t))\Big)_{\mathcal{H}}. \tag{6.20}$$

Define a functional $\Psi : X \to \mathbb{R}$ by

$$\Psi(z) = \int_{\Gamma_3} \psi_\nu(z)\, d\Gamma \quad \forall\, z \in X, \tag{6.21}$$

and define $\Phi : V \to \mathbb{R}$ by

$$\Phi(\boldsymbol{v}) = \int_{\Gamma_3} F_b |\boldsymbol{v}_\tau|\, d\Gamma \quad \forall\, \boldsymbol{v} \in V. \tag{6.22}$$

Then we obtain the following result.

Theorem 6.3. *Assume (6.8)–(6.15). Then, there exists a unique solution $\boldsymbol{u}$ to Problem 6.2. Moreover, the solution $\boldsymbol{u}$ depends Lipschitz continuously on $\boldsymbol{f}$.*

Proof. Let us verify all the assumptions of Theorem 5.3 for Problem 6.2. By (6.8), $H(A)$ is valid with $\alpha_A = \alpha_1$. By (6.9), $H(B)$ is valid with $\alpha_B = \alpha_2$. It follows from (6.10) that the operator $\mathcal{S} : \mathcal{V} \to \mathcal{V}^*$ is history-dependent with $C_T = \|\mathcal{R}\|_{C(I;\mathcal{Q}_\infty)}$. It is easy to see $H(\Phi)$ is satisfied. According to (6.11), we know that $H(\Psi)$ is satisfied with $c_0 = \bar{c}_0\sqrt{|\Gamma_3|}$, $c_1 = \bar{c}_1$ and $\alpha_\Psi = \alpha_3$. Next, we check $\gamma_\nu : V \to X$ satisfies the conditions in $H(\gamma)$. Note that γ_ν is linear and continuous by the Sobolev trace theorem (cf. [3,25]). Moreover, for $\lambda \in (0, \frac{1}{2})$, the embedding $M^{2,2}(I; V, V^*) \subset L^2(I; H^{\frac{1}{2}+\lambda}(\Omega))$ is compact by Lemma 2.4. Let $\{\boldsymbol{w}_n\}$ be a bounded sequence in $M^{2,2}(I; V, V^*)$. Then, there exists a subsequence $\{\boldsymbol{w}_{n_i}\}$ of $\{\boldsymbol{w}_n\}$ and an element $\boldsymbol{w} \in L^2(I; H^{\frac{1}{2}+\lambda}(\Omega))$, such that $\boldsymbol{w}_{n_i}$ converges strongly to $\boldsymbol{w}$ in $L^2(I; H^{\frac{1}{2}+\lambda}(\Omega))$. Since the trace operator which maps $L^2(I; H^{\frac{1}{2}+\lambda}(\Omega))$ to $L^2(I; H^{\lambda}(\Gamma_3))$ is continuous, and $L^2(I; H^{\lambda}(\Gamma_3)) \subset \mathcal{X}$, we obtain $\widetilde{\gamma}_\nu \boldsymbol{w}_{n_i} \to \widetilde{\gamma}_\nu \boldsymbol{w}$ in $\mathcal{X}$ (cf. [21]), where $\widetilde{\gamma}_\nu$ is the Nemytskii operator of γ_ν. $H(f)$ and $H(P)$ are valid due to (6.12) and (6.14), respectively. By Theorem 5.3, we obtain Problem 6.2 has a unique solution $\boldsymbol{u}$. Finally, the Lipschitz continuous dependence of $\boldsymbol{u}$ on the right-hand side $\boldsymbol{f}$ is derived from Theorem 5.4. ■

Acknowledgment

The authors thank the anonymous referees whose comments and suggestions lead to an improvement of the presentation of the paper. The work of Wei Xu is supported by the scientific research project of Tongji Zhejiang College under grant N0. KY0221519.

Data availability

No data was used for the research described in the article.

References

[1] P.D. Panagiotopoulos, Nonconvex.energy. functions, Hemivariational inequalities and substationary principles, Acta Mech. 42 (1983) 160–183.
[2] S. Carl, V.K. Le, D. Motreanu, Nonsmooth Variational Problems and their Inequalities: Comparison Principles and Applications, Springer, New York, 2007.
[3] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Springer, New York, 2013.
[4] Z. Naniewicz, P.D. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications, Dekker, New York, 1995.
[5] M. Sofonea, S. Migórski, Variational–Hemivariational Inequalities with Applications, CRC Press, Boca Raton, FL, 2018.
[6] W. Han, Z. Huang, C. Wang, W. Xu, Nunerical analysis of elliptic hemivariational inequalities for semipermeable media, J. Comput. Math. 37 (2019) 545–562.
[7] W. Han, S. Migórski, M. Sofonea, A class of variational–hemivariational inequalities with applications to frictional contact problems, SIAM J. Math. Anal. 46 (2014) 3891–3912.
[8] W. Han, C. Wang, Numerical analysis of a parabolic hemivariational inequality for semipermeable media, J. Comput. Appl. Math. 389 (2021) 113326.
[9] J. Haslinger, M. Miettinen, P.D. Panagiotopoulos, Finite Element Method for Hemivariational Inequalities. Theory, Methods and Applications, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999.
[10] S. Wang, W. Xu, W. Han, W. Chen, Numerical analysis of history-dependent variational–hemivariational inequalities, Sci. China Math. 63 (11) (2020) 2207–2232.
[11] W. Xu, Z. Huang, W. Han, W. Chen, C. Wang, Numerical analysis of history-dependent variational–hemivariational inequalities with applications in contact mechanics, J. Comput. Appl. Math. 351 (2019) 364–377.
[12] W. Xu, Z. Huang, W. Han, W. Chen, C. Wang, Numerical approximation of an electro-elastic frictional contact problem modeled by hemivariational inequality, Comput. Appl. Math. 39 (2020) 265.
[13] W. Han, M. Sofonea, Numerical analysis of hemivariational inequalities in contact mechanics, Acta Numer. 28 (2019) 175–286.
[14] S. Migórski, S. Zeng, Rothe method and numerical analysis for history-dependent hemivariational inequalities with applications to contact mechanics, Numer. Algorithms 82 (2019) 423–450.
[15] W. Han, Minimization principles for elliptic hemivariational inequalities, Nonlinear Anal. Real World Appl. 54 (2020) 103114.
[16] W. Han, A revisit of elliptic variational–hemivariational inequalities, Numer. Funct. Anal. Optim. 42 (4) (2021) 371–395.
[17] P. Kalita, Convergence of the rothe scheme for hemivariational inequalities for parabolic type, Int. J. Numer. Anal. Model. 10 (2013) 445–465.
[18] K. Bartosz, X. Cheng, P. Kalita, Y. Yu, C. Zheng, Rothe method for parabolic variational–hemivariational inequalities, J. Math. Anal. Appl. 423 (2015) 841–862.
[19] K. Bartosz, M. Sofonea, The rothe method for variational–hemivariational inequalities with applications to contact mechanics, SIAM J. Math. Anal. 48 (2) (2016) 861–883.

[20] C. Carstensen, J. Gwinner, A theory of discretization for nonlinear evolution inequalities applied to parabolic signorini problems, Ann. Mat. Pura Appl. 177 (1999) 363–394.

[21] S. Migórski, W. Han, S. Zeng, A new class of hyperbolic variational–hemivariational inequalities driven by non-linear evolution equations, Eur. J. Appl. Math. 32 (2021) 59–88.

[22] S. Migórski, A. Ochal, M. Sofonea, History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics, Nonlinear Anal. Real World Appl. 12 (2011) 3384–3396.

[23] E. Zeidler, Nonlinear Functional Analysis and Its Applications. III: Variational Methods and Optimization, Springer-Verlag, New York, 1986.

[24] K. Atkinson, W. Han, Theoretical Numerical Analysis: A Functional Analysis Framework, Third Edition, Springer-Verlag, New York, 2009.

[25] W. Han, M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, AMS/IP, Providence, RI, 2002.

[26] E. Zeidler, Nonlinear Functional Analysis and Its Applications. II/A: Linear Monotone Operators, Springer-Verlag, New York, 1990.

[27] M. Sofonea, S. Migórski, A class of history-dependent variational–hemivariational inequalities, Nonlinear Differ. Equ. Appl. 23 (3) (2016) 1–23.