Journal of Computational and Applied Mathematics 477 (2026) 117179

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Analysis and numerical approximation of doubly-history dependent hemivariational inequalities in contact mechanics

Wei Xu [a], Wenbin Chen [b], Weimin Han [c], Yang Liu [d],*, Ziping Huang [d]

[a] Tongji Zhejiang College, Jiaxing 314051, China
[b] School of Mathematical Sciences, Fudan University, Shanghai 200433, China
[c] Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA
[d] School of Mathematical Sciences, Tongji University, Shanghai 200092, China

ARTICLE INFO

Keywords:
Hemivariational inequality
Existence
Uniqueness
Finite element method
Error estimate
Optimal order

ABSTRACT

This paper is devoted to studies of doubly-history dependent hemivariational inequalities in contact mechanics. Existence and uniqueness of a solution to the problem is proved by applying a basic well-posedness result combined with a Banach fixed-point argument. A fully discrete scheme is used to solve the problem, with temporal integrals approximated by rectangular rules and the spatial discretization done by the linear element method. Under suitable solution regularity assumptions, an optimal order error bound is proved for the numerical solutions. Finally, simulation results on a numerical example are reported to illustrate numerical convergence orders.

1. Introduction

Hemivariational inequalities are a family of mathematical models for application problems with non-smooth non-convex features. They have attracted ever growing attention from the research communities. Comprehensive references on mathematical theory and applications of hemivariational inequalities include [1–3] from the early years, and [4–9] more recently. In the majority of publications on mathematical analysis of hemivariational inequalities, abstract surjectivity results on pseudomonotone operators are applied, together with fixed-point arguments. An alternative and more accessible approach for the mathematical theory of hemivariational inequalities has been developed without the need of the theory of pseudomonotone operators. This approach was developed in a series of papers [10–13], and is documented in the book [14]. Since no closed-form solution formulas are available for hemivariational inequalities arising in applications, numerical methods are needed to solve the problems. On the numerical solution of hemivariational inequalities, the reader is referred to the monograph [15] for early development, and to the two survey papers [16,17] for more recent development.

History-dependent hemivariational inequalities are a class of hemivariational inequalities including so-called history-dependent operators, which usually represent integrations of physical quantities with respect to the time variable, reflecting the fact that the current values of physical quantities depend on their values in the past. For mathematical theories of history-dependent problems, cf. [18–23], and for the numerical approximations of the problems, cf. [24–28].

Recently, doubly-history dependent variational inequalities have been studied in [29], which contain terms with repeated time integration. In this paper, we extend this framework to doubly-history dependent hemivariational inequalities, analyzing both solution well-posedness and numerical methods. Existence and uniqueness of a solution to the problem is shown by applying a

* Corresponding author.
E-mail address: ly_tj@tongji.edu.cn (Y. Liu).

https://doi.org/10.1016/j.cam.2025.117179
Received 14 May 2025; Received in revised form 22 October 2025
Available online 27 October 2025

basic well-posedness result on hemivariational inequalities combined with a Banach fixed-point argument. A fully discrete scheme is introduced to solve the problem, and an optimal order error estimate is proved for the numerical solutions. We study the history-dependent hemivariational inequality in the context of a quasistatic contact problem for a viscoelastic material.

Descriptions and studies of hemivariational inequalities require the notions of the generalized directional derivatives and the generalized subdifferentials in the sense of Clarke. The reader is referred to [30,31] or [8] for detailed discussions of these notions and their properties.

The paper is organized as follows. In Section 2, a doubly-history dependent hemivariational inequality is introduced for a quasi-static contact problem for viscoelastic material. In Section 3, a solution existence and uniqueness result is proved. In Section 4, a fully-discrete scheme is studied and an optimal order error estimate is derived for the linear finite element solutions. In Section 5, a numerical example is provided, showing the performance of the numerical scheme.

2. A doubly-history dependent hemivariational inequality in viscoelastic contact

We consider a quasi-static contact problem for a viscoelastic material. The physical setting of the problem is as follows. The configuration of the viscoelastic body is represented by a Lipschitz domain $\Omega \subset \mathbb{R}^d$, d being the dimension of the spatial domain. The boundary $\Gamma = \partial\Omega$ is split into three measurable parts Γ_1, Γ_2 and Γ_3; the portion Γ_3 is further split into two parts: $\Gamma_{3,1}$ and $\Gamma_{3,2}$ where different contact conditions will be specified. We assume $\text{meas}\,(\Gamma_1) > 0$ and $\text{meas}\,(\Gamma_{3,1}) + \text{meas}\,(\Gamma_{3,2}) > 0$. The body is subject to the action of volume forces of a total density $\boldsymbol{f}_0$ in Ω and surface tractions of a total density $\boldsymbol{f}_2$ on Γ_2, and it is fixed on Γ_1. We assume a frictionless contact with unilateral constraint in the velocity variable on $\Gamma_{3,1}$, and a general normal damped response and friction law expressed in a subdifferential form on $\Gamma_{3,2}$. We are interested in the deformation of the body in a time interval $[0, T]$.

Next we introduce some notations in mechanics. Let $\mathbb{R}^d$ be a d-dimensional vector space. For $\boldsymbol{v} \in \mathbb{R}^d$, $v_i \in \mathbb{R}$ for $1 \le i \le d$. The canonical inner product and norm on $\mathbb{R}^d$ are defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i,\ \ \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$

where summation convention on a repeated index is adopted. The symbol $\boldsymbol{\nu}$ denotes the unit outward normal vector defined on Γ, $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ denotes the normal and tangential components of $\boldsymbol{v}$, respectively.

Let $\mathbb{S}^d$ be the space of symmetric matrices of order d. For $\boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d$, $\tau_{ij} = \tau_{ji} \in \mathbb{R}$ for $1 \le i, j \le d$. The canonical inner product and norm on the space $\mathbb{S}^d$ are defined by

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij},\ \ \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \quad \text{for } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

The notation $\boldsymbol{\sigma}$ stands for a stress tensor, $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ stands for the normal and tangential components of $\boldsymbol{\sigma}$, respectively.

The classical formulation of the contact problem is to find a displacement field $\boldsymbol{u} : \Omega \times [0, T]$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0, T]$ such that for $t \in [0, T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{2.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \tag{2.4}$$

$$u'_\nu(t) \le 0,\ \ \sigma_\nu(t) \le 0,\ \ \sigma_\nu(t)\,u'_\nu(t) = 0,\ \ \boldsymbol{\sigma}_\tau(t) = 0 \quad \text{on } \Gamma_{3,1}, \tag{2.5}$$

$$-\sigma_\nu(t) \in \partial\psi_\nu(u'_\nu(t)),\ \ -\boldsymbol{\sigma}_\tau(t) \in \partial\psi_\tau(\boldsymbol{u}'_\tau(t)) \quad \text{on } \Gamma_{3,2}, \tag{2.6}$$

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{2.7}$$

A brief interpretation of the above equations and relations is as follows. Eq. (2.1) is a general constitutive law for viscoelastic materials, in which, $\boldsymbol{\varepsilon}$ is a linear strain tensor, $\mathcal{A}$ is a viscosity tensor, $\mathcal{B}$ is an elasticity tensor, $\mathcal{R}$ is a relaxation tensor, and the integral term describes the long memory character of the materials (cf. [20]). Here, a prime on a variable denotes the time derivative of the variable. Formula (2.2) is the equilibrium equation, in which Div denotes the divergence operator, $\boldsymbol{f}_0$ is the density of volume forces acting on the deformable body. Eq. (2.3) is the displacement boundary condition on Γ_1, and Eq. (2.4) is the traction boundary condition on Γ_2. Relations (2.5) describe a frictionless unilateral constraint on $\Gamma_{3,1}$ (cf. [32]). Relations (2.6) are the multi-valued contact conditions on $\Gamma_{3,2}$, following [8, (6.42)] and [8, (6.47)]. The symbols $\partial\psi_\nu$ and $\partial\psi_\tau$ denote the Clarke subdifferential of the given functions ψ_ν and ψ_τ, respectively. Finally, formula (2.7) is an initial displacement field.

We will study the contact problem through its weak form. For this purpose, we first introduce some function spaces and sets. The function space for stress and strain fields at a fixed time is

$$Q = L^2(\Omega)^{d\times d}_{\text{sym}},$$

which is a Hilbert space with the canonical inner product

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_Q=\int_\Omega \sigma_{ij}\tau_{ij}\,dx,\quad \boldsymbol{\sigma},\boldsymbol{\tau}\in Q$$

and its induced norm. The function space for displacement and velocity fields at a fixed time is

$$V=\left\{\boldsymbol{v}\in H^1(\Omega;\mathbb{R}^d)\mid \boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_1\right\}.$$

Since $\operatorname{meas}(\Gamma_1)>0$, by the Korn inequality ([33, page 79]), V is a Hilbert space with the inner product $(\boldsymbol{u},\boldsymbol{v})_V=(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q$. Moreover, the associated norm $\|\boldsymbol{v}\|_V=\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q$ is equivalent to the standard $H^1(\Omega;\mathbb{R}^d)$-norm on V. The set of admissible velocity fields is

$$K=\left\{\boldsymbol{v}\in V\mid v_\nu\le 0\text{ a.e. on }\Gamma_{3,1}\right\}.$$

We further introduce the space of bounded, symmetric fourth-order tensor fields

$$Q_\infty=\left\{\mathcal{E}=(\mathcal{E}_{ijkl})_{1\le i,j,k,l\le d}\mid \mathcal{E}_{ijkl}=\mathcal{E}_{jikl}=\mathcal{E}_{klij}\in L^\infty(\Omega),\ 1\le i,j,k,l\le d\right\}.$$

Regarding the problem data, we make the following assumptions.

$H(\mathcal{A})$ $\mathcal{A}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d$,

$$\begin{cases}\text{(a) } \mathcal{A}(\cdot,\boldsymbol{\varepsilon})\text{ is measurable on }\Omega\text{ for all }\boldsymbol{\varepsilon}\in\mathbb{S}^d;\\ \text{(b) } \mathcal{A}(\boldsymbol{x},\boldsymbol{0})=\boldsymbol{0}\text{ for a.e. }\boldsymbol{x}\in\Omega;\\ \text{(c) for a constant } L_{\mathcal{A}}>0,\\ \qquad \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|_{\mathbb{S}^d}\le L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d}\ \forall\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. }\boldsymbol{x}\in\Omega;\\ \text{(d) for a constant } m_{\mathcal{A}}>0,\\ \qquad \big(\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\big)\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\ge m_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2_{\mathbb{S}^d}\ \forall\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. }\boldsymbol{x}\in\Omega.\end{cases}$$

$H(\mathcal{B})$ $\mathcal{B}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d$ is such that $\mathcal{B}(\cdot,\boldsymbol{0})\in Q$, $\mathcal{B}(\cdot,\boldsymbol{\varepsilon})$ is measurable on Ω for all $\boldsymbol{\varepsilon}\in\mathbb{S}^d$, and

$$\|\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|_{\mathbb{S}^d}\le L_{\mathcal{B}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d}\quad\forall\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. }\boldsymbol{x}\in\Omega\text{ with }L_{\mathcal{B}}>0.$$

$H_a(\mathcal{R})$ $\mathcal{R}\in C([0,T];Q_\infty)$.

$H_a(\boldsymbol{f})$ $\boldsymbol{f}_0\in C([0,T];L^2(\Omega;\mathbb{R}^d))$, $\boldsymbol{f}_2\in C([0,T];L^2(\Gamma_2;\mathbb{R}^d))$.

$H(\psi_\nu)$ For $\psi_\nu:\Gamma_{3,2}\times\mathbb{R}\to\mathbb{R}$,

$$\begin{cases}\text{(a) } \psi_\nu(\cdot,r)\text{ is measurable on }\Gamma_{3,2}\text{ for all }r\in\mathbb{R},\ \psi_\nu(\cdot,0)\in L^1(\Gamma_{3,2});\\ \text{(b) } \psi_\nu(\boldsymbol{x},\cdot)\text{ is Lipschitz continuous on }\mathbb{R}\text{ with a Lipschitz constant }\bar{c}_{0\nu}\ge 0\text{ for a.e. }\boldsymbol{x}\in\Gamma_{3,2};\\ \text{(c) } \psi_\nu^0(\boldsymbol{x},r_1;r_2-r_1)+\psi_\nu^0(\boldsymbol{x},r_2;r_1-r_2)\le\alpha_{\psi_\nu}|r_1-r_2|^2\\ \qquad\text{for a.e. }\boldsymbol{x}\in\Gamma_{3,2},\ \text{all }r_1,\ r_2\in\mathbb{R}\text{ with }\alpha_{\psi_\nu}\ge 0.\end{cases}$$

$H(\psi_\tau)$ For $\psi_\tau:\Gamma_{3,2}\times\mathbb{R}^d\to\mathbb{R}$,

$$\begin{cases}\text{(a) } \psi_\tau(\cdot,\boldsymbol{\xi})\text{ is measurable on }\Gamma_{3,2}\text{ for all }\boldsymbol{\xi}\in\mathbb{R}^d,\ \psi_\tau(\cdot,\boldsymbol{0})\in L^1(\Gamma_{3,2});\\ \text{(b) } \psi_\tau(\boldsymbol{x},\cdot)\text{ is Lipschitz continuous on }\mathbb{R}^d\text{ with a Lipschitz constant }\bar{c}_{0\tau}\ge 0\text{ for a.e. }\boldsymbol{x}\in\Gamma_{3,2};\\ \text{(c) } \psi_\tau^0(\boldsymbol{x},\boldsymbol{\xi}_1;\boldsymbol{\xi}_2-\boldsymbol{\xi}_1)+\psi_\tau^0(\boldsymbol{x},\boldsymbol{\xi}_2;\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\le\alpha_{\psi_\tau}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|^2_{\mathbb{R}^d}\\ \qquad\text{for a.e. }\boldsymbol{x}\in\Gamma_3,\ \text{all }\boldsymbol{\xi}_1,\ \boldsymbol{\xi}_2\in\mathbb{R}^d\text{ with }\alpha_{\psi_\tau}\ge 0.\end{cases}$$

$H(\boldsymbol{u}_0)$ $\boldsymbol{u}_0\in V$.

By the Riesz representation theorem, at any $t\in[0,T]$, we can define $\boldsymbol{f}(t)\in V^*$ through the relation

$$\langle\boldsymbol{f}(t),\boldsymbol{v}\rangle=(\boldsymbol{f}_0(t),\boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)}+(\boldsymbol{f}_2(t),\boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)}\quad \boldsymbol{v}\in V.$$

Due to $H_a(\boldsymbol{f})$, we have $\boldsymbol{f}\in C([0,T];V^*)$. By a standard procedure, one can obtain the following weak formulation of the contact problem (2.1)–(2.7).

Problem 2.1. Find a displacement field $\boldsymbol{u}:[0,T]\to V$ such that for $t\in[0,T]$,

$$\begin{aligned}&\boldsymbol{u}'(t)\in K,\quad \Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))+\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u}'(t))\Big)_Q\\ &\quad+\int_{\Gamma_{3,2}}\big[\psi_\nu^0(u'_\nu(t);v_\nu-u'_\nu(t))+\psi_\tau^0(\boldsymbol{u}'_\tau(t);\boldsymbol{v}_\tau-\boldsymbol{u}'_\tau(t))\big]\,da\ge\langle\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}'(t)\rangle\quad\forall\boldsymbol{v}\in K,\end{aligned}$$

and

$$\boldsymbol{u}(0)=\boldsymbol{u}_0\quad\text{in }\Omega.$$

It will be convenient to reformulate the problem in terms of the velocity variable

$$\boldsymbol{w}(t) = \boldsymbol{u}'(t).$$

Note that given $\boldsymbol{w}(t)$ and the initial displacement $\boldsymbol{u}_0$, we can recover the displacement $\boldsymbol{u}(t)$ by

$$\boldsymbol{u}(t) = \boldsymbol{I}\boldsymbol{w}(t) \equiv \boldsymbol{u}_0 + \int_0^t \boldsymbol{w}(s)\,ds.$$

Then Problem 2.1 can be reformulated in terms of $\boldsymbol{w}$.

Problem 2.2. Find a velocity field $\boldsymbol{w} : [0,T] \to V$ such that for $t \in [0,T]$,

$$\begin{aligned} &\boldsymbol{w}(t) \in K, \quad \Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{I}\boldsymbol{w}(t)) + \int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{I}\boldsymbol{w}(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}(t))\Big)_Q \\ &\quad + \int_{\Gamma_{3,2}} \big[\psi_\nu^0(w_\nu(t); v_\nu - w_\nu(t)) + \psi_\tau^0(\boldsymbol{w}_\tau(t); \boldsymbol{v}_\tau - \boldsymbol{w}_\tau(t))\big]\,da \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle \ \forall\, \boldsymbol{v} \in K. \end{aligned} \tag{2.8}$$

Note that the hemivariational inequality (2.8) contains a doubly-history dependent term

$$\int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{I}\boldsymbol{w}(s))\,ds.$$

For this reason, (2.8) is called a doubly-history dependent hemivariational inequality.

When $\Gamma_{3,2} = \emptyset$, Problem 2.2 is reduced to a variational inequality of the first kind.

3. Existence and uniqueness result of the hemivariational inequality

In this section, we prove the solution existence and uniqueness of Problem 2.2 by a Banach fixed-point argument.

We rewrite (2.8) as

$$\begin{aligned} &\boldsymbol{w}(t) \in K, \ \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}(t))\big)_Q + \int_{\Gamma_{3,2}} \big[\psi_\nu^0(w_\nu(t); v_\nu - w_\nu(t)) + \psi_\tau^0(\boldsymbol{w}_\tau(t); \boldsymbol{v}_\tau - \boldsymbol{w}_\tau(t))\big]\,da \\ &\quad \ge \langle \boldsymbol{f}(t), \boldsymbol{v}-\boldsymbol{w}(t)\rangle - \Big(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{I}\boldsymbol{w}(t)) + \int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{I}\boldsymbol{w}(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}(t))\Big)_Q \quad \forall\, \boldsymbol{v} \in K, \end{aligned}$$

and introduce an intermediate hemivariational inequality: with $\tilde{\boldsymbol{w}}(t) \in K$ given,

$$\begin{aligned} &\boldsymbol{w}(t) \in K, \ \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}(t))\big)_Q + \int_{\Gamma_{3,2}} \big[\psi_\nu^0(w_\nu(t); v_\nu - w_\nu(t)) + \psi_\tau^0(\boldsymbol{w}_\tau(t); \boldsymbol{v}_\tau - \boldsymbol{w}_\tau(t))\big]\,da \\ &\quad \ge \langle \boldsymbol{f}(t), \boldsymbol{v}-\boldsymbol{w}(t)\rangle - \Big(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{I}\tilde{\boldsymbol{w}}(t)) + \int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{I}\tilde{\boldsymbol{w}}(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}(t))\Big)_Q \quad \forall\, \boldsymbol{v} \in K. \end{aligned} \tag{3.1}$$

Also, we recall a result on stationary hemivariational inequalities (cf. [34, Theorem 3.1]).

Theorem 3.1. *Assume*

(A_1) *X is a reflexive Banach space, and K is a non-empty, closed and convex subset of X.*

(A_2) *For $i = 1, 2$, X_i is a Banach space, $\gamma_i \in \mathcal{L}(X, X_i)$: for a constant $c_i > 0$,*

$$\|\gamma_i v\|_{X_i} \le c_i \|v\|_X \quad \forall\, v \in X.$$

(A_3) *$A : X \to X^*$ is bounded, continuous and strongly monotone: for a constant $m_A > 0$,*

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \ge m_A \|v_1 - v_2\|_X^2 \quad \forall\, v_1, v_2 \in X.$$

(A_4) *For $i = 1, 2$, $J_i : X_i \to \mathbb{R}$ is locally Lipschitz continuous, and there exist constants $c_{i,0}, c_{i,1}, \alpha_i \ge 0$ such that*

$$\|\partial J_i(z)\|_{X_i^*} \le c_{i,0} + c_{i,1}\|z\|_{X_i} \quad \forall\, z \in X_i,$$

$$J_i^0(z_1; z_2 - z_1) + J_i^0(z_2; z_1 - z_2) \le \alpha_i \|z_1 - z_2\|_{X_i}^2 \quad \forall\, z_1, z_2 \in X_i.$$

(A_5)

$$\alpha_1 c_1^2 + \alpha_2 c_2^2 < m_A.$$

(A_6)

$$f \in X^*.$$

Then there is a unique $u \in K$ such that

$$\langle Au, v - u\rangle + J_1^0(\gamma_1 u; \gamma_1 v - \gamma_1 u) + J_2^0(\gamma_2 u; \gamma_2 v - \gamma_2 u) \ge \langle f, v - u\rangle \quad \forall\, v \in K.$$

In the study of Problem 2.2 as well as the auxiliary problem (3.1), we will make use of the following smallness condition

$H(s)$ $$\alpha_{\psi_\nu}\lambda_\nu^{-1}+\alpha_{\psi_\tau}\lambda_\tau^{-1}<m_{\mathcal{A}}.$$

Here, $\lambda_\nu>0$ is the smallest eigenvalue of the eigenvalue problem

$$\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx=\lambda\int_{\Gamma_{3,2}}u_\nu v_\nu\,da\quad\forall\,\boldsymbol{v}\in V,$$

whereas $\lambda_\tau>0$ is the smallest eigenvalue of the eigenvalue problem

$$\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx=\lambda\int_{\Gamma_{3,2}}\boldsymbol{u}_\tau\cdot\boldsymbol{v}_\tau\,da\quad\forall\,\boldsymbol{v}\in V.$$

Lemma 3.2. *Assume $H(\mathcal{A})$, $H(\mathcal{B})$, $H_a(\mathcal{R})$, $H_a(\boldsymbol{f})$, $H(\psi_\nu)$, $H(\psi_\tau)$, $H(\boldsymbol{u}_0)$, and $H(s)$. Then for any $\tilde{\boldsymbol{w}}\in C([0,T];K)$, there exists a unique solution $\boldsymbol{w}\in C([0,T];K)$ to the hemivariational inequality (3.1).*

Proof. We use Theorem 3.1 with the following setting: $X=V$, $X_1=L^2(\Gamma_{3,2})$, $X_2=L^2(\Gamma_{3,2};\mathbb{R}^d)$, $\gamma_1\boldsymbol{v}=v_\nu$ and $\gamma_2\boldsymbol{v}=\boldsymbol{v}_\tau$ for $\boldsymbol{v}\in V$, and

$$\begin{aligned}
&\langle A\boldsymbol{u},\boldsymbol{v}\rangle=\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})\big)_Q\quad\forall\,\boldsymbol{u},\boldsymbol{v}\in V,\\
&J_1(z)=\int_{\Gamma_{3,2}}\psi_\nu(z)\,da\quad\forall\,z\in X_1,\\
&J_2(\boldsymbol{z})=\int_{\Gamma_{3,2}}\psi_\tau(\boldsymbol{z})\,da\quad\forall\,\boldsymbol{z}\in X_2,\\
&\langle f,\boldsymbol{v}\rangle=\langle\boldsymbol{f}(t),\boldsymbol{v}\rangle-\Big(\mathcal{B}\boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}(t))+\int_0^t\mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}(s))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v})\Big)_Q\quad\forall\,\boldsymbol{v}\in V,\ \forall\,t\in[0,T].
\end{aligned}$$

Then the assumptions (A_1)–(A_6) are satisfied with $c_1=\lambda_\nu^{-1/2}$, $c_2=\lambda_\tau^{-1/2}$, $m_A=m_{\mathcal{A}}$, $\alpha_1=\alpha_{\psi_\nu}$, and $\alpha_2=\alpha_{\psi_\tau}$. Moreover, (A_5) takes the form $H(s)$. Thus, by Theorem 3.1, for any $t\in[0,T]$, the problem

$$\begin{aligned}
&\boldsymbol{w}(t)\in K,\quad\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}(t))\big)_Q+J_1^0(w_\nu(t);v_\nu-w_\nu(t))+J_2^0(\boldsymbol{w}_\tau(t);\boldsymbol{v}_\tau-\boldsymbol{w}(t))\\
&\quad\geq\langle\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{w}(t)\rangle-\Big(\mathcal{B}\boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}(t))+\int_0^t\mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}(s))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}(t))\Big)_Q\quad\forall\,\boldsymbol{v}\in K
\end{aligned}\tag{3.2}$$

has a unique solution. By a property of the generalized directional derivative of an integral functional (cf. [31, page 77], [8, Theorem 3.47 (iv)]),

$$\begin{aligned}
&J_1^0(w_\nu(t);v_\nu-w_\nu(t))\leq\int_{\Gamma_{3,2}}\psi_\nu^0(w_\nu(t);v_\nu-w_\nu(t))\,da,\\
&J_2^0(\boldsymbol{w}_\tau(t);\boldsymbol{v}_\tau-\boldsymbol{w}_\tau(t))\leq\int_{\Gamma_{3,2}}\psi_\tau^0(\boldsymbol{w}_\tau(t);\boldsymbol{v}_\tau-\boldsymbol{w}_\tau(t))\,da,
\end{aligned}$$

we see that the solution $\boldsymbol{w}(t)\in K$ of (3.2) is also a solution of (3.1). Uniqueness of the solution of (3.1) follows from the stated assumptions through a standard argument (cf. [16,35]).

To prove the continuity of $\boldsymbol{w}(t)$ with respect to the variable t, let $t_1,t_2\in(0,T)$, $t_1\neq t_2$. Take $t=t_1$ and $\boldsymbol{v}=\boldsymbol{w}(t_2)$ in (3.1); then take $t=t_2$ and $\boldsymbol{v}=\boldsymbol{w}(t_1)$ in (3.1). Add the two resulting inequalities and derive an inequality of the form

$$\|\boldsymbol{w}(t_1)-\boldsymbol{w}(t_2)\|_V\leq c_3\big(c_4\,\|\mathcal{R}(t_1-s)-\mathcal{R}(t_2-s)\|_{Q_\infty}+c_5\,|t_1-t_2|+\|\boldsymbol{f}(t_1)-\boldsymbol{f}(t_2)\|_{V^*}\big),\tag{3.3}$$

where $c_4=T\|\boldsymbol{u}_0\|_V+T^2\|\tilde{\boldsymbol{w}}\|_{C([0,T];V)}$, $c_5=\|\mathcal{R}\|_{C([0,T];Q_\infty)}(\|\boldsymbol{u}_0\|_V+T\|\tilde{\boldsymbol{w}}\|_{C([0,T];V)}+L_{\mathcal{B}}\|\tilde{\boldsymbol{w}}\|_{C([0,T];V)})$, and $c_3=(m_{\mathcal{A}}-\alpha_{\psi_\nu}\lambda_\nu^{-1}-\alpha_{\psi_\tau}\lambda_\tau^{-1})^{-1}$. By $H(s)$, $H_a(\mathcal{R})$ and $H_a(\boldsymbol{f})$, we have $\boldsymbol{w}\in C([0,T];K)$. ■

Based on Lemma 3.2, we prove an existence and uniqueness result on Problem 2.2.

Theorem 3.3. *Keep the assumptions stated in Lemma 3.2. Then, Problem 2.2 has a unique solution $\boldsymbol{w}\in C([0,T];K)$.*

Proof. For any $\tilde{\boldsymbol{w}}\in C([0,T];K)$, we define an operator $\boldsymbol{\Phi}$ by $\boldsymbol{\Phi}\tilde{\boldsymbol{w}}=\boldsymbol{w}$. Let us show that there exists a unique fixed point of $\boldsymbol{\Phi}$. For $i=1,2$, let $\boldsymbol{w}_i\in C([0,T];K)$ be the solution to (3.1) with fixed $\tilde{\boldsymbol{w}}_i\in C([0,T];K)$, i.e., for any $\boldsymbol{v}\in K$,

$$\begin{aligned}
&\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_1(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}_1(t))\big)_Q+\int_{\Gamma_{3,2}}\big[\psi_\nu^0(w_{1,\nu}(t);v_\nu-w_{1,\nu}(t))+\psi_\tau^0(\boldsymbol{w}_{1,\tau}(t);\boldsymbol{v}_\tau-\boldsymbol{w}_{1,\tau}(t))\big]\,da\\
&\quad\geq\langle\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{w}_1(t)\rangle-\Big(\mathcal{B}\boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}_1(t))+\int_0^t\mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}_1(s))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}_1(t))\Big)_Q,
\end{aligned}\tag{3.4}$$

and

$$\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_2(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}_2(t))\big)_Q+\int_{\Gamma_{3,2}}\big[\psi_\nu^0(w_{2,\nu}(t);v_\nu-w_{2,\nu}(t))+\psi_\tau^0(\boldsymbol{w}_{2,\tau}(t);\boldsymbol{v}_\tau-\boldsymbol{w}_{2,\tau}(t))\big]\,da$$

$$\geq \langle \boldsymbol{f}(t), \boldsymbol{v}-\boldsymbol{w}_2(t)\rangle - \big(\mathcal{B}\boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}_2(t)) + \int_0^t \mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}_2(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{w}_2(t))\big)_Q. \tag{3.5}$$

We take $\boldsymbol{v}=\boldsymbol{w}_2(t)$ in (3.4) and $\boldsymbol{v}=\boldsymbol{w}_1(t)$ in (3.5) respectively, and add the resulting inequalities to obtain

$$\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_1(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_2(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}_1(t)) - \boldsymbol{\varepsilon}(\boldsymbol{w}_2(t))\big)_Q \leq \sum_{i=1}^4 G_i, \tag{3.6}$$

where

$$G_1 = \int_{\Gamma_{3,2}} \big[\psi_\nu^0(w_{1,\nu}(t); w_{2,\nu}(t)-w_{1,\nu}(t)) + \psi_\nu^0(w_{2,\nu}(t); w_{1,\nu}(t)-w_{2,\nu}(t))\big]\,da,$$
$$G_2 = \int_{\Gamma_{3,2}} \big[\psi_\tau^0(\boldsymbol{w}_{1,\tau}(t); \boldsymbol{w}_{2,\tau}(t)-\boldsymbol{w}_{1,\tau}(t)) + \psi_\tau^0(\boldsymbol{w}_{2,\tau}(t); \boldsymbol{w}_{1,\tau}(t)-\boldsymbol{w}_{2,\tau}(t))\big]\,da,$$
$$G_3 = \big(\mathcal{B}\boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}_1(t)) - \mathcal{B}\boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}_2(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}_2(t)-\boldsymbol{w}_1(t))\big)_Q,$$
$$G_4 = \Big(\int_0^t \mathcal{R}(t-s)\,(\boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}_1(s)) - \boldsymbol{\varepsilon}(I\tilde{\boldsymbol{w}}_2(s)))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{w}_2(t)-\boldsymbol{w}_1(t))\Big)_Q.$$

From $H(\mathcal{A})$(d), we have

$$m_{\mathcal{A}}\|\boldsymbol{w}_1(t)-\boldsymbol{w}_2(t)\|_V^2 \leq \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_1(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_2(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}_1(t)) - \boldsymbol{\varepsilon}(\boldsymbol{w}_2(t))\big)_Q. \tag{3.7}$$

$H(\psi_\nu)$(c) is used on G_1 to obtain

$$G_1 \leq \int_{\Gamma_{3,2}} \alpha_{\psi_\nu} |w_{1,\nu}(t)-w_{2,\nu}(t)|^2 da \leq \alpha_{\psi_\nu}\lambda_\nu^{-1}\|\boldsymbol{w}_1(t)-\boldsymbol{w}_2(t)\|_V^2. \tag{3.8}$$

In addition, $H(\psi_\tau)$(c) is utilized on G_2 to get

$$G_2 \leq \int_{\Gamma_{3,2}} \alpha_{\psi_\tau} \|\boldsymbol{w}_{1,\tau}(t)-\boldsymbol{w}_{2,\tau}(t)\|_{\mathbb{R}^d}^2 da \leq \alpha_{\psi_\tau}\lambda_\tau^{-1}\|\boldsymbol{w}_1(t)-\boldsymbol{w}_2(t)\|_V^2. \tag{3.9}$$

On G_3, the Cauchy–Schwarz inequality and the Lipschitz continuity of $\mathcal{B}$ are applied to derive

$$G_3 \leq L_{\mathcal{B}}\|\boldsymbol{w}_1(t)-\boldsymbol{w}_2(t)\|_V \int_0^t \|\tilde{\boldsymbol{w}}_1(s)-\tilde{\boldsymbol{w}}_2(s)\|_V\,ds. \tag{3.10}$$

Similarly, $H_a(\mathcal{R})$ is used on G_4 to get

$$G_4 \leq T\|\mathcal{R}\|_{C([0,T];Q_\infty)}\|\boldsymbol{w}_1(t)-\boldsymbol{w}_2(t)\|_V \int_0^t \|\tilde{\boldsymbol{w}}_1(s)-\tilde{\boldsymbol{w}}_2(s)\|_V\,ds. \tag{3.11}$$

Combining (3.6)–(3.11), we derive

$$\|\Phi\tilde{\boldsymbol{w}}_1(t) - \Phi\tilde{\boldsymbol{w}}_2(t)\|_V \leq c_3(L_{\mathcal{B}} + T\|\mathcal{R}\|_{C([0,T];Q_\infty)}) \int_0^t \|\tilde{\boldsymbol{w}}_1(s)-\tilde{\boldsymbol{w}}_2(s)\|_V\,ds. \tag{3.12}$$

Then, we use [18, Theorem 1] on (3.12) to conclude that there exists a unique $\boldsymbol{w}$ satisfying $\Phi\boldsymbol{w}=\boldsymbol{w}$. Thus, we finish the proof of Theorem 3.3. ■

On numerical analysis of Problem 2.2, a solution regularity $\boldsymbol{w}\in W^{1,1}(0,T;K)$ is required, which can be deduced by stronger assumptions on $\mathcal{R}$ and f. Denote:

$H_b(\mathcal{R})$ $\mathcal{R}\in W^{1,1}(0,T;Q_\infty)$.

$H_b(\boldsymbol{f})$ $\boldsymbol{f}_0\in W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^d))$, $\boldsymbol{f}_2\in W^{1,1}(0,T;L^2(\Gamma_2;\mathbb{R}^d))$.

Lemma 3.4. *Assume $H(\mathcal{A})$, $H(\mathcal{B})$, $H_b(\mathcal{R})$, $H_b(\boldsymbol{f})$, $H(\psi_\nu)$, $H(\psi_\tau)$, $H(\boldsymbol{u}_0)$, and $H(s)$. Then, Problem 2.2 has a unique solution $\boldsymbol{w}\in W^{1,1}(0,T;K)$.*

The proof is similar to that of Theorem 3.3, and hence its details are omitted here. Using $H_b(\mathcal{R})$ and $H_b(\boldsymbol{f})$ on (3.3), one can show that $\boldsymbol{w}$ is absolutely continuous on $[0,T]$. Then the regularity $\boldsymbol{w}\in W^{1,1}(0,T;K)$ follows since V is reflexive (cf. [36, Lemma 2.31]).

4. Numerical analysis of the hemivariational inequality

We introduce and analyze a fully discrete scheme for Problem 2.2 in this section. For a positive integer N, let $k=T/N$ be the time step-size, and let $t_n=nk$, $0\leq n\leq N$, be the nodes. We will use the left endpoint rule for the numerical integration of a continuous function Z:

$$\int_0^{t_n} Z(s)ds \approx k\sum_{j=0}^{n-1} Z(t_j).$$

For simplicity, Ω is assumed to be a polygon/polyhedron. The boundary Γ is further divided as the following: $\overline{\Gamma_1} = \cup_{1\le l\le l_1}\Gamma_{1,l}$, $\overline{\Gamma_2} = \cup_{1\le l\le l_2}\Gamma_{2,l}$, $\overline{\Gamma_{3,1}} = \cup_{1\le l\le l_{3,1}}\Gamma_{3,1,l}$ and $\overline{\Gamma_{3,2}} = \cup_{1\le l\le l_{3,2}}\Gamma_{3,2,l}$, where each subset is a line segment (if $d = 2$) or a polygon (if $d = 3$). Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\overline{\Omega}$ into triangles/tetrahedrons such that if a side/face of an element has a non-trivial intersection with a boundary subset $\Gamma_{1,l}$, $\Gamma_{2,l}$, $\Gamma_{3,1,l}$ or $\Gamma_{3,2,l}$, then the side/face lies entirely on that boundary subset.

For each partition $\mathcal{T}^h$, define

$$V^h = \{\boldsymbol{v}^h \in C(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in \mathbb{P}_1(T;\mathbb{R}^d) \text{ for } T \in \mathcal{T}^h, \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \Gamma_1\}$$

and

$$K^h = \{\boldsymbol{v}^h \in V^h \mid v^h_\nu \le \boldsymbol{0} \text{ at node points on } \Gamma_{3,1}\}.$$

Note that V^h consists of continuous piecewise linear functions and K^h is a convex subset of K. The symbols $\boldsymbol{w}_n = \boldsymbol{w}(t_n)$, $\boldsymbol{\sigma}_n = \boldsymbol{\sigma}(t_n)$, $\boldsymbol{f}_n = \boldsymbol{f}(t_n)$, $\boldsymbol{f}_{0n} = \boldsymbol{f}_0(t_n)$, and $\boldsymbol{f}_{2n} = \boldsymbol{f}_2(t_n)$ are used below. Let $\boldsymbol{u}^h_0 \in V^h$ be an approximation of $\boldsymbol{u}_0$.

The fully discrete scheme for Problem 2.2 is as follows.

Problem 4.1. Find a discrete velocity field $\boldsymbol{w}^{kh} := \{\boldsymbol{w}^{kh}_n\}_{n=0}^N \subset V^h$ such that for $0 \le n \le N$, $\boldsymbol{w}^{kh}_n \in K^h$ and

$$\begin{aligned}
&\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}^{kh}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{w}^{kh}_n)\big)_Q + \int_{\Gamma_{3,2}} \Big[\psi^0_\nu(w^{kh}_{n,\nu}; v^h_\nu - w^{kh}_{n,\nu}) + \psi^0_\tau(\boldsymbol{w}^{kh}_{n,\tau}; \boldsymbol{v}^h_\tau - \boldsymbol{w}^{kh}_{n,\tau})\Big]\, da \\
&\ge \langle \boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{w}^{kh}_n\rangle - \big(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^h_0 + k\sum_{j=0}^{n-1}\boldsymbol{w}^{kh}_j), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{w}^{kh}_n)\big)_Q \\
&\quad - k\sum_{j=0}^{n-1}\big(\mathcal{R}(t_n - t_j)\,\boldsymbol{\varepsilon}(\boldsymbol{u}^h_0 + k\sum_{m=1}^{j}\boldsymbol{w}^{kh}_m), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{w}^{kh}_n)\big)_Q \quad \forall\, \boldsymbol{v}^h \in K^h.
\end{aligned} \tag{4.1}$$

With respect to the doubly-history dependent term, the integral of strain tensor is approximated by the left endpoint rule, and the integral of the velocity field is approximated by the right endpoint rule. The well-posedness result for Problem 4.1 is as follows.

Theorem 4.2. *Keep the assumptions stated in Lemma 3.2. Then, Problem 4.1 has a unique solution $\boldsymbol{w}^{kh} \subset K^h$.*

Proof. The inequality (4.1) for $n = 0$ is to find $\boldsymbol{w}^{kh}_0 \in K^h$ such that

$$\begin{aligned}
&\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}^{kh}_0), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{w}^{kh}_0)\big)_Q + \int_{\Gamma_{3,2}} \Big[\psi^0_\nu(w^{kh}_{0,\nu}; v^h_\nu - w^{kh}_{0,\nu}) + \psi^0_\tau(\boldsymbol{w}^{kh}_{0,\tau}; \boldsymbol{v}^h_\tau - \boldsymbol{w}^{kh}_{0,\tau})\Big]\, da \\
&\ge \langle \boldsymbol{f}_0, \boldsymbol{v}^h - \boldsymbol{w}^{kh}_0\rangle - \big(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^h_0), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{w}^{kh}_0)\big)_Q \quad \forall\, \boldsymbol{v}^h \in K^h.
\end{aligned} \tag{4.2}$$

Similar to the proof of Lemma 3.2, it can be shown that the inequality (4.2) admits a unique solution $\boldsymbol{w}^{kh}_0$.

For $1 \le n \le N$, assuming that $\{\boldsymbol{w}^{kh}_j\}_{j\le n-1}$ are known, it can be shown similarly that (4.1) has a unique solution $\boldsymbol{w}^{kh}_n$. Thus, through an induction argument, we prove that Problem 4.1 has a unique solution $\boldsymbol{w}^{kh} \subset K^h$. ■

An optimal order error estimate for the numerical solutions of Problem 4.1 is presented in the next result.

Theorem 4.3. *Keep the assumptions stated in Lemma 3.4. Assume the solution regularity $\boldsymbol{w} \in C([0,T]; H^2(\Omega;\mathbb{R}^d))$, $\boldsymbol{w} \in C([0,T]; H^2(\Gamma_{3,1,l};\mathbb{R}^d))$ for $1 \le l \le l_{3,1}$, $\boldsymbol{w} \in C([0,T]; H^2(\Gamma_{3,2,l};\mathbb{R}^d))$ for $1 \le l \le l_{3,2}$, and $\boldsymbol{\sigma} \in C([0,T]; H^1(\Omega;\mathbb{S}^d))$. Moreover, assume $\boldsymbol{u}_0 \in H^2(\Omega;\mathbb{R}^d)$, and let $\boldsymbol{u}^h_0$ be the finite element interpolation or the $L^2(\Omega)$-projection of $\boldsymbol{u}_0$. Then, the following error bound holds for the numerical solution of Problem 4.1:*

$$\max_{0\le n\le N} \|\boldsymbol{w}_n - \boldsymbol{w}^{kh}_n\|_V \le c\,(k + h). \tag{4.3}$$

Proof. We begin with

$$\begin{aligned}
m_{\mathcal{A}}\|\boldsymbol{w}_n - \boldsymbol{w}^{kh}_n\|^2_V &\le \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}^{kh}_n), \boldsymbol{\varepsilon}(\boldsymbol{w}_n) - \boldsymbol{\varepsilon}(\boldsymbol{w}^{kh}_n)\big)_Q \\
&= \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_n), \boldsymbol{\varepsilon}(\boldsymbol{w}_n - \boldsymbol{w}^{kh}_n)\big)_Q + \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}^{kh}_n), \boldsymbol{\varepsilon}(\boldsymbol{w}^{kh}_n - \boldsymbol{v}^h)\big)_Q \\
&\quad + \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}^{kh}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{w}_n)\big)_Q.
\end{aligned} \tag{4.4}$$

In (2.8), let $t = t_n$ and $\boldsymbol{v}^h = \boldsymbol{w}^{kh}_n$,

$$\begin{aligned}
\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_n), \boldsymbol{\varepsilon}(\boldsymbol{w}_n - \boldsymbol{w}^{kh}_n)\big)_Q &\le \big(\mathcal{B}\boldsymbol{\varepsilon}(I\boldsymbol{w}(t_n)) + \int_0^{t_n}\mathcal{R}(t_n - s)\,\boldsymbol{\varepsilon}(I\boldsymbol{w}(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{w}^{kh}_n - \boldsymbol{w}_n)\big)_Q \\
&\quad + \int_{\Gamma_{3,2}} \Big[\psi^0_\nu(w_{n,\nu}; w^{kh}_{n,\nu} - w_{n,\nu}) + \psi^0_\tau(\boldsymbol{w}_{n,\tau}; \boldsymbol{w}^{kh}_{n,\tau} - \boldsymbol{w}_{n,\tau})\Big]\, da \\
&\quad - \langle \boldsymbol{f}_n, \boldsymbol{w}^{kh}_n - \boldsymbol{w}_n\rangle.
\end{aligned} \tag{4.5}$$

Rewrite (4.1) into the following:

$$\begin{aligned}\big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_n^{kh}),\boldsymbol{\varepsilon}(\boldsymbol{w}_n^{kh}-\boldsymbol{v}^h)\big)_Q &\le -\langle \boldsymbol{f}_n,\boldsymbol{v}^h-\boldsymbol{w}_n^{kh}\rangle + \big(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h+k\sum_{j=0}^{n-1}\boldsymbol{w}_j^{kh}),\boldsymbol{\varepsilon}(\boldsymbol{v}^h-\boldsymbol{w}_n^{kh})\big)_Q\\ &\quad+\big(k\sum_{j=0}^{n-1}\mathcal{R}(t_n-t_j)\,\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h+k\sum_{m=1}^{j}\boldsymbol{w}_m^{kh}),\boldsymbol{\varepsilon}(\boldsymbol{v}^h-\boldsymbol{w}_n^{kh})\big)_Q\\ &\quad+\int_{\Gamma_{3,2}}\Big[\psi_\nu^0(w_{n,\nu}^{kh};v_\nu^h-w_{n,\nu}^{kh})+\psi_\tau^0(\boldsymbol{w}_{n,\tau}^{kh};\boldsymbol{v}_\tau^h-\boldsymbol{w}_{n,\tau}^{kh})\Big]\,da. \end{aligned}\tag{4.6}$$

Then, we substitute (4.5) and (4.6) into (4.4), and reform the resulting inequality to get

$$m_{\mathcal{A}}\|\boldsymbol{w}_n-\boldsymbol{w}_n^{kh}\|_V^2 \le \sum_{i=1}^5 I_i + R(\boldsymbol{v}^h,\boldsymbol{w}_n),\tag{4.7}$$

where

$$\begin{aligned}I_1 &= \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_n^{kh})-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_n),\boldsymbol{\varepsilon}(\boldsymbol{v}^h-\boldsymbol{w}_n)\big)_Q,\\ I_2 &= \big(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h+k\sum_{j=0}^{n-1}\boldsymbol{w}_j^{kh})-\mathcal{B}\boldsymbol{\varepsilon}(I\boldsymbol{w}(t_n)),\boldsymbol{\varepsilon}(\boldsymbol{v}^h-\boldsymbol{w}_n^{kh})\big)_Q,\\ I_3 &= \big(k\sum_{j=0}^{n-1}\mathcal{R}(t_n-t_j)\,\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h+k\sum_{m=1}^{j}\boldsymbol{w}_m^{kh})-\int_0^{t_n}\mathcal{R}(t_n-s)\,\boldsymbol{\varepsilon}(I\boldsymbol{w}(s))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v}^h-\boldsymbol{w}_n^{kh})\big)_Q,\\ I_4 &= \int_{\Gamma_{3,2}}\Big[\psi_\nu^0(w_{n,\nu};w_{n,\nu}^{kh}-v_\nu^h)+\psi_\nu^0(w_{n,\nu}^{kh};v_\nu^h-w_{n,\nu}^{kh})\Big]\,da,\\ I_5 &= \int_{\Gamma_{3,2}}\Big[\psi_\tau^0(\boldsymbol{w}_{n,\tau};\boldsymbol{w}_{n,\tau}^{kh}-\boldsymbol{v}_\tau^h)+\psi_\tau^0(\boldsymbol{w}_{n,\tau}^{kh};\boldsymbol{v}_\tau^h-\boldsymbol{w}_{n,\tau}^{kh})\Big]\,da,\end{aligned}$$

and

$$\begin{aligned}R(\boldsymbol{v}^h,\boldsymbol{w}_n) &= \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)+\mathcal{B}\boldsymbol{\varepsilon}(I\boldsymbol{w}(t_n))+\int_0^{t_n}\mathcal{R}(t_n-s)\,\boldsymbol{\varepsilon}(I\boldsymbol{w}(s))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v}^h-\boldsymbol{w}_n)\big)_Q\\ &\quad+\int_{\Gamma_{3,2}}\big[\psi_\nu^0(w_{n,\nu};v_\nu^h-w_{n,\nu})+\psi_\tau^0(\boldsymbol{w}_{n,\tau};\boldsymbol{v}_\tau^h-\boldsymbol{w}_{n,\tau})\big]\,da-\langle\boldsymbol{f}_n,\boldsymbol{v}^h-\boldsymbol{w}_n\rangle.\end{aligned}$$

Here, $R(\boldsymbol{v}^h,\boldsymbol{w}_n)$ denotes a residue term. By (2.8), it is nonnegative. Below, we will use c for a generic constant in places, which depends on $\|\mathcal{R}\|_{C([0,T];Q_\infty)}$, $\|\mathcal{R}'\|_{L^1(0,T;Q_\infty)}$, $\|\boldsymbol{u}_0\|_V$, $\|\boldsymbol{w}\|_{C([0,T];V)}$, $\|\boldsymbol{w}'\|_{L^1(0,T;V)}$, $\|\boldsymbol{\sigma}_n\boldsymbol{\nu}\|_{L^2(\Gamma_3;\mathbb{R}^d)}$, $L_{\mathcal{A}}$, $L_{\mathcal{B}}$, T, $\bar{c}_{0\nu}$, $\bar{c}_{0\tau}$, α_{ψ_ν}, α_{ψ_τ}, and an arbitrary small $\epsilon>0$.

Now we proceed to bound each. By the Cauchy–Schwarz inequality and the Lipschitz continuity of $\mathcal{A}$,

$$I_1 \le L_{\mathcal{A}}\|\boldsymbol{w}_n^{kh}-\boldsymbol{w}_n\|_V\|\boldsymbol{v}^h-\boldsymbol{w}_n\|_V.$$

Applying the modified Cauchy–Schwarz inequality

$$a\,b\le\epsilon\,a^2+\frac{1}{4\,\epsilon}\,b^2\quad\forall\,a,b\in\mathbb{R},\ \forall\,\epsilon>0,$$

we derive that

$$I_1\le\epsilon\,\|\boldsymbol{w}_n-\boldsymbol{w}_n^{kh}\|_V^2+\frac{L_{\mathcal{A}}^2}{4\,\epsilon}\|\boldsymbol{w}_n-\boldsymbol{v}^h\|_V^2.\tag{4.8}$$

On I_2, the error bound

$$\|\int_0^{t_n}\boldsymbol{w}(s)\,ds-k\sum_{j=0}^{n-1}\boldsymbol{w}_j\|_V\le k\,\|\boldsymbol{w}'\|_{L^1(0,T;V)}$$

is satisfied, due to $\boldsymbol{w}\in W^{1,1}(0,T;K)$. Then, an analogous argument with I_1 is utilized to get

$$\begin{aligned}I_2 &\le L_{\mathcal{B}}\,\|\boldsymbol{u}_0^h+k\sum_{j=0}^{n-1}\boldsymbol{w}_j^{kh}-(\boldsymbol{u}_0+\int_0^{t_n}\boldsymbol{w}(s)\,ds)\|_V\|\boldsymbol{v}^h-\boldsymbol{w}_n^{kh}\|_V\\ &\le L_{\mathcal{B}}\,\big(\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V+k\sum_{j=0}^{n-1}\|\boldsymbol{w}_j-\boldsymbol{w}_j^{kh}\|_V+\|\int_0^{t_n}\boldsymbol{w}(s)\,ds-k\sum_{j=0}^{n-1}\boldsymbol{w}_j\|_V\big)\|\boldsymbol{v}^h-\boldsymbol{w}_n^{kh}\|_V\\ &\le L_{\mathcal{B}}\,\big(k\,\|\boldsymbol{w}'\|_{L^1(0,T;V)}+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V+k\sum_{j=0}^{n-1}\|\boldsymbol{w}_j-\boldsymbol{w}_j^{kh}\|_V\big)\|\boldsymbol{v}^h-\boldsymbol{w}_n^{kh}\|_V.\end{aligned}$$

From the triangle inequality, the relation $\|\boldsymbol{v}^h-\boldsymbol{w}_n^{kh}\|_V \le \|\boldsymbol{w}_n-\boldsymbol{w}_n^{kh}\|_V+\|\boldsymbol{w}_n-\boldsymbol{v}^h\|_V$ holds. Then, we use the modified Cauchy–Schwarz inequality to derive that

$$I_2 \le \epsilon\,\|\boldsymbol{w}_n-\boldsymbol{w}_n^{kh}\|_V^2+\|\boldsymbol{w}_n-\boldsymbol{v}^h\|_V^2+c\,\big(k+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V+k\sum_{j=0}^{n-1}\|\boldsymbol{w}_j-\boldsymbol{w}_j^{kh}\|_V\big)^2. \tag{4.9}$$

To analyze I_3, we focus on the errors of the repeated numerical integrals. Observe that

$$\begin{aligned}&\|k\sum_{j=0}^{n-1}\mathcal{R}(t_n-t_j)\,\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h+k\sum_{m=1}^{j}\boldsymbol{w}_m^{kh})-k\sum_{j=0}^{n-1}\mathcal{R}(t_n-t_j)\,\boldsymbol{\varepsilon}(\boldsymbol{u}_0+k\sum_{m=1}^{j}\boldsymbol{w}_m)\|_Q\\&\quad\le T\,\|\mathcal{R}\|_{C([0,T];Q_\infty)}(\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V+k\sum_{j=0}^{n-1}\|\boldsymbol{w}_j-\boldsymbol{w}_j^{kh}\|_V).\end{aligned} \tag{4.10}$$

Moreover, the following bound holds:

$$\begin{aligned}&\|k\sum_{j=0}^{n-1}\mathcal{R}(t_n-t_j)\,\boldsymbol{\varepsilon}(\boldsymbol{u}_0+k\sum_{m=1}^{j}\boldsymbol{w}_m)-\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\mathcal{R}(t_n-t_j)\,\boldsymbol{\varepsilon}(I\boldsymbol{w}(s))\,ds\|_Q\\&\quad\le\|\mathcal{R}\|_{C([0,T];Q_\infty)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\|k\sum_{m=1}^{j}\boldsymbol{w}_m-\int_0^s\boldsymbol{w}(z)\,dz\|_V\,ds\\&\quad\le k\,T\,\|\mathcal{R}\|_{C([0,T];Q_\infty)}\|\boldsymbol{w}'\|_{L^1(0,T;V)}.\end{aligned} \tag{4.11}$$

To proceed further, $H_b(\mathcal{R})$ is used to get

$$\begin{aligned}&\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\mathcal{R}(t_n-t_j)\,\boldsymbol{\varepsilon}(I\boldsymbol{w}(s))\,ds-\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\mathcal{R}(t_n-s)\,\boldsymbol{\varepsilon}(I\boldsymbol{w}(s))\,ds\|_Q\\&\quad=\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\big(\int_{t_n-s}^{t_n-t_j}\mathcal{R}'(z)\,dz\,\boldsymbol{\varepsilon}(I\boldsymbol{w}(s))\big)\,ds\|_Q\\&\quad\le k\,\|\mathcal{R}'\|_{L^1(0,T;Q_\infty)}(\|\boldsymbol{u}_0\|_V+T\|\boldsymbol{w}\|_{C([0,T];V)}).\end{aligned} \tag{4.12}$$

Together with (4.10)–(4.12), we obtain

$$\begin{aligned}I_3&\le c\,\big(k+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V+k\sum_{j=0}^{n-1}\|\boldsymbol{w}_j-\boldsymbol{w}_j^{kh}\|_V\big)(\|\boldsymbol{w}_n-\boldsymbol{w}_n^{kh}\|_V+\|\boldsymbol{w}_n-\boldsymbol{v}^h\|_V)\\&\le\epsilon\,\|\boldsymbol{w}_n-\boldsymbol{w}_n^{kh}\|_V^2+\|\boldsymbol{w}_n-\boldsymbol{v}^h\|_V^2+c\,\big(k+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V+k\sum_{j=0}^{n-1}\|\boldsymbol{w}_j-\boldsymbol{w}_j^{kh}\|_V\big)^2.\end{aligned} \tag{4.13}$$

In the estimation of I_4, the sub-additivity of Clarke directional derivative [8] is used to get

$$\int_{\Gamma_{3,2}}\psi_\nu^0(w_{n,\nu};w_{n,\nu}^{kh}-v_\nu^h)\,da\le\int_{\Gamma_{3,2}}\Big[\psi_\nu^0(w_{n,\nu};w_{n,\nu}^{kh}-w_{n,\nu})+\psi_\nu^0(w_{n,\nu};w_{n,\nu}-v_\nu^h)\Big]\,da,$$

and

$$\int_{\Gamma_{3,2}}\psi_\nu^0(w_{n,\nu}^{kh};v_\nu^h-w_{n,\nu}^{kh})\,da\le\int_{\Gamma_{3,2}}\Big[\psi_\nu^0(w_{n,\nu}^{kh};v_\nu^h-w_{n,\nu})+\psi_\nu^0(w_{n,\nu}^{kh};w_{n,\nu}-w_{n,\nu}^{kh})\Big]\,da.$$

Add the inequalities and use $H(\psi_\nu)$ (b), (c) to derive

$$I_4\le\int_{\Gamma_{3,2}}(\alpha_{\psi_\nu}|w_{n,\nu}-w_{n,\nu}^{kh}|^2+2\bar{c}_{0\nu}|w_{n,\nu}-v_\nu^h|)\,da.$$

Then, we apply Hölder's inequality to obtain

$$I_4\le\alpha_{\psi_\nu}\lambda_\nu^{-1}\,\|\boldsymbol{w}_n-\boldsymbol{w}_n^{kh}\|_V^2+c\,\|w_{n,\nu}-v_\nu^h\|_{L^2(\Gamma_{3,2})}. \tag{4.14}$$

By using an analogous approach of (4.14),

$$I_5\le\alpha_{\psi_\tau}\lambda_\tau^{-1}\,\|\boldsymbol{w}_n-\boldsymbol{w}_n^{kh}\|_V^2+c\,\|\boldsymbol{w}_{n,\tau}-\boldsymbol{v}_\tau^h\|_{L^2(\Gamma_{3,2};\mathbb{R}^d)}. \tag{4.15}$$

To bound the residual term $R(\boldsymbol{v}^h,\boldsymbol{w}_n)$, we deduce point-wise equations by (2.8). A function space $\widetilde{U}$ is defined by

$$\widetilde{U}=\{\boldsymbol{v}\in C^\infty(\overline{\Omega};\mathbb{R}^d)\mid\boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_1\cup\Gamma_3\}.$$

Let

$$\boldsymbol{\sigma}(t)=\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}(t))+\mathcal{B}\boldsymbol{\varepsilon}(I\boldsymbol{w}(t))+\int_0^t\mathcal{R}(t-s)\,\boldsymbol{\varepsilon}(I\boldsymbol{w}(s))\,ds\text{ in }\Omega.$$

Taking $\boldsymbol{v} = \boldsymbol{w}(t) \pm \widetilde{\boldsymbol{v}}$ in (2.8), where $\widetilde{\boldsymbol{v}} \in \widetilde{U}$,

$$\left(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\pm\widetilde{\boldsymbol{v}})\right)_Q + \int_{\Gamma_{3,2}} \left[\psi_\nu^0(w_\nu(t); \pm\widetilde{v}_\nu) + \psi_\tau^0(\boldsymbol{w}_\tau(t); \pm\widetilde{\boldsymbol{v}}_\tau)\right] da \geq \langle \boldsymbol{f}(t), \pm\widetilde{\boldsymbol{v}}\rangle.$$

Thus, $\left(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{v}})\right)_Q = \langle \boldsymbol{f}(t), \widetilde{\boldsymbol{v}}\rangle$ is derived, due to $\widetilde{\boldsymbol{v}} = \boldsymbol{0}$ on $\Gamma_{3,2}$. From Green's formula

$$\left(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{v}})\right)_Q + \langle \mathrm{Div}\, \boldsymbol{\sigma}(t), \widetilde{\boldsymbol{v}}\rangle = \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \widetilde{\boldsymbol{v}}\, da,$$

we have

$$\langle \mathrm{Div}\, \boldsymbol{\sigma}(t), \widetilde{\boldsymbol{v}}\rangle + \left(\boldsymbol{f}_0(t), \widetilde{\boldsymbol{v}}\right)_{L^2(\Omega;\mathbb{R}^d)} = \int_{\Gamma_2} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \widetilde{\boldsymbol{v}}\, da - \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \widetilde{\boldsymbol{v}}\, da.$$

Since $\widetilde{\boldsymbol{v}}$ is arbitrary, we utilize the technique in [36, Section 8.1] to get

$$\mathrm{Div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{a.e. in } \Omega, \tag{4.16}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{a.e. on } \Gamma_2. \tag{4.17}$$

By taking the inner product of (4.16) and $\boldsymbol{v}^h - \boldsymbol{w}_n$, integrating over the domain Ω, and setting $t = t_n$, we obtain

$$\int_\Omega \mathrm{Div}\, \boldsymbol{\sigma}_n \cdot (\boldsymbol{v}^h - \boldsymbol{w}_n)\, dx = -\int_\Omega \boldsymbol{f}_{0n} \cdot (\boldsymbol{v}^h - \boldsymbol{w}_n)\, dx.$$

Then, Green's formula and (4.17) are applied to derive

$$\begin{aligned}&\int_\Omega \boldsymbol{\sigma}_n \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{w}_n)\, dx - \int_\Omega \boldsymbol{f}_{0n} \cdot (\boldsymbol{v}^h - \boldsymbol{w}_n)\, dx - \int_{\Gamma_2} \boldsymbol{f}_{2n} \cdot (\boldsymbol{v}^h - \boldsymbol{w}_n)\, da\\ &\quad = \int_{\Gamma_3} \boldsymbol{\sigma}_n \boldsymbol{\nu} \cdot (\boldsymbol{v}^h - \boldsymbol{w}_n)\, da.\end{aligned}$$

Hence, the residual term is simplified to that

$$R(\boldsymbol{v}^h, \boldsymbol{w}_n) = \int_{\Gamma_3} \boldsymbol{\sigma}_n \boldsymbol{\nu} \cdot (\boldsymbol{v}^h - \boldsymbol{w}_n)\, da + \int_{\Gamma_{3,2}} \left[\psi_\nu^0(w_{n,\nu}; v_\nu^h - w_{n,\nu}) + \psi_\tau^0(\boldsymbol{w}_{n,\tau}; \boldsymbol{v}_\tau^h - \boldsymbol{w}_{n,\tau})\right] da.$$

Based on the solution regularity $\boldsymbol{\sigma} \in C([0,T]; H^1(\Omega;\mathbb{S}^d))$, $H(\psi_\nu)$(b) and $H(\psi_\tau)$(b), we use Hölder's inequality to get

$$R(\boldsymbol{v}^h, \boldsymbol{w}_n) \leq c\, \|\boldsymbol{w}_n - \boldsymbol{v}^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}. \tag{4.18}$$

Use the bounds (4.8), (4.9), (4.13), (4.14), (4.15) and (4.18) in (4.7) to get

$$\begin{aligned}m_{\mathcal{A}} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{kh}\|_V^2 &\leq (\alpha_{\psi_\nu} \lambda_\nu^{-1} + \alpha_{\psi_\tau} \lambda_\tau^{-1} + 3\epsilon)\, \|\boldsymbol{w}_n - \boldsymbol{w}_n^{kh}\|_V^2 + (2 + \frac{L_{\mathcal{A}}^2}{4\epsilon})\, \|\boldsymbol{w}_n - \boldsymbol{v}^h\|_V^2\\ &\quad + c\, \|\boldsymbol{w}_n - \boldsymbol{v}^h\|_{L^2(\Gamma_3;\mathbb{R}^d)} + c\left(k + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + k \sum_{j=0}^{n-1} \|\boldsymbol{w}_j - \boldsymbol{w}_j^{kh}\|_V\right)^2.\end{aligned} \tag{4.19}$$

By $H(s)$, $m_{\mathcal{A}} - (\alpha_{\psi_\nu} \lambda_\nu^{-1} + \alpha_{\psi_\tau} \lambda_\tau^{-1}) - 3\epsilon > 0$ is satisfied for sufficiently small $\epsilon > 0$. Rearrange the terms of (4.19) and take the square root of its both sides to derive

$$\begin{aligned}\|\boldsymbol{w}_n - \boldsymbol{w}_n^{kh}\|_V &\leq c\left(\|\boldsymbol{w}_n - \boldsymbol{v}^h\|_V + \|\boldsymbol{w}_n - \boldsymbol{v}^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}^{\frac{1}{2}} + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + k\right)\\ &\quad + c\, k \sum_{j=0}^{n-1} \|\boldsymbol{w}_j - \boldsymbol{w}_j^{kh}\|_V.\end{aligned} \tag{4.20}$$

Thus, we apply a discrete Gronwall's inequality ([36, Lemma 7.25]) on (4.20) to get

$$\begin{aligned}\max_{0\leq n\leq N} \|\boldsymbol{w}_n - \boldsymbol{w}_n^{kh}\|_V &\leq c \max_{0\leq n\leq N} \left\{ \inf_{\boldsymbol{v}^h \in K^h} \left(\|\boldsymbol{w}_n - \boldsymbol{v}^h\|_V + \|\boldsymbol{w}_n - \boldsymbol{v}^h\|_{L^2(\Gamma_3;\mathbb{R}^d)}^{1/2}\right)\right\}\\ &\quad + c\,(\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + k).\end{aligned} \tag{4.21}$$

Based on the standard finite element interpolation error estimates (cf. [37,38]), $\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \leq c\, h$, and with $\boldsymbol{v}^h$ the linear element interplant of $\boldsymbol{w}_n$,

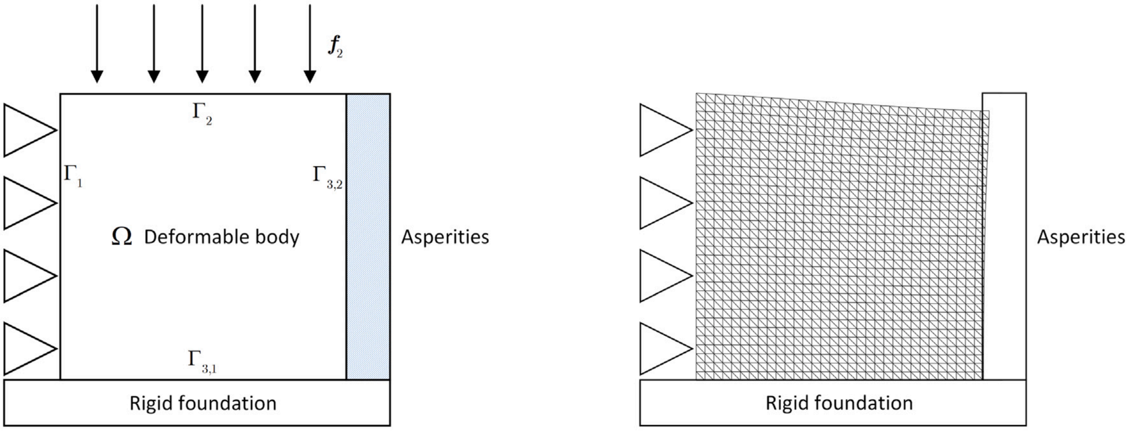

Fig. 1. The contact model (left) and the configuration of Ω (right).

$$\|\boldsymbol{w}_n - \boldsymbol{v}^h\|_V + \|\boldsymbol{w}_n - \boldsymbol{v}^h\|^{1/2}_{L^2(\Gamma_3;\mathbb{R}^d)} \le c\, h$$

under the stated solution regularity assumptions. Then, we derive (4.3) from (4.21). ■

5. A numerical example

In this section, we present a numerical experiment for the viscoelastic contact problem. As depicted in Fig. 1 (left), Ω denotes the cross section of the deformable body, a rigid foundation is located horizontally, and a layer of elastic asperities is laid vertically. Let $\Omega = (0, L_1) \times (0, L_2)$, and the whole boundary Γ is divided as follows:

$$\Gamma_1 = \{0\} \times \left(0, L_2\right), \Gamma_2 = \left[0, L_1\right] \times \{L_2\},\ \Gamma_{3,1} = \left[0, L_1\right] \times \{0\},\ \Gamma_{3,2} = \{L_1\} \times \left(0, L_2\right).$$

The body is fixed on Γ_1, and surface traction of a total density $\boldsymbol{f}_2$ acts on Γ_2. In the constitutive law, the viscosity tensor $\mathcal{A}$ and the elasticity tensor $\mathcal{B}$ are defined by

$$(\mathcal{A}\tau)_{ij} = 2\theta\tau_{ij} + \zeta(\tau_{11} + \tau_{22})\delta_{ij},\quad 1 \le i, j \le 2,$$
$$(\mathcal{B}\tau)_{ij} = \frac{E}{1+\kappa}\tau_{ij} + \frac{E\kappa}{1-\kappa^2}(\tau_{11} + \tau_{22})\delta_{ij},\ 1 \le i, j \le 2,$$

respectively, where $\theta > 0$ and $\zeta \ge 0$ are the viscosity coefficients, δ_{ij} denotes the Kronecker symbol, E means the Young modulus, and κ represents the Poisson ratio of the material. The relaxation tensor $\mathcal{R}(s) = e^{-s}\mathbb{I}$, where $\mathbb{I}$ is an identity matrix. On $\Gamma_{3,2}$, the normal damped response condition is expressed as

$$-\sigma_\nu(u'_\nu) = \begin{cases} 0, & u'_\nu \le 0, \\ 0.01u'_\nu, & 0 < u'_\nu \le 0.02, \\ 0.0002 - 0.01(u'_\nu - 0.02), & 0.02 < u'_\nu \le 0.03, \\ 0.0001 - 0.01(u'_\nu - 0.03), & u'_\nu > 0.03, \end{cases}$$

and the friction law is described as

$$|\boldsymbol{\sigma}_\tau| \le 0.04F_b \text{ if } \boldsymbol{u}'_\tau = \boldsymbol{0},\quad -\boldsymbol{\sigma}_\tau = 0.02(e^{-|\boldsymbol{u}'_\tau|} + 1)F_b\frac{\boldsymbol{u}'_\tau}{|\boldsymbol{u}'_\tau|} \text{ if } \boldsymbol{u}'_\tau \ne \boldsymbol{0}.$$

where F_b denotes the friction bound.

The domain Ω is divided uniformly, triangular finite element partitions are applied, and continuous linear finite element spaces are used for computation. The following parameters are used in the experiment:

$$L_1 = L_2 = 1 \text{ m},\quad \theta = 0.5,\quad \zeta = 0.5,\quad E = 10 \text{ GN/m}^2,\quad \kappa = 0.3,\quad F_b = 1,\quad T = 1 \text{ s},$$
$$\boldsymbol{f}_2 = \left(0, -0.6(e^t - 1)x\right) \text{ GN/m on } \Gamma_2,\quad \boldsymbol{f}_0 = \boldsymbol{0}\ in\ \Omega,\quad \boldsymbol{u}_0 = \boldsymbol{0} \text{ in } \Omega.$$

The graphic of the deformable body Ω at $t = 1$ with $k = 1/512$ and $h = 1/32$ is presented in Fig. 1 (right). The relative errors are calculated by $\|\boldsymbol{w}_N^{kh} - \boldsymbol{w}_N\|_V / \|\boldsymbol{w}_N\|_V$, where $\boldsymbol{w}_N$ is the reference solution with $k = 1/512$ and $h = 1/256$. Finally, convergence orders of the numerical solutions for fixed spatial mesh-size and fixed temporal step-size are reported in Tables 1 and 2, respectively. The data shows an optimal first order of the numerical solutions, which is predicted by the theoretical analysis.

Table 1
Convergence orders with fixed spatial mesh-size.

h	k	Error	Order
1/256	1/8	0.054171	–
1/256	1/16	0.026299	1.0425
1/256	1/32	0.012671	1.0535
1/256	1/64	0.005905	1.1014
1/256	1/128	0.002532	1.2220

Table 2
Convergence orders with fixed temporal step-size.

k	h	Error	Order
1/512	1/4	0.263533	–
1/512	1/8	0.146693	0.8452
1/512	1/16	0.078339	0.9050
1/512	1/32	0.040600	0.9483
1/512	1/64	0.020115	1.0132

Data availability

Data will be made available on request.

References

[1] D. Motreanu, P.D. Panagiotopoulos, Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities, Kluwer Academic Publishers, Berlin, 1999.
[2] Z. Naniewicz, P.D. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications, Dekker, New York, 1995.
[3] P.D. Panagiotopoulos, Hemivariational Inequalities, Applications in Mechanics and Engineering, Springer-Verlag, Berlin, 1993.
[4] S. Carl, V.K. Le, Multi-Valued Variational Inequalities and Inclusions, Springer, New York, 2021.
[5] S. Carl, V.K. Le, D. Motreanu, Nonsmooth Variational Problems and their Inequalities: Comparison Principles and Applications, Springer, New York, 2007.
[6] D. Goeleven, D. Motreanu, Variational and Hemivariational Inequalities: Theory, Methods and Applications. Vol. I. Unilateral Analysis and Unilateral Mechanics, Kluwer Academic Publishers, Boston, MA, 2003.
[7] D. Goeleven, D. Motreanu, Y. Dumont, M. Rochdi, Variational and Hemivariational Inequalities: Theory, Methods and Applications. Vol. II. Unilateral Problems, Kluwer Academic Publishers, Boston, MA, 2003.
[8] S. Migórski, A. Ochal, M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Springer, New York, 2013.
[9] M. Sofonea, S. Migórski, Variational–Hemivariational Inequalities with Applications, CRC Press, Boca Raton, FL, 2018.
[10] W. Han, Minimization principles for elliptic hemivariational inequalities, Nonlinear Anal. Real World Appl. 54 (2020) 103114.
[11] W. Han, A revisit of elliptic variational–hemivariational inequalities, Numer. Funct. Anal. Optim. 42 (2021) 371–395.
[12] W. Han, A. Matei, Minimax principles for elliptic mixed hemivariational-variational inequalities, Nonlinear Anal. Real World Appl. 64 (2022) 103448.
[13] W. Han, A. Matei, Well-posedness of a general class of elliptic mixed hemivariational-variational inequalities, Nonlinear Anal. Real World Appl. 66 (2022) 103553.
[14] W. Han, An Introduction To Theory and Applications of Stationary Variational–Hemivariational Inequalities, Springer, New York, 2024.
[15] J. Haslinger, M. Miettinen, P.D. Panagiotopoulos, Finite Element Method for Hemivariational Inequalities: Theory, Methods and Applications, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999.
[16] W. Han, M. Sofonea, Numerical analysis of hemivariational inequalities in contact mechanics, Acta Numer. 28 (2019) 175–286.
[17] W. Han, F. Feng, F. Wang, J. Huang, Numerical analysis of hemivariational inequalities with applications in contact mechanics, Adv. Appl. Mech. 60 (2025) 113–178.
[18] M. Sofonea, A. Matei, History-dependent quasivariational inequalities arising in contact mechanics, European J. Appl. Math. 22 (2011) 471–491.
[19] X. Cheng, S. Migórski, A. Ochal, M. Sofonea, Analysis of two quasistatic history-dependent contact models, Discret. Contin. Dyn. Systerms Ser. B 19 (8) (2014) 2425–2445.
[20] S. Migórski, A. Ochal, M. Sofonea, History-dependent variational–hemivariational inequalities in contact mechanics, Nonlinear Anal. Real Word Appl. 22 (2015) 604–618.
[21] S. Migórski, M. Sofonea, S. Zeng, Well-posedness of history-dependent sweeping processes, SIAM J. Math. Anal. 51 (2) (2019) 1082–1107.
[22] M. Sofonea, S. Migórski, A class of history-dependent variational-hemi-va-ria-tional inequalities, Nonlinear Differ. Equ. Appl. 23 (3) (2016) 1–23.
[23] W. Xu, W. Han, T. Li, Z. Huang, Well-posedness of a class of evolutionary variational–hemivariational inequalities in contact mechanics, J. Comput. Appl. Math. 459 (2025) 116366.
[24] K. Kazmi, M. Barboteu, W. Han, M. Sofonea, Numerical analysis of history-dependent quasivariational inequalities with applications in contact mechanics, ESAIM Math. Model. Numer. Anal. 48 (2014) 919–942.
[25] M. Sofonea, W. Han, S. Migórski, Numerical analysis of history-dependent variational–hemivariational inequalities with applications to contact problems, European J. Appl. Math. 26 (2015) 427–452.
[26] S. Wang, W. Xu, W. Han, W. Chen, Numerical analysis of history-dependent variational–hemivariational inequalities, Sci. China Math. 63 (2020) 2207–2232.
[27] W. Xu, Z. Huang, W. Han, W. Chen, C. Wang, Numerical analysis of history-dependent variational–hemivariational inequalities with applications in contact mechanics, J. Comput. Appl. Math. 351 (2019) 364–377.
[28] W. Xu, Z. Huang, W. Han, W. Chen, C. Wang, Numerical analysis of history-dependent hemivariational inequalities and applications to viscoelastic contact problems with normal penetration, Comput. Math. Appl. 77 (2019) 2596–2607.
[29] W. Xu, C. Wang, M. He, W. Chen, W. Han, Z. Huang, Numerical analysis of doubly-history dependent variational inequalities in contact mechanics, Fixed Point Theory Algorithms Sci. Eng. (2021) 2021:24.

[30] F.H. Clarke, Generalized gradients and applications, Trans. Amer. Math. Soc. 205 (1975) 247–262.
[31] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[32] C. Eck, J. Jarusek, M. Krbec, Unilateral Contact Problems: Variational Methods and Existence Theorems, Chapman/CRC Press, New York, 2005.
[33] J. Nečas, I. Hlavaček, Mathematical Theory of Elastic and Elastoplastic Bodies: An Introduction, Elsevier, Amsterdam, 1981.
[34] W. Han, Z. Huang, C. Wang, W. Xu, Numerical analysis of elliptic hemivariational inequalities for semipermeable media, J. Comput. Math. 37 (2019) 543–560.
[35] W. Han, M. Sofonea, M. Barboteu, Numerical analysis of elliptic hemivariational inequalities, SIAM J. Numer. Anal. 55 (2) (2017) 640–663.
[36] W. Han, M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, in: Studies in Advanced Mathematics, vol. 30, Americal Mathematical Society, Providence, RI–International Press, Somerville, MA, 2002.
[37] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North Holland, Amsterdam, 1978.
[38] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, third ed., Springer-Verlag, New York, 2008.