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# Well-posedness of viscoelastic contact problems with modified Signorini, Tresca-friction, and Clarke-subdifferential type contact conditions incorporating both velocity and displacement

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**Abstract**
We propose three modified contact boundary conditions incorporating both the velocity and the displacement with a parameter $\delta$ for the viscoelastic problem. As $\delta$ approaches 0, these conditions formally reduce to the conventional Signorini, Tresca-friction, and Clarke-subdifferential type boundary conditions, respectively. Consequently, the modified conditions, as a generalization of the conventional ones, can be viewed as contact conditions in the displacement with a dynamic setting. We derive weak formulations for the viscoelastic contact model under three modified contact conditions and explore their well-posedness. Additionally, we provide bounds on the weak solutions with respect to the parameter $\delta$.

**Keywords** Viscoelastic contact problem · Signorini condition · Tresca-friction · Clarke-subdifferential · (hemi-)variational inequality

**Mathematics Subject Classification** 49J40 · 70E18 · 74M15 · 74D99

## 1 Introduction

Various contact boundary conditions, such as the Signorini-type, friction-type, and Clarke-subdifferential-type conditions, have been proposed for the elastic and viscoelastic contact problems, which have a wide range of applications in mathematical modeling in the materials science and engineering simulation and have drawn considerable attention from numerous scholars in mathematical and numerical analysis [1–5]. Such boundary conditions describe various contact mechanics and are crucial for mathematical modeling of the complex dynamics of (visco)elastic body and fluid

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flow interacting with the boundary [6–12]. In a typical contact problem (Fig. 1), we are interested in the deformation and/or stress distribution in a deformable body subject to the action of body forces and surface tractions on a part of the boundary $\Gamma_N$, a displacement constraint on another part of the boundary $\Gamma_D$, and contact conditions on the remaining part of the boundary $\Gamma_C$, called the contact boundary. Numerous studies exist on the well-posedness and numerical methods for contact problems with different contact boundary conditions.

For convenience, we use "SC", "TC" and "CS" to refer to contact boundary conditions of the Signorini-type, friction-type, and Clarke-subdifferential-type. Moreover, we use the labels (SCD), (TCD), (CSD) for contact conditions expressed on the displacement $\boldsymbol{u}$ in studies of stationary/quasi-static/nonstationary contact problems, and use the labels (SCV), (TCV), (CSV) for contact conditions expressed on the velocity $\boldsymbol{u}'$ in studies of quasi-static/nonstationary contact problems. We label the contact conditions in our models by (SDV$_\delta$), (TDV$_\delta$), (CDV$_\delta$) for contact conditions expressed on $\boldsymbol{u} + \delta\boldsymbol{u}'$ in studies of quasi-static/nonstationary contact problems. Formally, we recover (SCD), (TCD), (CSD) from (SDV$_\delta$), (TDV$_\delta$), (CDV$_\delta$) as $\delta \to 0$, and recover (SCV), (TCV), (CSV) from (SDV$_\delta$), (TDV$_\delta$), (CDV$_\delta$) as $\delta \to \infty$.

The Signorini condition is commonly used to describe the non-penetration phenomenon in contact mechanics [8, 13]. For the unilateral condition, the normal components of the traction vector $\boldsymbol{\sigma n}$ and the displacement $\boldsymbol{u}$ are required to satisfy the following condition (called the Signorini condition in displacement):

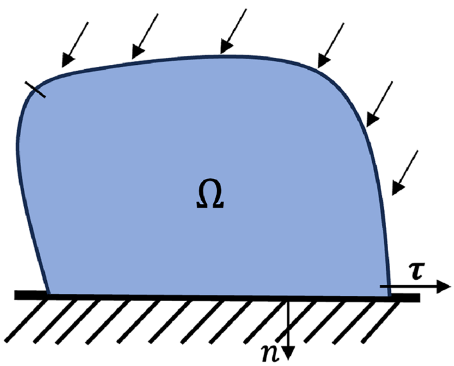
$$\boldsymbol{\sigma n} \cdot \boldsymbol{n} \leq 0, \quad \boldsymbol{u} \cdot \boldsymbol{n} \leq 0, \quad (\boldsymbol{\sigma n} \cdot \boldsymbol{n})(\boldsymbol{u} \cdot \boldsymbol{n}) = 0, \tag{SCD}$$

where $\boldsymbol{n}$ is the unit normal outward vector, and $\boldsymbol{u} \cdot \boldsymbol{n} \leq 0$ means no penetration occurs. The unilateral condition (SCD) is usually used in static contact problems. For dynamic contact problems or quasi-static contact problems, a different version of the Signorini condition (called the Signorini condition on the velocity) has also been proposed:

$$\boldsymbol{\sigma n} \cdot \boldsymbol{n} \leq 0, \quad \boldsymbol{u}' \cdot \boldsymbol{n} \leq 0, \quad (\boldsymbol{\sigma n} \cdot \boldsymbol{n})(\boldsymbol{u}' \cdot \boldsymbol{n}) = 0, \tag{SCV}$$

where $\boldsymbol{u}'$ is the velocity.

**Fig. 1** contact model



For some physicists and engineers, (SCD) appears both natural and accurate, as it effectively encapsulates the non-penetration constraint. However, from a mathematical perspective, the unique solvability of dynamic or quasi-static elastic and viscoelastic contact problems that incorporate the condition (SCD) remains inadequately examined. The well-posedness of the static elastic problem with the (SCD) condition has been established [8]. Additionally, the well-posedness of the wave equation under the (SCD) condition in a half-space has been demonstrated [14]. In more general domains, while existence has been confirmed [15], the aspect of uniqueness is still unresolved. Extending these results to dynamic and quasi-static elasticity equations is an ongoing challenge. Regarding the viscoelastic problem associated with (SCD), the existence of a weak solution has been validated in [16] using a penalty approach. Moreover, well-posedness of dynamic elastic and viscoelastic contact problems with the (SCV) condition has been established in [7, 17].

In [18], a modified Signorini condition was proposed for the elastic problem, which involves both displacement and velocity with a parameter $\delta > 0$:

$$\boldsymbol{\sigma n}\cdot\boldsymbol{n} \le 0, \quad (\boldsymbol{u}+\delta\boldsymbol{u}')\cdot\boldsymbol{n} \le 0, \quad (\boldsymbol{\sigma n}\cdot\boldsymbol{n})((\boldsymbol{u}+\delta\boldsymbol{u}')\cdot\boldsymbol{n}) = 0. \tag{SDV$_\delta$}$$

The inclusion of both displacement and velocity in the sum (i.e., $\boldsymbol{u}+\delta\boldsymbol{u}'$) may appear somewhat peculiar. However, one can think of the parameter $\delta > 0$ as a physical quantity measuring the time effect on the displacement. Notably, a similar issue regarding physical units arises in the variational form for the quasi-static or dynamic elastic contact problems under the Tresca/Coulomb-friction condition (see [7, Page 205] for clarification). The existence of a strong solution for the elastic problem under (SDV$_\delta$) has been confirmed in [18] using regularization/penalty methods alongside Galerkin's approach. As $\delta$ approaches 0, (SDV$_\delta$) formally reduces to (SCD), while (SCV) can be formally derived from (SDV$_\delta$) by considering the limit as $\delta \to \infty$, or by substituting $\boldsymbol{u}+\delta\boldsymbol{u}'$ with $\delta\boldsymbol{u}+\boldsymbol{u}'$ and allowing $\delta \to 0$. It is important to note that a rigorous mathematical justification for the limit as $\delta \to 0$ has not yet been established.

We now shift our focus to the interaction involving friction. The Tresca-friction condition, which functions as a simplified form of the Coulomb-friction law, is frequently utilized to model contact mechanics on rough surfaces. For static contact problems, the Tresca friction condition in displacement is given by

$$|\boldsymbol{\sigma}_\tau| \le g, \quad \boldsymbol{\sigma}_\tau\cdot\boldsymbol{u}_\tau + g|\boldsymbol{u}_\tau| = 0, \tag{TCD}$$

where $\boldsymbol{\sigma}_\tau$ and $\boldsymbol{u}_\tau$ represent the tangential component of the traction and the displacement, respectively, and $g > 0$ is the maximum static friction. For dynamic or the quasi-static elastic/viscoelastic contact problem, it seems more popular to impose the Tresca-friction condition in terms of the tangential velocity [6, 7],

$$|\boldsymbol{\sigma}_\tau| \le g, \quad \boldsymbol{\sigma}_\tau\cdot\boldsymbol{u}'_\tau + g|\boldsymbol{u}'_\tau| = 0, \tag{TCV}$$

where $\boldsymbol{u}'_\tau$ is the tangential component of the velocity. Well-posedness of the static elastic contact problems defined by (TCD) has been established [8]. In the case of a linearized elastodynamic body under the (TCV) condition, the proof of well-posedness is provided in [6, 19]. Nonetheless, the existence of a solution for the dynamic problems with (TCD) remains an open problem, even in the viscoelastic case [7]. Moreover, to the authors' knowledge, (TCD) is seldomly considered in the context of quasi-static contact problems. For further exploration of the mathematical and numerical analysis pertaining to (visco)elastic and fluid problems involving various Tresca-friction conditions, consult [7, 12, 20, 21] and the references cited therein.

We shall now focus on the contact condition of the Clarke-subdifferential type in the normal direction. This condition has been proposed to model a more general and complex penetration phenomenon in contact mechanics [5, 22, 23]. For instance, the condition describes a deformable foundation and assigns a reactive normal pressure depending on the interpenetration of the asperities on the body surface and the foundation (the Clarke normal compliance):

$$-\boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{n} \in \partial j_n(\boldsymbol{u}\cdot\boldsymbol{n}), \tag{CSD}$$

where $j_n$ is a nonsmooth and nonconvex function and $\partial j_n$ denotes the Clarke subdifferential of $j_n$. When modeling contact with a lubricated foundation (such as oil), the reactive normal pressure depends on the normal velocity on the contact surface, leading to the following condition (the Clarke normal damped response):

$$-\boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{n} \in \partial j_n(\boldsymbol{u}'\cdot\boldsymbol{n}). \tag{CSV}$$

For elastic and viscoelastic contact problems with (CSD), well-posedness theories have been explored in [23, 24]. The unique solvability for the viscoelastic contact problem characterized by (CSV) has been investigated in [25].

As previously indicated, various types of contact boundary conditions have been designed to model different contact mechanisms in the literature. In certain cases, even for the same contact mechanism involving elastic or viscoelastic bodies, the contact boundary conditions for static, quasi-static, and dynamic scenarios may be formulated in different expressions, depending on whether they pertain to the displacement or the velocity. Is there a single condition that can be regarded as a generalization of the contact conditions on the displacement and the velocity? The modified Signorini condition (SDV$_\delta$) introduced in [18] represents a first attempt to generalize the Signorini condition concerning displacement and velocity for elastic problems. Drawing inspiration from the aforementioned research, the motivation of this paper is to examine viscoelastic contact problems with the modified Signorini boundary condition (SDV$_\delta$), as well as the modified Tresca-friction condition (TDV$_\delta$) and the Clarke-subdifferential-type condition (CDV$_\delta$):

$$|\boldsymbol{\sigma}_\tau| \le g, \quad \boldsymbol{\sigma}_\tau\cdot(\boldsymbol{u}_\tau+\delta\boldsymbol{u}'_\tau)+g|\boldsymbol{u}_\tau+\delta\boldsymbol{u}'_\tau| = 0, \tag{TDV$_\delta$}$$

$$-\boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{n} \in \partial j_n((\boldsymbol{u}+\delta\boldsymbol{u}')\cdot\boldsymbol{n}). \tag{CDV$_\delta$}$$

These modified conditions appear to serve as intermediaries between the corresponding conventional conditions on either the displacement or the velocity. As $\delta$ approaches 0, the modified contact conditions formally reduce to the conventional ones in terms of the displacement, i.e., (SCD), (TCD), and (CSD). In contrast, as $\delta$ approaches $\infty$ (or equivalently, by replacing $\boldsymbol{u} + \delta \boldsymbol{u}'$ with $\delta \boldsymbol{u} + \boldsymbol{u}'$ and allowing $\delta \to 0$), they reduce to the corresponding conventional conditions in terms of the velocity, i.e., (SCV), (TCV), and (CSV). Thus, these modified conditions can be interpreted as a perturbation of the conventional contact conditions. It is believed that investigating these modified contact conditions will contribute to the modeling and mathematical theories of contact mechanisms while also facilitating a better understanding of the relationships and distinctions between conventional contact conditions concerning displacement and velocity. As an initial endeavor, the present work aims to develop the well-posedness theory for the linear viscoelastic problem incorporating three modified contact conditions.

The main results of this paper are summarized as follows. We introduce the transform (3.3) (or, equivalently $\boldsymbol{u}(t) + \delta \boldsymbol{u}'(t) = \boldsymbol{w}(t)$) and $\boldsymbol{z}(t) = \boldsymbol{w}(t) e^{-\frac{t}{\delta}}$ to reformulate the original second-order temporal system of $\boldsymbol{u}$ into a parabolic history-dependent variational problem of $\boldsymbol{z}$. Then, we apply the operator theory to study the well-posedness of the problem instead of using Galerkin's method. We establish the unique existence of the viscoelastic contact problem under the modified Signorini, Tresca-friction, and Clarke-subdifferential type boundary conditions, respectively. Additionally, we bound the solution with respect to the parameter $\delta$.

The rest of this paper is structured as follows. In Section 2, we state the viscoelastic problem with the three modified contact conditions. In Section 3, we present the transformation technique and show the well-posedness of the viscoelastic problem with the modified Signorini contact condition. Section 4 and 5 are devoted to the unique existence of solutions of the viscoelastic contact problems under the modified Tresca-friction and Clarke-subdifferential-type boundary conditions, respectively.

## 2 Viscoelastic contact problems with three modified contact conditions

We introduce the model problem and assumptions in this section. Let $\Omega$ be an open, bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz boundary $\partial\Omega = \Gamma$. The boundary $\Gamma$ is partitioned into three disjoint measurable parts $\Gamma_C, \Gamma_D$ and $\Gamma_N$ with $\text{meas}(\Gamma_D) > 0$ and $\text{meas}(\Gamma_C) > 0$. Let $I = [0, T]$ be a time interval with $0 < T < \infty$. Here and below, we use boldface letters for vectors and tensors. We denote by $\boldsymbol{n}$ the unit normal outward vector to the boundary. The normal and tangential components of a vector $\boldsymbol{w}$ on the boundary are denoted by $w_n = \boldsymbol{w} \cdot \boldsymbol{n}$ and $\boldsymbol{w}_\tau = \boldsymbol{w} - w_n \boldsymbol{n}$, respectively. For a stress tensor $\boldsymbol{\sigma}$, we set $\sigma_n = (\boldsymbol{\sigma}\boldsymbol{n}) \cdot \boldsymbol{n}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{n} - \sigma_n \boldsymbol{n}$. For simplicity, we use $|\cdot|$ to represent the Euclidean norm of $\mathbb{R}$, $\mathbb{R}^d$ and $\mathbb{S}^d$. We use the notation

$$\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top), \quad \nabla \cdot \boldsymbol{\sigma} = (\sigma_{ij,j}), \quad i, j = 1, \cdots, d,$$

to define the linearized strain tensor and the divergence operator, respectively. Note that the summation convention over a repeated index is adopted.

The viscoelastic contact model is stated as follows.

***Problem P*** Find a displacement field $\boldsymbol{u} : \Omega \times I \to \mathbb{R}^d$ such that for a.e. $t \in I$,

$$\boldsymbol{\sigma}(t) = \mathscr{A}\mathbb{D}(\boldsymbol{u}(t)) + \mathscr{B}\mathbb{D}(\boldsymbol{u}'(t)) \qquad \text{in} \quad \Omega, \tag{2.1a}$$

$$\boldsymbol{u}''(t) - \nabla \cdot \boldsymbol{\sigma}(t) = \boldsymbol{f}_0(t) \qquad \text{in} \quad \Omega, \tag{2.1b}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_D, \tag{2.1c}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{n} = \boldsymbol{f}_1(t) \qquad \text{on} \quad \Gamma_N, \tag{2.1d}$$

$$\boldsymbol{u}(0) = \boldsymbol{0}, \quad \boldsymbol{u}'(0) = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{2.1e}$$

and on $\Gamma_C$, one of the three types of contact conditions:

- The modified Signorini condition ($\text{SDV}_\delta$):

$$-\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0}, \sigma_n(t) \le 0,\ u_n(t) + \delta u_n'(t) \le 0,\ \sigma_n(t)(u_n(t) + \delta u_n'(t)) = 0. \tag{2.2}$$

- The modified Tresca-friction condition ($\text{TDV}_\delta$):

$$u_n(t) = 0,\ -\boldsymbol{\sigma}_\tau(t) \in g\partial \left| \boldsymbol{u}_\tau(t) + \delta \boldsymbol{u}_\tau'(t) \right|. \tag{2.3}$$

- The modified contact condition of Clarke-subdifferential-type ($\text{CDV}_\delta$):

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0},\ -\sigma_n(t) \in \partial j_n(u_n(t) + \delta u_n'(t)), \tag{2.4}$$

where $j_n : \Gamma_C \times \mathbb{R} \to \mathbb{R}$.

In the description of Problem P, (2.1a) represents the constitutive law of the viscoelastic body; (2.1b) is the equation of motion, $\boldsymbol{f}_0$ being the density of the body force; (2.1c) specifies the displacement boundary condition on $\Gamma_D$ where for simplicity, the zero boundary value is used without loss of generality; (2.1d) describes the surface traction condition on $\Gamma_N$, $\boldsymbol{f}_1$ being the density of the surface traction; (2.1e) reflects the initial values of the displacement and the velocity, both initial values are taken to be zero without loss of generality. In the contact conditions (2.2)–(2.4), the parameter $\delta \in (0, \infty)$.

**Remark 2.1** *The parameter $\delta$ can be interpreted as indicative of a short time period, ensuring that the physical unit of $u + \delta u'$ remains meaningful. The modified contact conditions can thus be seen as conventional conditions on the displacement that possess a degree of dynamism. As $\delta$ approaches 0, the three modified conditions formally reduce to the contact conditions associated with the displacement. By allowing $\delta \to \infty$, or by substituting $u + \delta u'$ with $u' + \delta u$ and subsequently letting $\delta \to 0$, we formally derive the contact conditions on the velocity. Therefore, the three modified* *contact conditions that encompass both displacement and velocity can be regarded as a generalization of the traditional contact conditions that address either the displacement or the velocity.*

To study Problem P, we need to introduce function spaces, bilinear forms and the assumptions. We set

$$H = L^2(\Omega;\mathbb{R}^d), \quad V = \{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \mathbf{0} \text{ a.e. on } \Gamma_D\}.$$

The inner product and norm of $V$ are defined by

$$(\boldsymbol{v},\boldsymbol{u})_V = \int_\Omega \mathbb{D}(\boldsymbol{v}) : \mathbb{D}(\boldsymbol{u}) dx = (\mathbb{D}(\boldsymbol{v}),\mathbb{D}(\boldsymbol{u})),\ \|\boldsymbol{v}\|_V^2 = (\boldsymbol{v},\boldsymbol{v})_V \quad \forall \boldsymbol{v},\boldsymbol{u} \in V.$$

Since $meas(\Gamma_D) > 0$, it follows from Korn's inequality that $\|\cdot\|_{H^1}$ and $\|\cdot\|_V$ are equivalent norms of $V$. Note that the embedding $V \subset H$ is compact, and $(V, H, V^*)$ constitutes an evolution triple. For brevity, we set $\mathcal{V} = L^2(I;V)$, $\mathcal{V}^* = L^2(I;V^*)$ and $\mathcal{W} = \{\boldsymbol{v} \in \mathcal{V} \mid \boldsymbol{v}' \in \mathcal{V}^*\}$. Note that $\mathcal{W}$ is a separable and reflexive Banach space and its norm is defined through the equality $\|\boldsymbol{v}\|_{\mathcal{W}}^2 = \|\boldsymbol{v}\|_{\mathcal{V}}^2 + \|\boldsymbol{v}'\|_{\mathcal{V}^*}^2$. The distributional derivative operator $L : D(L) \subset \mathcal{V} \to \mathcal{V}^*$ is defined by

$$\langle L\boldsymbol{u},\boldsymbol{v}\rangle = \int_0^T \langle \boldsymbol{u}'(t),\boldsymbol{v}(t)\rangle dt \quad \forall \boldsymbol{v} \in \mathcal{V},$$

where the domain $D(L) = \{\boldsymbol{v} \in \mathcal{W} \mid \boldsymbol{v}(0) = \mathbf{0}\}$. The operator $L$ is linear, densely defined, and maximal monotone.

Next, we state hypotheses on the data of Problem P.

$\underline{H(\mathscr{A})}$: The function $\mathscr{A} = (\mathscr{A}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(i) $\mathscr{A}_{ijkl} \in L^\infty(\Omega)$, $1 \le i,j,k,l \le d$;

(ii) $\mathscr{A}(\boldsymbol{x},\cdot)$ is symmetric, for a.e. $\boldsymbol{x} \in \Omega$, i.e., $\mathscr{A}_{ijkl} = \mathscr{A}_{jikl} = \mathscr{A}_{klij}$, $1 \le i,j,k,l \le d$;

(iii) there exists $m_{\mathscr{A}} > 0$ such that for all $\boldsymbol{\epsilon} \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$,

$$(\mathscr{A}(\boldsymbol{x})\boldsymbol{\epsilon}) : \boldsymbol{\epsilon} \ge m_{\mathscr{A}} |\boldsymbol{\epsilon}|^2 .$$

$\underline{H(\mathscr{B})}$: The function $\mathscr{B} = (\mathscr{B}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(i) $\mathscr{B}_{ijkl} \in L^\infty(\Omega)$, $1 \le i,j,k,l \le d$;

(ii) there exists $m_{\mathscr{B}} > 0$ such that, for all $\boldsymbol{\epsilon} \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$,

$$(\mathscr{B}(\boldsymbol{x})\boldsymbol{\epsilon}) : \boldsymbol{\epsilon} \ge m_{\mathscr{B}} |\boldsymbol{\epsilon}|^2 .$$

$\underline{H(\boldsymbol{f})}$: $\boldsymbol{f}_0 \in C(I; L^2(\Omega;\mathbb{R}^d))$, $\boldsymbol{f}_1 \in C(I; L^2(\Gamma_N;\mathbb{R}^d))$.

We define $\boldsymbol{f} \in C(I; V^*)$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} dx + \int_{\Gamma_N} \boldsymbol{f}_1(t) \cdot \boldsymbol{v} \, d\Gamma \quad \forall \boldsymbol{v} \in V. \tag{2.5}$$

Further, define operators $A : V \to V^*$ and $B : V \to V^*$: for all $\boldsymbol{u} \in V$,

$$\langle A\boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega (\mathscr{A}\mathbb{D}(\boldsymbol{u})) : \mathbb{D}(\boldsymbol{v}) \, dx \quad \forall \boldsymbol{v} \in V; \tag{2.6}$$

$$\langle B\boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega (\mathscr{B}\mathbb{D}(\boldsymbol{u})) : \mathbb{D}(\boldsymbol{v}) \, dx \quad \forall \boldsymbol{v} \in V. \tag{2.7}$$

## 3 The modified Signorini condition

In this section, we consider Problem P with the modified Signorini contact condition (2.2). To state the weak formulation, we set

$$K = \{\boldsymbol{v} \in V \mid v_n \leq 0 \text{ a.e. on } \Gamma_C\},$$

which is a nonempty, closed, convex subset of $V$.

### 3.1 Weak formulation of the problem (2.1) and (2.2)

We derive the weak formulation of Problem P with the modified Signorini contact condition (2.2). Assume that the problem has a sufficiently smooth solution $\boldsymbol{u}$. For any $\boldsymbol{v} \in K$, testing (2.1b) by $\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))$ and using the integration by parts, we have

$$\int_\Omega \boldsymbol{u}''(t) \cdot (\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))dx + \int_\Omega \boldsymbol{\sigma}(t) : \mathbb{D}(\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))dx$$
$$- \int_\Gamma (\boldsymbol{\sigma}\boldsymbol{n})(t) \cdot (\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))d\Gamma = \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))dx.$$

Utilizing (2.1c), (2.1d) and (2.2), we see that

$$\int_\Gamma (\boldsymbol{\sigma}\boldsymbol{n})(t) \cdot (\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))d\Gamma \geq \int_{\Gamma_N} \boldsymbol{f}_1(t) \cdot (\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))d\Gamma.$$

Hence, we get

$$\int_\Omega \boldsymbol{u}''(t) \cdot (\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))dx + \int_\Omega \boldsymbol{\sigma}(t) : \mathbb{D}(\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))dx$$
$$\geq \int_{\Gamma_N} \boldsymbol{f}_1(t) \cdot (\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))d\Gamma + \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))dx.$$

Together with (2.1a), the weak formulation of the problem (2.1) and (2.2) is stated as follows.

**Problem 3.1** Find $\boldsymbol{u} : I \to V$ with $\boldsymbol{u}'' : I \to V^*$, and $\boldsymbol{u} + \delta\boldsymbol{u}' : I \to K$, such that $\boldsymbol{u}(0) = \mathbf{0}, \boldsymbol{u}'(0) = \mathbf{0}$, and for all $\boldsymbol{v} \in K$, a.e. $t \in I$,

$$\begin{aligned}\langle \boldsymbol{u}''(t), \boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))\rangle + \langle A\boldsymbol{u}(t), \boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))\rangle \\ + \langle B\boldsymbol{u}'(t), \boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))\rangle \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))\rangle.\end{aligned} \tag{3.1}$$

## 3.2 Transformations of Problem 3.1

In this part, we perform some transformations (see Fig. 2) on Problem 3.1 to make it easier to establish the well-posedness. By taking $\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t) = \boldsymbol{w}(t)$, $\boldsymbol{z}(t) = \boldsymbol{w}(t)e^{-\frac{t}{\delta}}$ and defining the history-dependent $S$, Problem 3.1 is transformed into Problem 3.2. Then, Problem 3.2 can be transformed into Problem 3.3 by fixing the history-dependent $S$. Next, by introducing the regularization term (i.e., the penalty term), we transform Problem 3.3 into Problem 3.4.

First, we set

$$\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t) = \boldsymbol{w}(t). \tag{3.2}$$

It follows from $\boldsymbol{u}(0) = \mathbf{0}$ and $\boldsymbol{u}'(0) = \mathbf{0}$ that $\boldsymbol{w}(0) = \mathbf{0}$. We see that

$$\boldsymbol{u}(t) = \frac{1}{\delta}e^{-\frac{1}{\delta}t}\int_0^t \boldsymbol{w}(s)e^{\frac{1}{\delta}s}ds. \tag{3.3}$$

Note that (3.3) implies $\boldsymbol{u}(0) = \mathbf{0}$. In addition, we get

$$\boldsymbol{u}'(t) = \frac{1}{\delta}\boldsymbol{w}(t) - \frac{1}{\delta}R\boldsymbol{w}(t), \tag{3.4a}$$

$$\boldsymbol{w}(t) = \delta(R\boldsymbol{w})'(t) + R\boldsymbol{w}(t), \tag{3.4b}$$

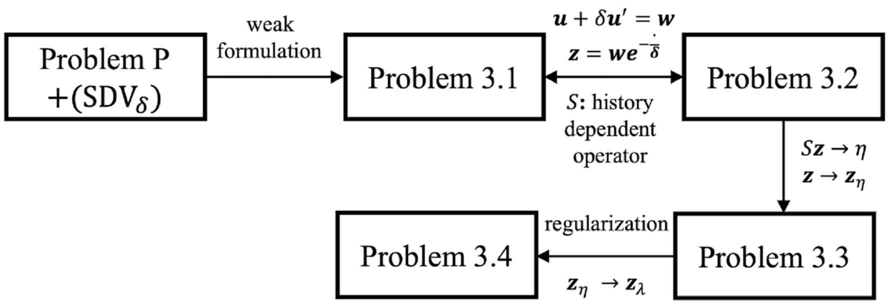



**Fig. 2** transformation process

$$\boldsymbol{u}''(t)=\frac{1}{\delta}\boldsymbol{w}'(t)-\frac{1}{\delta^2}\boldsymbol{w}(t)+\frac{1}{\delta^2}R\boldsymbol{w}(t), \tag{3.4c}$$

where the operator $R:\mathcal{V}\to\mathcal{V}$ is defined by

$$R\boldsymbol{v}(t)=\frac{1}{\delta}e^{-\frac{1}{\delta}t}\int_0^t \boldsymbol{v}(s)e^{\frac{1}{\delta}s}ds \quad \forall\boldsymbol{v}\in\mathcal{V},\ t\in I. \tag{3.5}$$

The operator $R$ is linear and continuous, satisfying

$$\|R\boldsymbol{v}(t)\|_V\le\delta^{-1}e^{-\frac{1}{\delta}t}\int_0^t e^{\frac{1}{\delta}s}\|\boldsymbol{v}(s)\|_V\,ds\le\delta^{-1}\int_0^t\|\boldsymbol{v}(s)\|_V\,ds \quad \forall\boldsymbol{v}\in\mathcal{V}. \tag{3.6}$$

Based on (3.3)–(3.4c), Problem 3.1 is equivalently reformulated as a problem for $\boldsymbol{w}$:

Find $\boldsymbol{w}:I\to K$ with $\boldsymbol{w}':I\to V^*$ such that $\boldsymbol{w}(0)=\mathbf{0}$ and for all $v\in K$, a.e. $t\in I$,

$$\begin{aligned}&\langle\boldsymbol{w}'(t)-\frac{1}{\delta}\boldsymbol{w}(t),\boldsymbol{v}-\boldsymbol{w}(t)\rangle+\langle(\frac{1}{\delta}I-B)R\boldsymbol{w}(t),\boldsymbol{v}-\boldsymbol{w}(t)\rangle\\&\quad+\delta\langle AR\boldsymbol{w}(t),\boldsymbol{v}-\boldsymbol{w}(t)\rangle+\langle B\boldsymbol{w}(t),\boldsymbol{v}-\boldsymbol{w}(t)\rangle\ge\delta\langle\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{w}(t)\rangle.\end{aligned} \tag{3.7}$$

In view of

$$\boldsymbol{w}'(t)-\frac{1}{\delta}\boldsymbol{w}(t)=e^{\frac{t}{\delta}}(\boldsymbol{w}'(t)e^{-\frac{t}{\delta}}-\frac{1}{\delta}\boldsymbol{w}(t)e^{-\frac{t}{\delta}})=e^{\frac{t}{\delta}}(\boldsymbol{w}(t)e^{-\frac{t}{\delta}})',$$

we introduce the transformation

$$\boldsymbol{z}(t)=\boldsymbol{w}(t)e^{-\frac{t}{\delta}} \tag{3.8}$$

and refomulate (3.7) as a problem for $\boldsymbol{z}$:

Find $\boldsymbol{z}:I\to K$ with $\boldsymbol{z}':I\to V^*$, such that $\boldsymbol{z}(0)=\mathbf{0}$ and for all $\boldsymbol{v}\in K$, a.e. $t\in I$,

$$\begin{aligned}&\langle\boldsymbol{z}'(t),\boldsymbol{v}-\boldsymbol{z}(t)\rangle+\langle B\boldsymbol{z}(t),\boldsymbol{v}-\boldsymbol{z}(t)\rangle+e^{-\frac{t}{\delta}}\langle(\frac{1}{\delta}I-B)R(e^{\frac{t}{\delta}}\boldsymbol{z}(t)),\boldsymbol{v}-\boldsymbol{z}(t)\rangle\\&\quad+\delta e^{-\frac{t}{\delta}}\langle AR(e^{\frac{t}{\delta}}\boldsymbol{z}(t)),\boldsymbol{v}-\boldsymbol{z}(t)\rangle\ge\delta e^{-\frac{t}{\delta}}\langle\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{z}(t)\rangle,\end{aligned} \tag{3.9}$$

where we have applied

$$\boldsymbol{w}'(t)-\frac{1}{\delta}\boldsymbol{w}(t)=e^{\frac{t}{\delta}}\boldsymbol{z}'(t). \tag{3.10}$$

We define the operator $S:\mathcal{V}\to\mathcal{V}^*$ by

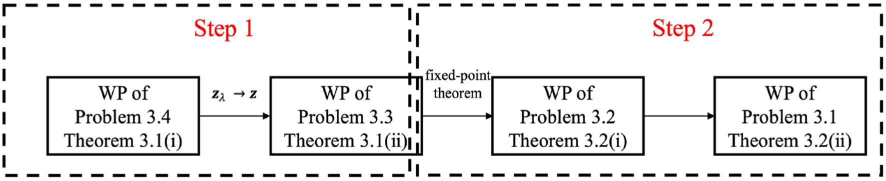



**Fig. 3** solution process

$$S\boldsymbol{v}(t) = e^{-\frac{t}{\delta}}(\frac{1}{\delta}I - B)R(e^{\frac{t}{\delta}}\boldsymbol{v}(t)) + \delta e^{-\frac{t}{\delta}}AR(e^{\frac{t}{\delta}}\boldsymbol{v}(t)) \quad \forall \boldsymbol{v} \in \mathcal{V}, t \in I. \tag{3.11}$$

It is easy to see that the operator $S$ is a history-dependent operator with the Lipschitz constant $L_S = \frac{1}{\delta}(\frac{1}{\delta} + \|B\|) + \|A\|$. In fact, using the linearity and continuity of $B$ and $A$, for any $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{V}$,

$$\begin{aligned} \|S\boldsymbol{v}_1(t) - S\boldsymbol{v}_2(t)\|_{V^*} &\le e^{-\frac{t}{\delta}}\|(\frac{1}{\delta}I - B)R(e^{\frac{t}{\delta}}\boldsymbol{v}_1(t)) - (\frac{1}{\delta}I - B)R(e^{\frac{t}{\delta}}\boldsymbol{v}_2(t))\|_{V^*} \\ &\quad + \delta\|e^{-\frac{t}{\delta}}AR(e^{\frac{t}{\delta}}\boldsymbol{v}_1(t)) - e^{-\frac{t}{\delta}}AR(e^{\frac{t}{\delta}}\boldsymbol{v}_2(t))\|_{V^*} \\ &\le (\frac{1}{\delta} + \|B\| + \delta\|A\|)\|R\boldsymbol{v}_1(t) - R\boldsymbol{v}_2(t)\|_V \\ &\le L_S \int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V ds \quad \forall t \in I \quad (\text{by } (3.6)). \end{aligned} \tag{3.12}$$

Now we can rewrite (3.9) into the following problem.

**Problem 3.2** Find $\boldsymbol{z} : I \to K$ with $\boldsymbol{z}' : I \to V^*$ such that $\boldsymbol{z}(0) = \boldsymbol{0}$ and for all $\boldsymbol{v} \in K$, a.e. $t \in I$,

$$\langle \boldsymbol{z}'(t), \boldsymbol{v} - \boldsymbol{z}(t)\rangle + \langle S\boldsymbol{z}(t), \boldsymbol{v} - \boldsymbol{z}(t)\rangle + \langle B\boldsymbol{z}(t), \boldsymbol{v} - \boldsymbol{z}(t)\rangle \ge \langle \boldsymbol{g}(t), \boldsymbol{v} - \boldsymbol{z}(t)\rangle,$$

where $\boldsymbol{g}(t) = \delta e^{-\frac{t}{\delta}}\boldsymbol{f}(t)$.

### 3.3 The well-posedness of Problem 3.1

The unique existence of a solution to Problem 3.1 (or equivalently, Problem 3.2) is obtained in two steps (see Fig. 3). First, we introduce an auxiliary problem (see Problem 3.3) and the associated penalty problem with the parameter $\lambda$ (see Problem 3.4). We demonstrate the well-posedness of the penalty problem using the operator theory and show that the solution of the penalty problem is bounded independent of the parameter $\lambda$. Taking a convergent subsequence of the solutions of the penalty problem, and passing to the limit $\lambda \to 0$, we demonstrate the well-posedness of the auxiliary problem. Then, by the fixed-point theory, we show the well-posedness (WP) of Problem 3.1.

### 3.3.1 The well-posedness of the auxiliary problem

Given $\eta \in \mathcal{V}^*$, we introduce the following auxiliary problem.

**Problem 3.3** Find $\boldsymbol{z}_\eta : I \to K$ with $\boldsymbol{z}_\eta'(t) \in V^*$ and $\boldsymbol{z}_\eta(0) = \mathbf{0}$ such that for all $\boldsymbol{v} \in K$, a.e. $t \in I$,

$$\langle \boldsymbol{z}_\eta'(t), \boldsymbol{v} - \boldsymbol{z}_\eta(t)\rangle + \langle B\boldsymbol{z}_\eta(t), \boldsymbol{v} - \boldsymbol{z}_\eta(t)\rangle \geq \langle \boldsymbol{g}(t) - \eta(t), \boldsymbol{v} - \boldsymbol{z}_\eta(t)\rangle. \quad (3.13)$$

According to [26, Lemma 16], we have the following equivalent form of Problem 3.3:

Find $\boldsymbol{z}_\eta \in D(L) \cap \mathcal{K}$ such that for all $\boldsymbol{v} \in \mathcal{K}$,

$$\int_0^T \langle \boldsymbol{z}_\eta'(t) + B\boldsymbol{z}_\eta(t), \boldsymbol{v}(t) - \boldsymbol{z}_\eta(t)\rangle dt \geq \int_0^T \langle \boldsymbol{g}(t) - \eta(t), \boldsymbol{v}(t) - \boldsymbol{z}_\eta(t)\rangle dt,$$

where $\mathcal{K} = \{\boldsymbol{v} \in \mathcal{V} \mid \boldsymbol{v}(t) \in K \text{ a.e. } t \in I\}$. We introduce the penalty oprator $P = J(I - P_K)$ ([27, Definition 23]) where $J : V \to V^*$ is the duality map, $I : V \to V$ is the identity map on $V$ and $P_K : V \to K$ is the projection operator. We also introduce the Nemytski operators $\mathcal{B} : \mathcal{V} \to \mathcal{V}^*$ and $\mathcal{P} : \mathcal{V} \to \mathcal{V}^*$ of $B$ and $P$ respectively, i.e.,

$$(\mathcal{B}\boldsymbol{v})(t) = B\boldsymbol{v}(t), \quad (\mathcal{P}\boldsymbol{v})(t) = P\boldsymbol{v}(t) \quad \forall\, \boldsymbol{v} \in \mathcal{V}. \quad (3.14)$$

**Lemma 3.1** *([28, Lemma 3.1]) Assume that $H(\mathscr{B})$ holds and $P$ satisfies $\|P\boldsymbol{u}\|_{V^*} \leq a + b\|\boldsymbol{u}\|_V$ with $a \geq 0$ and $b \geq 0$, for all $\boldsymbol{u} \in V$. Then we have:*

*(i) $\mathcal{B}$ is linear, continuous and strongly monotone with $m_{\mathscr{B}}$;*
*(ii) $\mathcal{P}$ is bounded, demicontinuous, monotone and $\boldsymbol{v} \in \mathcal{K}$ if and only if $\mathcal{P}\boldsymbol{v} = \mathbf{0}$.*
For any $\lambda > 0$, we consider the following penalty problem.

**Problem 3.4** Find $\boldsymbol{z}_\lambda : I \to V$ with $\boldsymbol{z}_\lambda'(t) \in V^*$ for all $\boldsymbol{v} \in V$, a.e. $t \in I$ and $\boldsymbol{z}_\lambda(0) = \mathbf{0}$, such that

$$\langle \boldsymbol{z}_\lambda'(t), \boldsymbol{v} - \boldsymbol{z}_\lambda(t)\rangle + \langle B\boldsymbol{z}_\lambda(t), \boldsymbol{v} - \boldsymbol{z}_\lambda(t)\rangle + \frac{1}{\lambda}\langle P\boldsymbol{z}_\lambda(t), \boldsymbol{v} - \boldsymbol{z}_\lambda(t)\rangle \geq \langle \boldsymbol{g}(t) - \eta(t), \boldsymbol{v} - \boldsymbol{z}_\lambda(t)\rangle. \quad (3.15)$$

Equivalently, the problem is to find $\boldsymbol{z}_\lambda \in D(L)$ such that for all $\boldsymbol{v} \in \mathcal{V}$,

$$\langle L\boldsymbol{z}_\lambda, \boldsymbol{v} - \boldsymbol{z}_\lambda\rangle + \langle \mathcal{B}\boldsymbol{z}_\lambda, \boldsymbol{v} - \boldsymbol{z}_\lambda\rangle + \frac{1}{\lambda}\langle \mathcal{P}\boldsymbol{z}_\lambda, \boldsymbol{v} - \boldsymbol{z}_\lambda\rangle \geq \langle \boldsymbol{g} - \eta, \boldsymbol{v} - \boldsymbol{z}_\lambda\rangle, \quad (3.16)$$

where the operator $L$ is defined in Section 2.

**Theorem 3.1** *Under the assumptions of Lemma 3.1, together with hypotheses $H(\mathscr{A})$ and $H(\boldsymbol{f})$, we have:*

(i) *for each $\lambda > 0$, there exists a unique solution $\boldsymbol{z}_\lambda \in \mathcal{W}$ of Problem 3.4 satisfying*

$$\|\boldsymbol{z}_\lambda\|_{\mathcal{W}} \le \big(m_{\mathscr{B}}^{-1} + 2(\|\mathcal{B}\| m_{\mathscr{B}}^{-1} + 1)\big)\|\boldsymbol{g} - \eta\|_{\mathcal{V}^*}. \tag{3.17}$$

(ii) *Problem 3.3 has a unique solution $\boldsymbol{z}_\eta \in \mathcal{W}$, and the bound (3.17) holds with $\boldsymbol{z}_\lambda$ replaced by $\boldsymbol{z}_\eta$.*

***Proof*** We start with the proof of (i). It follows from Lemma 3.1 that the operators $\mathcal{B}$ and $\mathcal{P}$ are pseudomonotone which implies that $\mathcal{B} + \mathcal{P}$ is pseudomonotone. Moreover, by taking advantage of the strong monotonicity of $\mathcal{B}$ and the monotonicity of $\mathcal{P}$, we establish that $\mathcal{B} + \mathcal{P}$ is 0-coercive. Additionally, since $\mathcal{B}$ is linear continuous and $\mathcal{P}$ is bounded, $\mathcal{B} + \mathcal{P}$ is bounded. Since $L$ is a maximal monotone operator, we can apply [27, Theorem 74] to know that the operator $L + \mathcal{B} + \frac{1}{\lambda}\mathcal{P}$ is surjective. Thus, for each $\lambda > 0$, Problem 3.4 has a solution. Also we have the uniqueness of solution to Problem 3.4 by using a standard argument.

Next we show that $\|\boldsymbol{z}_\lambda\|_{\mathcal{W}}$ is bounded independent of $\lambda$. Testing (3.15) by $\boldsymbol{v} = \boldsymbol{0} \in K$, we have

$$\langle \boldsymbol{z}_\lambda'(t), \boldsymbol{z}_\lambda(t)\rangle + \langle B\boldsymbol{z}_\lambda(t), \boldsymbol{z}_\lambda(t)\rangle + \frac{1}{\lambda}\langle P\boldsymbol{z}_\lambda(t), \boldsymbol{z}_\lambda(t)\rangle \le \langle \boldsymbol{g}(t) - \eta(t), \boldsymbol{z}_\lambda(t)\rangle.$$

By the monotonicity of $P$ and the strong monotonicity of $B$, we derive from the above inequality that

$$\|\boldsymbol{z}_\lambda\|_{\mathcal{V}} \le m_{\mathscr{B}}^{-1}\|\boldsymbol{g} - \eta\|_{\mathcal{V}^*}. \tag{3.18}$$

Note that the inequality (3.16) is equivalent to

$$\langle L\boldsymbol{z}_\lambda, \boldsymbol{v}\rangle + \langle \mathcal{B}\boldsymbol{z}_\lambda, \boldsymbol{v}\rangle + \frac{1}{\lambda}\langle \mathcal{P}\boldsymbol{z}_\lambda, \boldsymbol{v}\rangle = \langle \boldsymbol{g} - \eta, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \mathcal{V}, \tag{3.19}$$

from which we deduce that

$$\begin{aligned} \|L\boldsymbol{z}_\lambda\|_{\mathcal{V}^*} &\le \|\mathcal{B}\boldsymbol{z}_\lambda\|_{\mathcal{V}^*} + \|\boldsymbol{g} - \eta\|_{\mathcal{V}^*} + \frac{1}{\lambda}\|\mathcal{P}\boldsymbol{z}_\lambda\|_{\mathcal{V}^*} \\ &\le (\|\mathcal{B}\| m_{\mathscr{B}}^{-1} + 1)\|\boldsymbol{g} - \eta\|_{\mathcal{V}^*} + \frac{1}{\lambda}\|\mathcal{P}\boldsymbol{z}_\lambda\|_{\mathcal{V}^*}. \end{aligned} \tag{3.20}$$

According to the embedding theorem, we have $\boldsymbol{z}_\lambda \in \mathcal{W} \subset C(I; H)$. To pass to the limit $\lambda \to 0$, we need to show that $L\boldsymbol{z}_\lambda$ is bounded independent of $\lambda^{-1}$. In view of (3.18) and (3.20), our task next is to bound $\frac{1}{\lambda}\|\mathcal{P}\boldsymbol{z}_\lambda\|_{\mathcal{V}^*}$.

Since

$$\langle P\boldsymbol{z}_\lambda(t), (I-P_K)\boldsymbol{z}_\lambda(t)\rangle = \langle J(I-P_K)\boldsymbol{z}_\lambda(t), (I-P_K)\boldsymbol{z}_\lambda(t)\rangle = \|(I-P_K)\boldsymbol{z}_\lambda(t)\|_V^2,$$

we set $\tilde{\boldsymbol{v}}(t) = (I-P_K)\boldsymbol{z}_\lambda(t)$ for all $t \in I$ and substitute $\boldsymbol{v} = \tilde{\boldsymbol{v}} \in \mathcal{V}$ into (3.19) to obtain

$$\begin{aligned}
&\frac{1}{\lambda}\int_0^T \langle P\boldsymbol{z}_\lambda(t), (I-P_K)\boldsymbol{z}_\lambda(t)\rangle = \frac{1}{\lambda}\int_0^T \|(I-P_K)\boldsymbol{z}_\lambda(t)\|_V^2 dt \\
&\quad = \int_0^T \langle \boldsymbol{g}(t) - \eta(t) - B\boldsymbol{z}_\lambda(t) - \boldsymbol{z}_\lambda'(t), (I-P_K)\boldsymbol{z}_\lambda(t)\rangle dt \\
&\quad \le (\|\boldsymbol{g}-\eta\|_{\mathcal{V}^*} + \|\mathcal{B}\boldsymbol{z}_\lambda\|_{\mathcal{V}^*})(\int_0^T \|(I-P_K)\boldsymbol{z}_\lambda(t)\|_V^2 dt)^{\frac{1}{2}} - \int_0^T \langle \boldsymbol{z}_\lambda'(t), (I-P_K)\boldsymbol{z}_\lambda(t)\rangle dt.
\end{aligned}$$

It remains to bound $R_1 := \int_0^T \langle \boldsymbol{z}_\lambda'(t), (I-P_K)\boldsymbol{z}_\lambda(t)\rangle dt$. There exists a sequence of step functions $\boldsymbol{z}_\lambda^n := \sum_{i=1}^n \boldsymbol{z}_\lambda(t_i)\mathbb{I}_{[t_{i-1},t_i)}$ such that $\boldsymbol{z}_\lambda^n \to \boldsymbol{z}_\lambda$ in $\mathcal{V}$ as $n \to \infty$, where $0 = t_0 < t_1 < \cdots < t_n = T$ and $\mathbb{I}_{[t_{i-1},t_i)}$ represents the characteristic function of $[t_{i-1}, t_i)$. Write

$$R_1 = \int_0^T \langle \boldsymbol{z}_\lambda'(t), (I-P_K)(\boldsymbol{z}_\lambda(t) - \boldsymbol{z}_\lambda^n(t))\rangle dt + \int_0^T \langle \boldsymbol{z}_\lambda'(t), (I-P_K)\boldsymbol{z}_\lambda^n(t)\rangle dt =: R_{11} + R_{12}.$$

Since the projection operator $P_K$ is nonexpansive, we have

$$R_{11} \to 0 \quad \text{as} \quad n \to \infty.$$

For $R_{12}$, we calculate as follows:

$$\begin{aligned}
R_{12} &= \int_0^T \langle \boldsymbol{z}_\lambda'(t), (I-P_K)\sum_{i=1}^n \boldsymbol{z}_\lambda(t_i)\mathbb{I}_{[t_{i-1},t_i)}(t)\rangle dt = \sum_{i=1}^n (\boldsymbol{z}_\lambda(t_i) - \boldsymbol{z}_\lambda(t_{i-1}), (I-P_K)\boldsymbol{z}_\lambda(t_i))_H \\
&= \sum_{i=1}^n \big((I-P_K)\boldsymbol{z}_\lambda(t_i) - (I-P_K)\boldsymbol{z}_\lambda(t_{i-1}), (I-P_K)\boldsymbol{z}_\lambda(t_i)\big)_H \\
&\quad - \sum_{i=1}^n \big(P_K\boldsymbol{z}_\lambda(t_{i-1}) - P_K\boldsymbol{z}_\lambda(t_i), (I-P_K)\boldsymbol{z}_\lambda(t_i)\big)_H \\
&\ge \frac{1}{2}\Big(\|(I-P_K)\boldsymbol{z}_\lambda(t_n)\|_H^2 + \sum_{i=1}^n \|(I-P_K)(\boldsymbol{z}_\lambda(t_i) - \boldsymbol{z}_\lambda(t_{i-1}))\|_H^2\Big),
\end{aligned}$$

where we have used the facts that $\boldsymbol{z}_\lambda(0) = \mathbf{0}$ and

$$(\boldsymbol{v} - P_K\boldsymbol{w}, \boldsymbol{w} - P_K\boldsymbol{w})_H \le 0 \quad \forall\, \boldsymbol{v} \in K,\ \boldsymbol{w} \in V.$$

We conclude that $R_1 \ge 0$ and consequently,

$$\frac{1}{\lambda}\Big(\int_0^T \|(I-P_K)\boldsymbol{z}_\lambda(t)\|_V^2 dt\Big)^{\frac{1}{2}} \le \|\boldsymbol{g}-\eta\|_{\mathcal{V}^*} + \|\mathcal{B}\boldsymbol{z}_\lambda\|_{\mathcal{V}^*}. \tag{3.21}$$

Since $\|J\| \le 1$,

$$\|\mathcal{P}\boldsymbol{z}_\lambda\|_{\mathcal{V}^*} \leq (\int_0^T \|(I-P_K)\boldsymbol{z}_\lambda(t)\|_V^2 dt)^{\frac{1}{2}}.$$

Hence, from (3.20) and (3.21), we deduce that

$$\|L\boldsymbol{z}_\lambda\|_{\mathcal{V}^*} \leq 2(\|\boldsymbol{g}-\eta\|_{\mathcal{V}^*} + \|\mathcal{B}\boldsymbol{z}_\lambda\|_{\mathcal{V}^*}) \leq 2(\|\mathcal{B}\|m_{\mathscr{B}}^{-1}+1)\|\boldsymbol{g}-\eta\|_{\mathcal{V}^*}. \quad (3.22)$$

Then, (3.17) follows from (3.18) and (3.22), i.e., $\{\boldsymbol{z}_\lambda\}_\lambda$ is uniformly bounded in $\mathcal{W}$.

Now we turn to prove (ii). Since $\{\|\boldsymbol{z}_\lambda\|_{\mathcal{W}}\}_\lambda$ is bounded and $\mathcal{W}$ is reflexive, there exists a subsequence, also denoted by $\{\boldsymbol{z}_\lambda\}_\lambda$, such that

$$\boldsymbol{z}_\lambda \rightharpoonup \boldsymbol{z} \text{ in } \mathcal{W}.$$

Let us show that $\boldsymbol{z} \in \mathcal{K}$ is a solution to Problem 3.3. It follows from Lemma 3.1 and (3.16) that, for all $\boldsymbol{v} \in \mathcal{V}$,

$$\begin{aligned}\frac{1}{\lambda}\langle\mathcal{P}\boldsymbol{z}_\lambda, \boldsymbol{z}_\lambda-\boldsymbol{v}\rangle &\leq \langle L\boldsymbol{z}_\lambda, \boldsymbol{v}-\boldsymbol{z}_\lambda\rangle + \langle\mathcal{B}\boldsymbol{z}_\lambda, \boldsymbol{v}-\boldsymbol{z}_\lambda\rangle + \langle\boldsymbol{g}-\eta, \boldsymbol{z}_\lambda-\boldsymbol{v}\rangle \\ &\leq C\|\boldsymbol{g}-\eta\|_{\mathcal{V}^*}\|\boldsymbol{v}-\boldsymbol{z}_\lambda\|_{\mathcal{V}} \quad (\text{by } (3.17)).\end{aligned}$$

Taking $\boldsymbol{v}=\boldsymbol{z}$ in the above inequality, we get

$$\limsup\langle\mathcal{P}\boldsymbol{z}_\lambda, \boldsymbol{z}_\lambda-\boldsymbol{z}\rangle \leq 0. \quad (3.23)$$

By the pseudomonotonicity of $\mathcal{P}$, we have

$$\langle\mathcal{P}\boldsymbol{z}, \boldsymbol{z}-\boldsymbol{v}\rangle \leq \liminf\langle\mathcal{P}\boldsymbol{z}_\lambda, \boldsymbol{z}_\lambda-\boldsymbol{v}\rangle \quad \forall \boldsymbol{v}\in\mathcal{V},$$

which, together with (3.23), implies that $\langle\mathcal{P}\boldsymbol{z}, \boldsymbol{z}-\boldsymbol{v}\rangle \leq 0$ for all $\boldsymbol{v}\in\mathcal{V}$. Therefore, we conclude $\mathcal{P}\boldsymbol{z}=\boldsymbol{0}$, which means that $\boldsymbol{z}\in\mathcal{K}$ by Lemma 3.1(ii).

Moreover, taking $\boldsymbol{v}\in\mathcal{K}$ in (3.16) and using the monotonicity of $\mathcal{P}$, we have

$$\langle L\boldsymbol{z}_\lambda, \boldsymbol{v}-\boldsymbol{z}_\lambda\rangle + \langle\mathcal{B}\boldsymbol{z}_\lambda, \boldsymbol{v}-\boldsymbol{z}_\lambda\rangle \geq \langle\boldsymbol{g}-\eta, \boldsymbol{v}-\boldsymbol{z}_\lambda\rangle. \quad (3.24)$$

By the maximal monotonicity of $L$,

$$\begin{aligned}\lim\langle L\boldsymbol{z}_\lambda, \boldsymbol{v}-\boldsymbol{z}_\lambda\rangle &= \lim(\langle L\boldsymbol{z}_\lambda, \boldsymbol{v}-\boldsymbol{z}\rangle + \langle L\boldsymbol{z}_\lambda - L\boldsymbol{z}, \boldsymbol{z}-\boldsymbol{z}_\lambda\rangle + \langle L\boldsymbol{z}, \boldsymbol{z}-\boldsymbol{z}_\lambda\rangle) \\ &\leq \langle L\boldsymbol{z}, \boldsymbol{v}-\boldsymbol{z}\rangle.\end{aligned} \quad (3.25)$$

Similarly, since $\mathcal{B}$ is linear, continuous and strongly monotone, we derive

$$\begin{aligned}\lim\langle\mathcal{B}\boldsymbol{z}_\lambda, \boldsymbol{v}-\boldsymbol{z}_\lambda\rangle &= \lim(\langle\mathcal{B}\boldsymbol{z}_\lambda, \boldsymbol{v}-\boldsymbol{z}\rangle + \langle\mathcal{B}\boldsymbol{z}_\lambda - \mathcal{B}\boldsymbol{z}, \boldsymbol{z}-\boldsymbol{z}_\lambda\rangle + \langle\mathcal{B}\boldsymbol{z}, \boldsymbol{z}-\boldsymbol{z}_\lambda\rangle) \\ &\leq \langle\mathcal{B}\boldsymbol{z}, \boldsymbol{v}-\boldsymbol{z}\rangle.\end{aligned} \quad (3.26)$$

Passing to the limit $\lambda \to 0$ in (3.24), together with (3.25) and (3.26), we obtain

$$\langle L\boldsymbol{z}, \boldsymbol{v} - \boldsymbol{z}\rangle + \langle \mathcal{B}\boldsymbol{z}, \boldsymbol{v} - \boldsymbol{z}\rangle \geq \langle \boldsymbol{g} - \eta, \boldsymbol{v} - \boldsymbol{z}\rangle \quad \forall\, \boldsymbol{v} \in \mathcal{K}.$$

Hence, $\boldsymbol{z}$ is a solution of Problem 3.3. Since $\boldsymbol{z}$ is obtained by taking the weak convergence of a subsequence $\{\boldsymbol{z}_\lambda\}$, the same estimate (3.17) also holds for $\boldsymbol{z}$.

It remains to show the uniqueness of Problem 3.3. Let $\boldsymbol{z}_i$ $(i = 1, 2)$ be two solutions of Problem 3.3. It follows from (3.13) that

$$\langle \boldsymbol{z}_1'(t) - \boldsymbol{z}_2'(t), \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\rangle + \langle B\boldsymbol{z}_1(t) - B\boldsymbol{z}_2(t), \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\rangle \leq 0.$$

By the strong monotonicity of $B$, we have

$$\|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\|_H^2 + m_{\mathscr{B}} \int_0^t \|\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)\|_V^2 ds \leq 0,$$

which implies that

$$\boldsymbol{z}_1 = \boldsymbol{z}_2 \text{ in } \mathcal{V}.$$

Thus, the solution of Problem 3.3 is unique. □

#### 3.3.2 Well-posedness of Problem 3.1

For the existence of a unique solution to Problem 3.1 and Problem 3.2, we have the following result.

**Theorem 3.2** *Assume $H(\mathscr{A})$, $H(\mathscr{B})$ and $H(\boldsymbol{f})$. Then, we have:*

(i) *Problem 3.2 admits a unique solution $\boldsymbol{z} \in \mathcal{W}$;*
(ii) *Problem 3.1 admits a unique solution $\boldsymbol{u} \in C(I; V)$ satisfying $\boldsymbol{u}' \in \mathcal{W}$.*

***Proof*** We first provide the proof of (i). By Theorem 3.1(ii), Problem 3.3 has a unique solution $\boldsymbol{z}_\eta$, which allows us to introduce a mapping $\Lambda : L^2(I; V^*) \to L^2(I; V^*)$ defined by

$$\Lambda\eta = S\boldsymbol{z}_\eta.$$

We need to show that $\Lambda$ has a unique fixed point. For $i = 1, 2$, let $\boldsymbol{z}_i$ be the unique solution to Problem 3.3 associated with $\eta_i \in \mathcal{V}^*$. It follows from (3.13) that

$$\langle \boldsymbol{z}_1'(t) - \boldsymbol{z}_2'(t), \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\rangle + \langle B\boldsymbol{z}_1(t) - B\boldsymbol{z}_2(t), \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\rangle \leq \langle \eta_1(t) - \eta_2(t), \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\rangle.$$

Utilizing the strong monotonicity of $B$, we deduce that

$$\int_0^t \|\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)\|_V^2 ds \leq \frac{1}{m_{\mathscr{B}}^2} \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V^*}^2 ds. \tag{3.27}$$

By (3.12), we have

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{V^*}^2 = \|S\boldsymbol{z}_1(t) - S\boldsymbol{z}_2(t)\|_{V^*}^2 \leq L_S^2 t \int_0^t \|\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)\|_V^2 ds,$$

where $L_S = \frac{1}{\delta}(\frac{1}{\delta} + \|\mathscr{B}\|) + \|A\|$. We obtain from (3.27) that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{V^*}^2 \leq \frac{L_S^2 t}{m_{\mathscr{B}}^2} \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V^*}^2 ds.$$

Applying [28, Theorem 2.3], we conclude that $\Lambda$ has a unique fixed point, which implies that Problem 3.2 has a unique solution.

Now we prove (ii). In view of (3.8) and (3.4), it is easy to check that $\boldsymbol{u}$ defined by (3.3) indeed solves Problem 3.1. We need to verify that the solution of Problem 3.1 (or equivalently, (3.7)) is unique. To this end, we divide the interval $I$ into $N = T/k$ subintervals $[t_i, t_{i+1}]$ $(i = 0, 1, \cdots, N-1)$, where the time-step size $k$ satisfying $k < m_{\mathscr{B}}\delta/\|\mathscr{B}\|$. If we prove that the solution of (3.7) is unique in each subinterval, then we establish the global uniqueness.

Assume that (3.7) admits two solution $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. We set $\boldsymbol{e} = \boldsymbol{w}_1 - \boldsymbol{w}_2$. Testing the inequality of $\boldsymbol{w}_2(t)$ by $\boldsymbol{v} = \boldsymbol{w}_1(t)$, and testing the inequality of $\boldsymbol{w}_1(t)$ by $\boldsymbol{v} = \boldsymbol{w}_2(t)$, and adding the two resulting inequalities, we get

$$\begin{aligned} &\langle \boldsymbol{e}'(t), \boldsymbol{e}(t)\rangle + \frac{1}{\delta}(R\boldsymbol{e}(t), \boldsymbol{e}(t)) + \delta\langle AR\boldsymbol{e}(t), \boldsymbol{e}(t)\rangle \\ &\qquad + \langle B\boldsymbol{e}(t), \boldsymbol{e}(t)\rangle \leq \frac{1}{\delta}(\boldsymbol{e}(t), \boldsymbol{e}(t)) + \langle BR\boldsymbol{e}(t), \boldsymbol{e}(t)\rangle. \end{aligned} \tag{3.28}$$

We show that $\boldsymbol{e} = 0$ in $[0, t_1]$. It follows from (3.4b) and $R\boldsymbol{w}_i(0) = \mathbf{0}$, $i = 1, 2$, that

$$\begin{aligned} \int_0^t (R\boldsymbol{e}(s), \boldsymbol{e}(s)) ds &= \int_0^t (R\boldsymbol{e}(s), R\boldsymbol{e}(s)) ds + \delta \int_0^t (R\boldsymbol{e}(s), (R\boldsymbol{e})'(s)) ds \\ &= \|R\boldsymbol{e}\|_{L^2(0,t;H)}^2 + \frac{\delta}{2}\|R\boldsymbol{e}(t)\|_H^2, \\ \int_0^t \langle AR\boldsymbol{e}(s), \boldsymbol{e}(s)\rangle ds &= \int_0^t \langle AR\boldsymbol{e}(s), R\boldsymbol{e}(s)\rangle ds + \delta \int_0^t \langle AR\boldsymbol{e}(s), (R\boldsymbol{e})'(s)\rangle ds \\ &\geq m_{\mathscr{A}}\|R\boldsymbol{w}_1 - R\boldsymbol{w}_2\|_{L^2(0,t;V^*)}^2 + \frac{m_{\mathscr{A}}}{2}\delta\|R\boldsymbol{w}_1(t) - R\boldsymbol{w}_2(t)\|_V^2. \end{aligned}$$

It follows from (3.28) that

$$\|\boldsymbol{e}(t)\|_H^2 + 2m_{\mathscr{B}} \int_0^t \|\boldsymbol{e}(s)\|_V^2 ds \le \frac{2}{\delta}\int_0^t \|\boldsymbol{e}(s)\|_H^2 ds + 2\|\mathscr{B}\| \int_0^t \|R\boldsymbol{e}(s)\|_V \|\boldsymbol{e}(s)\|_V ds$$
$$\le \frac{2}{\delta}\int_0^t \|\boldsymbol{e}(s)\|_H^2 ds + \frac{2\|\mathscr{B}\|t}{\delta}\int_0^t \|\boldsymbol{e}(s)\|_V^2 ds \quad \text{(by (3.6))}.$$

Since $|t-t_0| < m_{\mathscr{B}}\delta/\|\mathscr{B}\|$ for all $t\in[t_0,t_1]$, we have $m_{\mathscr{B}} - \|\mathscr{B}\|t/\delta > 0$ and

$$\|\boldsymbol{e}(t)\|_H^2 \le \frac{2}{\delta}\int_0^t \|\boldsymbol{e}(s)\|_H^2 ds.$$

Apply the Gronwall inequality to conclude

$$\boldsymbol{e}(t) = \boldsymbol{0} \text{ i.e. } \boldsymbol{w}_1(t) = \boldsymbol{w}_2(t) \quad \forall\, t\in[t_0,t_1]\,. \tag{3.29}$$

We proceed to show that (3.7) admits a unique solution in $(t_1,t_2]$, i.e., $\boldsymbol{e}(t)=0$ in $(t_1,t_2]$. Keeping in mind that $\boldsymbol{e}=0$ in $[t_0,t_1]$, from (3.5) and (3.6), we have, for any $t\in(t_1,t_2]$,

$$\|R\boldsymbol{e}(s)\|_V \le \delta^{-1}e^{-\frac{1}{\delta}t}\int_0^t e^{\frac{1}{\delta}s}\|\boldsymbol{e}(s)\|_V\, ds \le \delta^{-1}\int_{t_1}^t \|\boldsymbol{e}(s)\|_V\, ds.$$

Integrating (3.28) on $[0,t]$ for any $t\in(t_1,t_2]$ and noting that $\boldsymbol{e}=0$ in $[0,t_1]$, we get

$$\|\boldsymbol{e}(t)\|_H^2 + 2\Big(m_{\mathscr{B}} - \frac{\|\mathscr{B}\||t-t_1|}{\delta}\Big)\int_{t_1}^t \|\boldsymbol{e}(s)\|_V^2 ds \le \frac{2}{\delta}\int_{t_1}^t \|\boldsymbol{e}(s)\|_H^2 ds.$$

Since $|t-t_1| < \frac{m_{\mathscr{B}}\delta}{\|\mathscr{B}\|}$ for all $t\in[t_1,t_2]$, we conclude

$$\|\boldsymbol{e}(t)\|_H^2 \le \frac{2}{\delta}\int_{t_1}^t \|\boldsymbol{e}(s)\|_H^2 ds,$$

together with $\boldsymbol{e}(t_1)=0$, which implies that $\boldsymbol{e}=0$ in $(t_1,t_2]$. Applying the induction argument, we assert that (3.7) has a unique solution. □

For the dependence of the solution on $\delta$, we present the following theorem that demonstrates the boundedness of the solution to Problem 3.1.

**Theorem 3.3** *Under the assumptions stated in Theorem 3.2, we have the following bound on the solution $\boldsymbol{u}$:*

$$\|\boldsymbol{u}\|_{L^\infty(I;H)}^2 + \delta\|\boldsymbol{u}'\|_{L^2(I;H)}^2 + \delta m_{\mathscr{A}}\|\boldsymbol{u}\|_{\mathcal{V}}^2 + \delta m_{\mathscr{B}}T\|\boldsymbol{u}'\|_{\mathcal{V}}^2 \le C_1(\delta,T)\|\boldsymbol{f}\|_{\mathcal{V}^*}^2, \tag{3.30}$$

*where $C_1(\delta,T) = \alpha_\delta(e^{2T\alpha_\delta^{-1}} - 1)((2m_{\mathscr{A}})^{-1} + \delta m_{\mathscr{B}}^{-1})$ and*
$\alpha_\delta = \min\{\delta, \|B\|^{-2}(\delta^2\ m_{\mathscr{A}} m_{\mathscr{B}})\}$.

*Moreover, if $\mathscr{B}$ is symmetric, then*

$$\|\boldsymbol{u}\|^2_{L^\infty(I;H)} + \delta\|\boldsymbol{u}'\|^2_{L^2(I;H)} + (\delta m_{\mathscr{A}} + m_{\mathscr{B}})\|\boldsymbol{u}\|^2_{\mathcal{V}} + \delta m_{\mathscr{B}} T\|\boldsymbol{u}'\|^2_{\mathcal{V}} \le C_2(\delta, T)\|\boldsymbol{f}\|^2_{\mathcal{V}^*}, \tag{3.31}$$

*where* $C_2(\delta, T) = \delta(e^{2T\delta^{-1}} - 1)((2m_{\mathscr{A}})^{-1} + \delta(2m_{\mathscr{B}})^{-1})$.

***Proof*** It follows from Theorem 3.2 that Problem 3.1 has a unique solution. Taking $\boldsymbol{v} = \boldsymbol{0}_V$ in (3.1), we get

$$\langle \boldsymbol{u}''(t) + A\boldsymbol{u}(t) + B\boldsymbol{u}'(t), \boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\rangle \le \langle \boldsymbol{f}(t), \boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\rangle. \tag{3.32}$$

Using integration by parts, we have

$$\int_0^t \langle \boldsymbol{u}''(s), \boldsymbol{u}(s)\rangle ds = -\int_0^t \|\boldsymbol{u}'(s)\|^2_H ds + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|^2_H, \tag{3.33a}$$

$$\int_0^t \langle \boldsymbol{u}''(s), \delta\boldsymbol{u}'(s)\rangle ds = \frac{\delta}{2}(\|\boldsymbol{u}'(t)\|^2_H - \|\boldsymbol{u}'(0)\|^2_H). \tag{3.33b}$$

It follows from $H(\mathscr{A})$ and $\boldsymbol{u}(0) = \boldsymbol{0}$ that

$$\int_0^t \langle A\boldsymbol{u}(s), \boldsymbol{u}(s)\rangle ds \ge m_{\mathscr{A}} \int_0^t \|\boldsymbol{u}(s)\|^2 ds, \tag{3.34a}$$

$$\int_0^t \langle A\boldsymbol{u}(s), \delta\boldsymbol{u}'(s)\rangle ds \ge \frac{\delta m_{\mathscr{A}}}{2}\|\boldsymbol{u}(t)\|^2_V. \tag{3.34b}$$

By virtue of hypothesis $H(\mathscr{B})$, we see that

$$\int_0^t \langle B\boldsymbol{u}'(s), \delta\boldsymbol{u}'(s)\rangle ds \ge \delta m_{\mathscr{B}} \int_0^t \|\boldsymbol{u}'(s)\|^2_V ds, \tag{3.35a}$$

$$\int_0^t \langle B\boldsymbol{u}'(s), \boldsymbol{u}(s)\rangle ds \le \|B\| \int_0^t \|\boldsymbol{u}'(s)\|_V \|u(s)\|_V ds. \tag{3.35b}$$

Integrating (3.32) on (0, $t$) with $t \in (0, T]$, together with (3.33a)-(3.35b) and $\boldsymbol{u}'(0) = \boldsymbol{u}(0) = \boldsymbol{0}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|_H^2 - \int_0^t \|\boldsymbol{u}'(s)\|_H^2 ds + \frac{\delta}{2}\|\boldsymbol{u}'(t)\|_H^2 \\ + m_{\mathscr{A}} \int_0^t \|\boldsymbol{u}(s)\|_V^2 ds + \frac{\delta m_{\mathscr{A}}}{2}\|\boldsymbol{u}(t)\|_V^2 + \delta m_{\mathscr{B}} \int_0^t \|\boldsymbol{u}'(s)\|_V^2 ds \\ \le \Big(\frac{1}{2m_{\mathscr{A}}} + \frac{\delta}{m_{\mathscr{B}}}\Big) \int_0^t \|\boldsymbol{f}(s)\|_{V^*}^2 ds + \frac{\delta m_{\mathscr{B}}}{2} \int_0^t \|\boldsymbol{u}'(s)\|_V^2 ds \\ + \Big(\frac{m_{\mathscr{A}}}{2} + \frac{\|B\|^2}{\delta m_{\mathscr{B}}}\Big) \int_0^t \|\boldsymbol{u}(s)\|_V^2 ds.\end{aligned} \tag{3.36}$$

Setting $\alpha_\delta = \min\{\delta, \|B\|^{-2}(\delta^2 m_{\mathscr{A}} m_{\mathscr{B}})\}$ and

$$\begin{aligned}a(t) &:= \|\boldsymbol{u}(t)\|_H^2 + \delta \int_0^t \|\boldsymbol{u}'(s)\|_H^2 ds + \delta m_{\mathscr{A}} \int_0^t \|\boldsymbol{u}(s)\|_V^2 ds \\ &\quad + 2\, m_{\mathscr{A}} \int_0^t \int_0^s \|\boldsymbol{u}(r)\|_V^2 dr\, ds + 2\, \delta m_{\mathscr{B}} \int_0^t \int_0^s \|\boldsymbol{u}'(r)\|_V^2 dr\, ds, \\ c(t) &:= \Big(\frac{1}{2m_{\mathscr{A}}} + \frac{\delta}{m_{\mathscr{B}}}\Big) \int_0^t \|\boldsymbol{f}(s)\|_{V^*}^2 ds,\end{aligned}$$

we obtain from (3.36) that

$$\frac{1}{2}\frac{d}{dt}a(t) \le \alpha_\delta^{-1} a(t) + c(t),$$

By the Gronwall inequality and $a(0) = 0$, we obtain

$$a(t) \le \alpha_\delta (e^{2\alpha_\delta^{-1} t} - 1) c(t).$$

Hence, we have the bound (3.30).

If $\mathscr{B}$ is symmetric, then, instead of (3.35b), we use (by $H_{\mathscr{B}}$(iii) and $\boldsymbol{u}(0) = \boldsymbol{0}$)

$$\int_0^t \langle B\boldsymbol{u}'(s), \boldsymbol{u}(s)\rangle ds \ge \frac{m_{\mathscr{B}}}{2}\|\boldsymbol{u}(t)\|_V^2$$

to derive the following inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\boldsymbol{u}(t)\|_H^2 + \delta \int_0^t \|\boldsymbol{u}'(s)\|_H^2 ds + \delta m_{\mathscr{B}} \int_0^t \int_0^s \|\boldsymbol{u}'(r)\|_V drds \\ + m_{\mathscr{A}} \int_0^t \int_0^s \|\boldsymbol{u}(r)\|_V^2 drds + (\delta m_{\mathscr{A}} + m_{\mathscr{B}}) \int_0^t \|\boldsymbol{u}(s)\|_V^2 ds\Big) \\ \le \Big(\frac{1}{2m_{\mathscr{A}}} + \frac{\delta}{2m_{\mathscr{B}}}\Big) \int_0^t \|\boldsymbol{f}(s)\|_{V^*}^2 ds + \int_0^t \|\boldsymbol{u}'(s)\|_H^2 ds.\end{aligned} \tag{3.37}$$

Now, we set

$$\tilde{a}(t) := a(t) + m_{\mathscr{B}} \int_0^t \|\boldsymbol{u}(s)\|_V^2 ds,$$
$$\tilde{c}(t) := \Big(\frac{1}{2m_{\mathscr{A}}} + \frac{\delta}{2m_{\mathscr{B}}}\Big) \int_0^t \|\boldsymbol{f}(s)\|_{V^*}^2 ds.$$

It follows from (3.37) that

$$\frac{1}{2}\frac{d}{dt}\tilde{a}(t) \le \delta^{-1}\tilde{a}(t) + \tilde{c}(t).$$

By the Gronwall inequality and $\tilde{a}(0) = 0$, we conclude (3.31). □

## 4 The modified Tresca-friction contact condition

In this section, we consider Problem P with the modified Tresca-friction contact condition (2.3). We set

$$V_1 = \{\boldsymbol{v} \in V \mid v_n = 0 \ \text{ a.e. on } \ \Gamma_C\},$$

and introduce a convex functional $j$ defined by

$$j(\boldsymbol{v}) = \int_{\Gamma_C} g\,|\boldsymbol{v}_\tau|\,dx \quad \forall\,\boldsymbol{v} \in V. \tag{4.1}$$

### 4.1 Weak formulation of the problem (2.1) and (2.3)

Suppose the problem (2.1) and (2.3) has a sufficiently smooth solution $\boldsymbol{u}$. For any $\boldsymbol{v} \in V_1$, multiplying (2.1b) by $\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))$ and integrating on $\Omega$, applying Green's formula, we get

$$\int_\Omega \boldsymbol{u}''(t)\cdot(\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))dx + \int_\Omega \boldsymbol{\sigma}(t) : \mathbb{D}(\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))dx$$
$$- \int_{\Gamma_C} \boldsymbol{\sigma}\boldsymbol{n}(t)\cdot(\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))d\Gamma = \int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))dx.$$

It follows from the boundary conditions (2.1c), (2.1d), and (2.3) that

$$\int_\Gamma (\boldsymbol{\sigma}\boldsymbol{n})(t)\cdot(\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))d\Gamma \ge \int_{\Gamma_N} \boldsymbol{f}_1(t)\cdot(\boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)))d\Gamma$$
$$+ \int_{\Gamma_C} g(-|\boldsymbol{v}_\tau| + |\boldsymbol{u}_\tau(t) + \delta\boldsymbol{u}_\tau'(t)|)d\Gamma.$$

Together with (2.1a), we get the following weak formulation of the contact problem.

**Problem 4.1** Find $\boldsymbol{u}: I \to V_1$ with $\boldsymbol{u}': I \to V_1$ and $\boldsymbol{u}'': I \to V_1^*$, such that $\boldsymbol{u}(0) = \mathbf{0}$, $\boldsymbol{u}'(0) = \mathbf{0}$, and for all $\boldsymbol{v} \in V_1$, a.e. $t \in I$,

$$\begin{aligned}\langle \boldsymbol{u}''(t) + A\boldsymbol{u}(t) + B\boldsymbol{u}'(t), \boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))\rangle + j(\boldsymbol{v})\\ - j(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)) \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - (\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))\rangle.\end{aligned} \tag{4.2}$$

### 4.2 Transformations of Problem 4.1

Similar to the transformations of Problem 3.1, let $\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t) = \boldsymbol{w}(t)$. Based on (3.3)–(3.4c), Problem 4.1 is equivalently transformed into the following problem.

**Problem 4.2** Find $\boldsymbol{w}: I \to V_1$ with $\boldsymbol{w}': I \to V_1^*$ such that $\boldsymbol{w}(0) = \mathbf{0}$, and for all $\boldsymbol{v} \in V_1$, a.e. $t \in I$,

$$\begin{aligned}&\langle \boldsymbol{w}'(t) - \frac{1}{\delta}\boldsymbol{w}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle + \langle (\frac{1}{\delta}I - B)R\boldsymbol{w}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle\\ &\quad + \delta\langle AR\boldsymbol{w}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle + \langle B\boldsymbol{w}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle + \delta j(\boldsymbol{v}) - \delta j(\boldsymbol{w}(t)) \geq \delta\langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle,\end{aligned}$$

where the operator $R$ is defined by (3.5).

Furthermore, by taking $\boldsymbol{z}(t) = e^{-\frac{t}{\delta}}\boldsymbol{w}(t)$, Problem 4.2 is transformed into the following problem.

**Problem 4.3** Find $\boldsymbol{z}: I \to V_1$ with $\boldsymbol{z}': I \to V_1^*$ such that $\boldsymbol{z}(0) = \mathbf{0}$ and for all $\boldsymbol{v} \in V_1$, a.e. $t \in I$,

$$\langle \boldsymbol{z}'(t), \boldsymbol{v} - \boldsymbol{z}(t)\rangle + \langle S\boldsymbol{z}(t)), \boldsymbol{v} - \boldsymbol{z}(t)\rangle + \langle B\boldsymbol{z}(t), \boldsymbol{v} - \boldsymbol{z}(t)\rangle + \tilde{j}(t, \boldsymbol{v}) - \tilde{j}(t, \boldsymbol{z}(t)) \geq \langle \boldsymbol{g}(t), \boldsymbol{v} - \boldsymbol{z}(t)\rangle,$$

where $\tilde{j}(t, \boldsymbol{v}) = \delta e^{-\frac{t}{\delta}} j(\boldsymbol{v})$, $\boldsymbol{g}(t) = \delta e^{-\frac{t}{\delta}} \boldsymbol{f}(t)$, and the operator $S$ is defined by (3.11).

### 4.3 Well-posedness of Problem 4.1

Here, we provide the solvability theorem which delivers the existence and uniqueness of a solution to Problem 4.3.

**Theorem 4.1** *Assume $H(\mathscr{A})$(i)(ii), $H(\mathscr{B})$ and $H(\boldsymbol{f})$. Then, Problem 4.3 has a unique solution.*

***Proof*** Note that $B$ is linear, continuous and strongly monotone, $\tilde{j}$ is convex, l.s.c., and

$$\|\partial_c \tilde{j}(t, \boldsymbol{v})\|_{V^*} \leq \delta g \cdot \mathrm{meas}(\Gamma_C) \quad \forall \boldsymbol{v} \in V_1,\ t \in I,$$

and $S$ is Lipschitz continuous (see (3.12)). Applying [27, Theorem 98], we know that Problem 4.3 has a unique solution. □

**Theorem 4.2** *Assume $H(\mathscr{A})$, $H(\mathscr{B})$ and $H(\boldsymbol{f})$. Then, Problem 4.1 has a unique solution $\boldsymbol{u} \in C(I; V)$ satisfying $\boldsymbol{u}' \in \mathcal{W}$.*

***Proof*** It is easy to check that every solution of Problem 4.3 solves Problem 4.1. Thus, Problem 4.1 has a solution by Theorem 4.1. The proof of uniqueness is similar to that of Theorem 3.2. □

The following result provides bounds on norms of the solution with respect to the parameter $\delta$.

**Theorem 4.3** *Under the same assumptions of Theorem 4.2, the following inequality holds:*

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{L^\infty(I;H)} + \delta\|\boldsymbol{u}'\|^2_{L^2(I;H)} + \delta m_{\mathscr{A}}\|\boldsymbol{u}\|^2_{\mathcal{V}} + \delta m_{\mathscr{B}} T\|\boldsymbol{u}'\|^2_{\mathcal{V}} \\ &\quad + 2g\|\boldsymbol{u}_\tau + \delta\boldsymbol{u}'_\tau\|_{L^1(I;L^1(\Gamma_C;\mathbb{R}))} \le C_1(\delta,T)\|\boldsymbol{f}\|^2_{\mathcal{V}^*}, \end{aligned} \tag{4.3}$$

*where $C_1(\delta, T)$ is defined in Theorem 3.3. Moreover, if the operator $\mathscr{B}$ is symmetric, then we have the bound*

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{L^\infty(I;H)} + \delta\|\boldsymbol{u}'\|^2_{L^2(I;H)} + (\delta m_{\mathscr{A}} + m_{\mathscr{B}})\|\boldsymbol{u}\|^2_{\mathcal{V}} \\ &\quad + \delta m_{\mathscr{B}} T\|\boldsymbol{u}'\|^2_{\mathcal{V}} + 2g\|\boldsymbol{u}_\tau + \delta\boldsymbol{u}'_\tau\|_{L^1(I;L^1(\Gamma_C;\mathbb{R}))} \le C_2(\delta,T)\|\boldsymbol{f}\|^2_{\mathcal{V}^*}, \end{aligned} \tag{4.4}$$

*where $C_2(\delta, T)$ is defined in Theorem 3.3.*

***Proof*** It follows from Theorem 4.2 that Problem 4.1 admits a unique solution. Taking $\boldsymbol{v} = \boldsymbol{0}_V$ and $\boldsymbol{v} = 2(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))$ in (4.2), respectively, we get

$$\langle \boldsymbol{u}''(t) + A\boldsymbol{u}(t) + B\boldsymbol{u}'(t), \boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\rangle + j(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)) = \langle \boldsymbol{f}(t), \boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\rangle.$$

By the definition of $j$,

$$j(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)) = g\int_{\Gamma_C} |\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)|\,dx = g\|\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\|_{L^1(\Gamma_C;\mathbb{R})}. \tag{4.5}$$

Similar to the proof of Theorem 3.3 and based on (4.5), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|^2_H - \int_0^t \|\boldsymbol{u}'(s)\|^2_H ds + \frac{\delta}{2}\|\boldsymbol{u}'(t)\|^2_H + m_{\mathscr{A}}\int_0^t \|\boldsymbol{u}(s)\|^2_V ds \\ &\quad + \frac{\delta}{2} m_{\mathscr{A}}\|\boldsymbol{u}(t)\|^2_V + \delta m_{\mathscr{B}}\int_0^t \|\boldsymbol{u}'(s)\|^2_V ds + g\|\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\|_{L^1(\Gamma_C;\mathbb{R})} ds \\ &\le \frac{\delta m_{\mathscr{B}}}{2}\int_0^t \|\boldsymbol{u}'(s)\|^2_V ds + (\frac{m_{\mathscr{A}}}{2} + \frac{\|B\|^2}{\delta m_{\mathscr{B}}})\int_0^t \|\boldsymbol{u}(s)\|^2_V ds + (\frac{1}{2m_{\mathscr{A}}} + \frac{\delta}{m_{\mathscr{B}}})\int_0^t \|\boldsymbol{f}(s)\|^2_{V^*} ds. \end{aligned}$$

Then we get

$$\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{u}(t)\|_H^2+\delta\int_0^t\|\boldsymbol{u}'(s)\|_H^2ds+2g\int_0^t\|\boldsymbol{u}(s)+\delta\boldsymbol{u}'(s)\|_{L^1(\Gamma_C;\mathbb{R})}ds)$$
$$+m_{\mathscr{A}}\int_0^t\int_0^s\|\boldsymbol{u}(r)\|_V^2drds+\delta m_{\mathscr{A}}\int_0^t\|\boldsymbol{u}(s)\|_V^2ds+\delta m_{\mathscr{B}}\int_0^t\int_0^s\|\boldsymbol{u}'(r)\|_V^2drds$$
$$\le(\frac{1}{2m_{\mathscr{A}}}+\frac{\delta}{m_{\mathscr{B}}})\int_0^t\|\boldsymbol{f}(s)\|_{V^*}^2ds+\frac{\|B\|^2}{\delta m_{\mathscr{B}}}\int_0^t\|\boldsymbol{u}(s)\|_V^2ds+\int_0^t\|\boldsymbol{u}'(s)\|_H^2ds.$$

Therefore, we deduce (4.3) through the derivation process of Theorem 3.3.

If the operator $B$ is symmetric, we have

$$\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{u}(t)\|_H^2+2g\int_0^t\|\boldsymbol{u}(s)+\delta\boldsymbol{u}'(s)\|_{L^1(\Gamma_C;\mathbb{R})}ds+m_{\mathscr{A}}\int_0^t\int_0^s\|\boldsymbol{u}(r)\|_V^2dr\,ds$$
$$+(\delta m_{\mathscr{A}}+m_{\mathscr{B}})\int_0^t\|\boldsymbol{u}(s)\|_V^2ds+\delta m_{\mathscr{B}}\int_0^t\int_0^s\|\boldsymbol{u}'(r)\|_V^2dr\,ds)+\delta\int_0^t\|\boldsymbol{u}'(s)\|_H^2$$
$$\le(\frac{1}{2m_{\mathscr{A}}}+\frac{\delta}{2m_{\mathscr{B}}})\int_0^t\|\boldsymbol{f}(s)\|_{V^*}^2ds+\int_0^t\|\boldsymbol{u}'(s)\|_H^2ds.$$

Then, we deduce (4.4). □

## 5 The modified contact condition of Clarke-subdifferential-type

In this section, we consider Problem P with the modified contact condition (2.4). First, we make the assumptions on the non-smooth and non-convex function $j_n:\Gamma_C\times\mathbb{R}\to\mathbb{R}$.

$\underline{H(j_n)}$: The following properties hold for $j_n$:

(i) $j_n(\cdot,r)$ is measurable on $\Gamma_C$ for all $r\in\mathbb{R}$ and there exists an $e\in L^2(\Gamma_C)$ such that $j_n(\cdot,e(\cdot))\in L^1(\Gamma_C)$;

(ii) $j_n(\boldsymbol{x},\cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $\boldsymbol{x}\in\Gamma_C$;

(iii) there exists $m_{j_n}\ge 0$ such that for all $r_1,r_2\in\mathbb{R}$, a.e. $\boldsymbol{x}\in\Gamma_C$,

$$j_n^\circ(\boldsymbol{x},r_1;r_2-r_1)+j_n^\circ(\boldsymbol{x},r_2;r_1-r_2)\le m_{j_n}|r_1-r_2|^2;$$

(iv) there exist $d_1,d_2\ge 0$ such that for all $r\in\mathbb{R}$, a.e. $\boldsymbol{x}\in\Gamma_C$,

$$|\partial j_n(\boldsymbol{x},r)|\le d_1+d_2|r|;$$

(v) either $j_n(\boldsymbol{x},\cdot)$ or $-j_n(\boldsymbol{x},\cdot)$ is regular for a.e. $\boldsymbol{x}\in\Gamma_C$.

Let $\gamma:V\to L^2(\Gamma_C,\mathbb{R}^d)$ be the trace operator. Then, $\gamma$ is linear, continuous and compact, the norm of which is denoted by $\|\gamma\|$. The functional $J:Z=\gamma(V)\to\mathbb{R}$ is defined by

$$J(\boldsymbol{v})=\int_{\Gamma_C}j_n(v_n)d\Gamma\quad\forall\boldsymbol{v}\in Z. \tag{5.1}$$

It follows from [23, Theorem 3.47] that $J$ satisfies:
(i) $J$ is locally Lipschitz continuous;
(ii) for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in Z$, we have

$$J^\circ(\boldsymbol{v}_1; \boldsymbol{v}_2 - \boldsymbol{v}_1) + J^\circ(\boldsymbol{v}_2; \boldsymbol{v}_1 - \boldsymbol{v}_2) \leq m_{j_n} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_Z^2; \tag{5.2}$$

(iii) for all $\boldsymbol{v} \in Z$, we have

$$\|\partial J(\boldsymbol{v})\|_{Z^*} \leq c_1 + c_2 \|\boldsymbol{v}\|_Z \tag{5.3}$$

with $c_1 = \sqrt{2meas(\Gamma_C)}d_1$ and $c_2 = \sqrt{2}d_2$.

### 5.1 Weak formulation of the problem (2.1) and (2.4)

We derive the weak formulation of Problem P with the modified contact condition (2.4). Assume the solution $\boldsymbol{u}$ of the contact problem is sufficiently smooth. Testing (2.1b) by $\boldsymbol{v}$ and using the integration by parts, we get

$$\int_\Omega \boldsymbol{u}''(t) \cdot \boldsymbol{v}\, dx + \int_\Omega \boldsymbol{\sigma}(t) : \mathbb{D}(\boldsymbol{v})\, dx - \int_\Gamma (\boldsymbol{\sigma n})(t) \cdot \boldsymbol{v} d\Gamma = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx.$$

Apply the boundary conditions (2.1c), (2.1d), and (2.4) to obtain

$$\int_\Gamma (\boldsymbol{\sigma n})(t) \cdot \boldsymbol{v} d\Gamma \geq \int_{\Gamma_N} \boldsymbol{f}_1(t) \cdot \boldsymbol{v} d\Gamma - \int_{\Gamma_C} j_n^\circ(u_n(t) + \delta u_n'(t); v_n) d\Gamma.$$

Together with (2.1a), we get the following weak formulation.

**Problem 5.1** Find $\boldsymbol{u} : I \to V$ with $\boldsymbol{u}' : I \to V$ and $\boldsymbol{u}'' : I \to V^*$, such that $\boldsymbol{u}(0) = \mathbf{0}$, $\boldsymbol{u}'(0) = \mathbf{0}$, and for all $\boldsymbol{v} \in V$, a.e. $t \in I$,

$$\langle \boldsymbol{u}''(t) + B\boldsymbol{u}'(t) + A\boldsymbol{u}(t) - \boldsymbol{f}(t), \boldsymbol{v}\rangle + J^\circ(\gamma(\boldsymbol{u}(t) + \delta \boldsymbol{u}'(t)); \gamma \boldsymbol{v}) \geq 0. \tag{5.4}$$

### 5.2 Transformations of Problem 5.1

Similar to the transformations of Problem 3.1, let $\boldsymbol{u}(t) + \delta \boldsymbol{u}'(t) = \boldsymbol{w}(t)$. Based on (3.3)–(3.4c), we transform Problem 5.1 into the following problem.

**Problem 5.2** Find $\boldsymbol{w} : I \to V$ with $\boldsymbol{w}' : I \to V^*$ such that $\boldsymbol{w}(0) = \mathbf{0}$ and for all $\boldsymbol{v} \in V$, a.e. $t \in I$,

$$\begin{aligned} \langle \boldsymbol{w}'(t) - \frac{1}{\delta}\boldsymbol{w}(t), \boldsymbol{v}\rangle + \langle (\frac{1}{\delta}I - B)R\boldsymbol{w}(t), \boldsymbol{v}\rangle + \delta \langle AR\boldsymbol{w}(t), \boldsymbol{v}\rangle \\ + \langle B\boldsymbol{w}(t), \boldsymbol{v}\rangle + \delta J^\circ(\gamma \boldsymbol{w}(t); \gamma \boldsymbol{v}) \geq \delta \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle, \end{aligned}$$

where the operator $R$ is defined by (3.5).
Define $\boldsymbol{z}(t) = \boldsymbol{w}(t)e^{-\frac{t}{\delta}}$. We can express Problem 5.2 in terms of $\boldsymbol{z}$.

**Problem 5.3** Find $\boldsymbol{z} : I \to V$ with $\boldsymbol{z}' : I \to V^*$ such that $\boldsymbol{z}(0) = \boldsymbol{0}$ and for all $\boldsymbol{v} \in V$, a.e. $t \in I$,

$$\langle \boldsymbol{z}'(t), \boldsymbol{v}\rangle + \langle S\boldsymbol{z}(t), \boldsymbol{v}\rangle + \langle B\boldsymbol{z}(t), \boldsymbol{v}\rangle + \delta e^{-\frac{t}{\delta}} J^\circ(\gamma e^{\frac{t}{\delta}} \boldsymbol{z}(t); \gamma \boldsymbol{v}) \geq \langle \boldsymbol{g}(t), \boldsymbol{v}\rangle,$$

where $\boldsymbol{g}(t) = \delta e^{-\frac{t}{\delta}} \boldsymbol{f}(t)$, and the operator $S$ is defined by (3.11).

### 5.3 Well-posedness of Problem 5.1

The existence of a unique solution of Problem 5.1 is shown in two steps. First, we introduce an auxiliary problem (see Problem 5.4) and show its well-posedness. Then, by the fixed-point theory, we demonstrate the well-posedness of Problem 5.1.

#### 5.3.1 The auxiliary problem of Problem 5.3

Let us consider the existence and uniqueness of a solution to the following auxiliary problem.

**Problem 5.4** Find $\boldsymbol{z} \in \mathcal{W}$ such that $\boldsymbol{z}(0) = \boldsymbol{0}$ and for a.e. $t \in I$,

$$\boldsymbol{g}(t) \in \boldsymbol{z}'(t) + B\boldsymbol{z}(t) + \delta e^{-\frac{t}{\delta}} \gamma^* \partial J(\gamma e^{\frac{t}{\delta}} \boldsymbol{z}(t)).$$

**Theorem 5.1** *Under the hypotheses $H(\mathscr{B})$, $H(\boldsymbol{f})$, $H(j_n)$ and the following smallness condition*

$$m_{\mathscr{B}} - \delta m_{j_n} \|\gamma\|^2 > 0, \tag{5.5}$$

*Problem 5.4 admits a unique solution $\boldsymbol{z} \in \mathcal{W}$.*

***Proof*** First, we prove the existence of solution. Recall the Nemytski operator $\mathcal{B} : \mathcal{V} \to \mathcal{V}^*$ defined by (3.14). Here we introduce a new Nemytski operator $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}^*}$ defined by

$$\mathcal{N}\boldsymbol{v} = \{\boldsymbol{\xi} \in \mathcal{V}^* \mid \boldsymbol{\xi}(t) \in \delta e^{-\frac{t}{\delta}} \gamma^* \partial J(\gamma e^{\frac{t}{\delta}} \boldsymbol{v}(t))\} \quad \forall \boldsymbol{v} \in \mathcal{V}.$$

Using the operators $\mathcal{B}, \mathcal{N}$ and the derivative operator $L$ which is defined in Section 2, Problem 5.4 is reformulated as an inclusion problem:

$$\text{Find } \boldsymbol{z} \in D(L) \text{ such that } L\boldsymbol{z} + \mathcal{T}\boldsymbol{z} \ni \boldsymbol{g}, \tag{5.6}$$

where $\mathcal{T}\boldsymbol{z} = \mathcal{B}\boldsymbol{z} + \mathcal{N}\boldsymbol{z}$.

**Claim 1.** $\mathcal{T}$ is a bounded operator. Taking $\boldsymbol{v} \in \mathcal{V}$ and $\boldsymbol{v}^* \in \mathcal{T}\boldsymbol{v}$, we have $\boldsymbol{v}^* \in \mathcal{B}\boldsymbol{v} + \mathcal{N}\boldsymbol{v}$. Note that $\mathcal{B}$ is linear and bounded. For any $\boldsymbol{\xi} \in \mathcal{N}\boldsymbol{v}$, we have $\boldsymbol{\xi}(t) \in \delta e^{\frac{-t}{\delta}}\gamma^*\partial J(\gamma e^{\frac{t}{\delta}}\boldsymbol{v}(t))$ for a.e. $t \in I$. It follows from [29, Lemma 13] that $\|\boldsymbol{\xi}\|_{\mathcal{V}^*} \le \bar{c}_0 + \bar{c}_1\|\boldsymbol{v}\|_{\mathcal{V}}$ with $\bar{c}_0, \bar{c}_1 \ge 0$. Thus, we obtain $\|\boldsymbol{v}^*\|_{\mathcal{V}^*} \le \bar{c}_0 + (\bar{c}_1 + \|\mathscr{B}\|)\|\boldsymbol{v}\|_{\mathcal{V}}$, i.e., $\mathcal{T}$ is bounded.

**Claim 2.** $\mathcal{T}$ is coercive. Let $\boldsymbol{v} \in \mathcal{V}$ and $\boldsymbol{v}^* \in \mathcal{T}\boldsymbol{v}$, i.e., $\boldsymbol{v}^* = \mathcal{B}\boldsymbol{v} + \boldsymbol{\xi}$ with $\boldsymbol{\xi} \in \mathcal{N}\boldsymbol{v}$. From Lemma 3.1(i), we have

$$\langle \mathcal{B}\boldsymbol{v}, \boldsymbol{v}\rangle \ge m_{\mathscr{B}}\|\boldsymbol{v}\|^2_{\mathcal{V}}. \tag{5.7}$$

Since $\boldsymbol{\xi} \in \mathcal{N}\boldsymbol{v}$, we have $\boldsymbol{\xi}(t) = \gamma^*\boldsymbol{z}(t)$ with $\boldsymbol{z}(t) \in \delta e^{\frac{-t}{\delta}}\partial J(\gamma e^{\frac{t}{\delta}}\boldsymbol{v}(t))$. It follows from (5.2) and (5.3) that

$$\begin{aligned}\langle \boldsymbol{\xi}, \boldsymbol{v}\rangle &= \int_0^T \langle \boldsymbol{z}(t), \gamma\boldsymbol{v}(t)\rangle dt = \int_0^T [\langle \boldsymbol{z}(t) - \boldsymbol{\theta}(t), \gamma\boldsymbol{v}(t)\rangle + \langle \boldsymbol{\theta}(t), \gamma\boldsymbol{v}(t)\rangle]\, dt \\ &\ge -\delta m_{j_n}\|\gamma\|^2\|\boldsymbol{v}\|^2_{\mathcal{V}} - c_1\sqrt{T}\delta\|\gamma\|\|\boldsymbol{v}\|_{\mathcal{V}},\end{aligned} \tag{5.8}$$

where $\boldsymbol{\theta}(t) \in \delta e^{\frac{-t}{\delta}}\partial J(\mathbf{0}_{\mathcal{V}}(t))$. Consequently,

$$\langle \mathcal{T}\boldsymbol{v}, \boldsymbol{v}\rangle \ge (m_{\mathscr{B}} - \delta m_{j_n}\|\gamma\|^2)\|\boldsymbol{v}\|^2_{\mathcal{V}} - c_1\sqrt{T}\delta\|\gamma\|\|\boldsymbol{v}\|_{\mathcal{V}} \quad \forall\, \boldsymbol{v} \in \mathcal{V},$$

which implies that the operator $\mathcal{T}$ is coercive thanks to the smallness condition (5.5).

**Claim 3.** $\mathcal{T}$ is $L$-pseudomonotone ([23, Definition 3.62]). Firstly, we show that $\mathcal{N}\boldsymbol{v}$ is nonempty, bounded, convex and closed for any $\boldsymbol{v} \in \mathcal{V}$. According to [30, Proposition 2.1.2], $\partial J(\cdot)$ is a nonempty, convex and weakly compact subset of $Z^*$. Claim 1 implies that $\mathcal{N}\boldsymbol{v}$ is bounded. Hence, $\mathcal{N}\boldsymbol{v}$ is nonempty, bounded and convex. Let us show that the set $\mathcal{N}\boldsymbol{v}$ is closed. Let $\{\boldsymbol{\xi}_n\}$ be any sequence of $\mathcal{N}\boldsymbol{v}$ such that $\boldsymbol{\xi}_n \to \boldsymbol{\xi}$. It follows from [23, Theorem 2.39] that $\boldsymbol{\xi}_n(t) \to \boldsymbol{\xi}(t)$ in $V^*$ for a.e. $t \in I$, passing to a subsequence if necessary. Since $\boldsymbol{\xi}_n(t) \in \delta e^{-\frac{t}{\delta}}\gamma^*\partial J(\gamma e^{\frac{t}{\delta}}\boldsymbol{v}(t))$ for any $n \in \mathbb{N}$, a.e. $t \in I$ and the latter is a closed subset of $V^*$, we get that $\boldsymbol{\xi}(t) \in \delta e^{-\frac{t}{\delta}}\gamma^*\partial J(\gamma e^{\frac{t}{\delta}}\boldsymbol{v}(t))$ for a.e. $t \in I$. Therefore, $\boldsymbol{\xi} \in \mathcal{N}\boldsymbol{v}$ and $\mathcal{N}\boldsymbol{v}$ is closed.

Secondly, we verify that $\mathcal{N}$ is upper semicontinuous from $\mathcal{V}$ to $2^{\mathcal{V}^*}$, where $\mathcal{V}^*$ is endowed with the weak topology. According to [31, Proposition 4.14], it suffices to prove that if a set $D$ is weakly closed in $\mathcal{V}^*$, then the set $\mathcal{N}^-(D) = \{\boldsymbol{v} \in \mathcal{V} \mid \mathcal{N}\boldsymbol{v} \cap D \ne \emptyset\}$ is closed in $\mathcal{V}$. Let $\{\boldsymbol{v}_n\}$ be any sequence of $\mathcal{N}^-(D)$ such that $\boldsymbol{v}_n \to \boldsymbol{v}$. By using [23, Theorem 2.39], passing to a subsequence if necessary, we have $\boldsymbol{v}_n(t) \to \boldsymbol{v}(t)$ in $V$ for a.e. $t \in I$. Let $\boldsymbol{\xi}_n \in \mathcal{N}\boldsymbol{v}_n \cap D$ for $n \in \mathbb{N}$. Since $\{\boldsymbol{v}_n\}$ is bounded and $\mathcal{N}$ is a bounded mapping, the sequence $\{\boldsymbol{\xi}_n\}$ is bounded in $\mathcal{V}^*$. Thus, there exist a subsequence, still denoted by $\{\boldsymbol{\xi}_n\}$, such that

$$\boldsymbol{\xi}_n \rightharpoonup \boldsymbol{\xi} \;\text{ in }\; \mathcal{V}^*, \tag{5.9}$$

which implies $e^{\frac{\cdot}{\delta}}\boldsymbol{\xi}_n \rightharpoonup e^{\frac{\cdot}{\delta}}\boldsymbol{\xi}$ in $\mathcal{V}^*$ and $\boldsymbol{\xi} \in D$ by the weak closedness of $D$ in $\mathcal{V}^*$. Also there exists $\boldsymbol{z}_n \in L^2(I; Z^*)$ for $n \in \mathbb{N}$ such that

$$e^{\frac{t}{\delta}}\boldsymbol{\xi}_n(t) = \gamma^* \boldsymbol{z}_n(t), \quad \boldsymbol{z}_n(t) \in \delta\partial J(\gamma e^{\frac{t}{\delta}} \boldsymbol{v}_n(t)) \text{ for a.e. } t \in I. \tag{5.10}$$

Using (5.3), we deduce that

$$\|\boldsymbol{z}_n\|_{L^2(I;Z^*)} \leq \sqrt{2T}\delta c_1 + \sqrt{2}\delta e^{\frac{T}{\delta}} \|\gamma\| \|\boldsymbol{v}_n\|_{\mathcal{V}},$$

which implies $\{\boldsymbol{z}_n\}$ is bounded. Then passing to a subsequence if necessary, we have

$$\boldsymbol{z}_n \rightharpoonup \boldsymbol{z} \quad \text{in} \quad L^2(I; Z^*). \tag{5.11}$$

Since $\delta\partial J(\cdot)$ is upper semicontinuous from $X$ to the closed convex subsets of $Z^*$ endowed with the weak topology and $\gamma e^{\frac{\beta t}{1-\beta}} \boldsymbol{v}_n(t) \to \gamma e^{\frac{\beta t}{1-\beta}} \boldsymbol{v}(t)$ in $Z$ for a.e. $t \in I$, we see that (by using the convergence theorem [32, Theorem 1])

$$\boldsymbol{z}(t) \in \delta\partial J(\gamma e^{\frac{t}{\delta}} \boldsymbol{v}(t)) \quad \text{for a.e.} \quad t \in I.$$

Combining (5.9), (5.10) and (5.11), we get $e^{\frac{t}{\delta}}\boldsymbol{\xi}(t) = \gamma^* \boldsymbol{z}(t)$ for a.e. $t \in I$ which means $\boldsymbol{\xi} \in \mathcal{N}\boldsymbol{v}$. Therefore, we have $\boldsymbol{\xi} \in \mathcal{N}\boldsymbol{v} \cap D$, i.e., $\boldsymbol{v} \in \mathcal{N}^-(D)$. This shows that $\mathcal{N}^-(D)$ is closed in $\mathcal{V}$, which means that $\mathcal{N}$ is upper semicontinuous.

Thirdly, we verify the last condition of the definition of $L$-pseudomonotonicity ([23, Definition 3.62]). Let $\{\boldsymbol{v}_n\}$ be any sequence of $D(L)$ such that $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ in $\mathcal{W}$, $\boldsymbol{\xi}_n \in \mathcal{N}\boldsymbol{v}_n$ with $\boldsymbol{\xi}_n \rightharpoonup \boldsymbol{\xi}$ in $\mathcal{V}^*$, and $\limsup\langle \boldsymbol{\xi}_n, \boldsymbol{v}_n - \boldsymbol{v}\rangle \leq 0$. We see that

$$e^{\frac{t}{\delta}}\boldsymbol{\xi}_n(t) \in \gamma^* \boldsymbol{z}_n(t), \quad \boldsymbol{z}_n(t) \in \delta\partial J(\gamma e^{\frac{t}{\delta}} \boldsymbol{v}_n(t)) \quad \text{for a.e.} \quad t \in I.$$

Since $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ in $\mathcal{W}$, we get $\{e^{\frac{\cdot}{\delta}}\boldsymbol{v}_n\}$ is bounded in $\mathcal{W}$, which implies that $e^{\frac{\cdot}{\delta}}\boldsymbol{v}_n \rightharpoonup e^{\frac{\cdot}{\delta}}\boldsymbol{v}$ in $\mathcal{W}$. It follows from [33, Theorem 2.18] that the Nemytski operator $\tilde{\gamma} : \mathcal{W} \to L^2(I; Z)$ of the trace operator $\gamma$ is linear continuous and compact. Thus we have $\tilde{\gamma} e^{\frac{\cdot}{\delta}}\boldsymbol{v}_n \to \tilde{\gamma} e^{\frac{\cdot}{\delta}}\boldsymbol{v}$ in $L^2(I; Z)$. And we can find a subsequence, still denoted by $\{e^{\frac{\cdot}{\delta}}\boldsymbol{v}_n\}$, such that

$$\gamma e^{\frac{t}{\delta}} \boldsymbol{v}_n(t) \to \gamma e^{\frac{t}{\delta}} \boldsymbol{v}(t) \text{ in } X \text{ for a.e. } t \in I. \tag{5.12}$$

Note that $\{\boldsymbol{z}_n\}$ is bounded in $L^2(I; Z^*)$ (by (5.3)). Passing to a subsequence if necessary, we have

$$\boldsymbol{z}_n \rightharpoonup \boldsymbol{z} \quad \text{in} \quad L^2(I; Z^*). \tag{5.13}$$

We have shown that $e^{\frac{t}{\delta}}\boldsymbol{\xi}(t) = \gamma^* \boldsymbol{z}(t)$ for a.e. $t \in I$ (see the proof of the upper semicontinuity of $\mathcal{N}$). It follows from (5.12), (5.13), and the convergence theorem that

$$\boldsymbol{z}(t) \in \delta\partial J(\gamma e^{\frac{t}{\delta}} \boldsymbol{v}(t)) \text{ for a.e. } t \in I,$$

which means that $\boldsymbol{\xi} \in \mathcal{N}\boldsymbol{v}$. By (5.13),

$$\langle \boldsymbol{\xi}_n, \boldsymbol{v}_n \rangle = \int_0^T e^{-\frac{t}{\delta}} \langle e^{\frac{t}{\delta}} \boldsymbol{\xi}_n(t), \boldsymbol{v}_n(t) \rangle dt = \int_0^T \langle \gamma^* \boldsymbol{z}_n(t), e^{-\frac{t}{\delta}} \boldsymbol{v}_n(t) \rangle dt$$
$$= \langle \boldsymbol{z}_n, \tilde{\gamma} e^{-\frac{\cdot}{\delta}} \boldsymbol{v}_n \rangle \to \langle \boldsymbol{z}, \tilde{\gamma} e^{-\frac{\cdot}{\delta}} \boldsymbol{v} \rangle = \int_0^T \langle \boldsymbol{z}(t), \gamma e^{-\frac{t}{\delta}} \boldsymbol{v}(t) \rangle dt = \langle \xi, \boldsymbol{v} \rangle.$$

Therefore, the mapping $\mathcal{N}$ is $L$-pseudomonotone.

It is clear that the operator $\mathcal{B}$ is $L$-pseudomonotone since $\mathcal{B}$ is linear, continuous and strongly monotone. Therefore, $\mathcal{T} = \mathcal{B} + \mathcal{N}$ is $L$-pseudomonotone (by [34, Proposition 2]).

According to [23, Theorem 3.63], **Claims 1-3** indicate that the problem (5.6) (equivalently, Problem 5.4) has a solution.

It remains to show the uniqueness. Suppose that Problem 5.4 admits two solutions $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$. Then there exist $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ such that for a.e. $t \in I$ and $i = 1, 2$,

$$\boldsymbol{g}_i(t) = \boldsymbol{z}_i'(t) + B\boldsymbol{z}_i(t) + \boldsymbol{\xi}_i(t), \tag{5.14}$$

where $\boldsymbol{\xi}_i(t) \in \delta e^{\frac{-t}{\delta}} \gamma^* \partial J(\gamma e^{\frac{t}{\delta}} \boldsymbol{z}_i(t))$ and $\boldsymbol{z}_i(0) = \mathbf{0}$. Testing (5.14) by $\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)$, one can derive that, by the strong monotonicity of $B$ and (5.2):

$$(m_{\mathscr{B}} - \delta m_{j_n} \|\gamma\|^2) \int_0^T \|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\|_V^2 dt \le 0.$$

Hence, the solution of Problem 5.4 is unique due to the smallness condition (5.5). □

### 5.3.2 Well-posedness of Problem 5.1

**Theorem 5.2** *Under the assumptions $H(\mathscr{A})$, $H(\mathscr{B})$, $H(\boldsymbol{f})$, $H(j_n)$ and the smallness condition (5.5), we have:*

(i) *Problem 5.3 has a unique solution $\boldsymbol{z} \in \mathcal{W}$;*

(ii) *Problem 5.1 has a unique solution $\boldsymbol{u} \in C(I; V)$ satisfying $\boldsymbol{u}' \in \mathcal{W}$.*

***Proof*** We start by proving (i). Given $\boldsymbol{\eta} \in \mathcal{V}^*$, we consider the following auxiliary problem.

**Problem 5.5** Find $\boldsymbol{z}_\eta : I \to V$ with $\boldsymbol{z}_\eta' : I \to V^*$ such that $\boldsymbol{z}(0) = \mathbf{0}$ and for all $\boldsymbol{v} \in V$, a.e. $t \in I$,

$$\langle \boldsymbol{z}_\eta'(t) + B\boldsymbol{z}_\eta(t), \boldsymbol{v} \rangle + \delta e^{-\frac{t}{\delta}} J^\circ(\gamma e^{\frac{t}{\delta}} \boldsymbol{z}_\eta(t); \gamma \boldsymbol{v}) \ge \langle \boldsymbol{g}(t) - \boldsymbol{\eta}(t), \boldsymbol{v} \rangle. \tag{5.15}$$

Problem 5.5 is equivalent to the following evolutionary inclusion problem:

Find $\boldsymbol{z}_\eta : I \to V$ with $\boldsymbol{z}_\eta' : I \to V^*$ such that, $\boldsymbol{z}_\eta(0) = \mathbf{0}$ and

$$\boldsymbol{g}(t)-\boldsymbol{\eta}(t)\in \boldsymbol{z}_\eta'(t)+B\boldsymbol{z}_\eta(t)+\delta e^{-\frac{t}{\delta}}\gamma^*\partial J(\gamma e^{\frac{t}{\delta}}\boldsymbol{z}_\eta(t)) \quad \text{for a.e.} \quad t\in(0,T). \tag{5.16}$$

It follows from Theorem 5.1 that (5.16) (also Problem 5.5) has a unique solution. To show the unique solvability of Problem 5.3, we define a mapping $\bar{\Lambda}:\mathcal{V}^*\to\mathcal{V}^*$ by

$$\bar{\Lambda}\boldsymbol{\eta}=S\boldsymbol{z}_\eta \quad \text{for all} \quad \boldsymbol{\eta}\in\mathcal{V}^*,$$

where $\boldsymbol{z}_\eta$ is the unique solution to Problem 5.5 associated with $\boldsymbol{\eta}$. We need show that the operator $\bar{\Lambda}$ has a unique fixed point. To this end, let $\boldsymbol{z}_i=\boldsymbol{z}_{\eta_i}$ be the solution of Problem 5.5 associated with $\boldsymbol{\eta}_i\in\mathcal{V}^*$ $(i=1,2)$. Setting $\boldsymbol{e}=\boldsymbol{z}_1-\boldsymbol{z}_2$, it follows from (5.15) that

$$\begin{aligned}&\langle \boldsymbol{e}'(t),\boldsymbol{e}(t)\rangle+\langle B\boldsymbol{e}(t),\boldsymbol{e}(t)\rangle\\ &\quad\le \delta e^{\frac{-2t}{\delta}}J^\circ(\gamma e^{\frac{t}{\delta}}\boldsymbol{z}_1(t);\gamma e^{\frac{t}{\delta}}(\boldsymbol{e}(t)))+\delta e^{\frac{-2t}{\delta}}J^\circ(\gamma e^{\frac{t}{\delta}}\boldsymbol{z}_2(t);\gamma e^{\frac{t}{\delta}}(\boldsymbol{e}(t)))+\langle\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t),\boldsymbol{e}(t)\rangle.\end{aligned}$$

Integrating the above inequality on (0, $t$), and by using the strong monotonicity of $B$, (5.2) and (5.5), we have

$$\int_0^t\|\boldsymbol{e}(s)\|_V^2ds\le\frac{1}{(m_{\mathscr{B}}-\delta m_{j_n}\|\gamma\|^2)^2}\int_0^t\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V^2ds,$$

which, together with (3.12), implies

$$\|\bar{\Lambda}\boldsymbol{\eta}_1(t)-\bar{\Lambda}\boldsymbol{\eta}_2(t)\|_{V^*}^2\le\frac{L_S^2t}{(m_{\mathscr{B}}-\delta m_{j_n}\|\gamma\|^2)^2}\int_0^t\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V^2ds,$$

where $L_S=\frac{1}{\delta}(\frac{1}{\delta}+\|\mathscr{B}\|)+\|A\|$. Hence, $\bar{\Lambda}$ has a unique fixed point (by [28, Theorem 2.3]), which says that Problem 5.3 has a unique solution.

Next, we proceed to prove (ii). It is easy to check that the solution to Problem 5.3 also solves Problems 5.2 and 5.1. Similar to the proof of Theorem 3.2, we can obtain the uniqueness of solution of Problem 5.1. □

The following result bounds the solution $\boldsymbol{u}$ in relation to the parameter $\delta$.

**Theorem 5.3** *Under the same assumptions of Theorem 5.2, the solution* $\boldsymbol{u}$ *of Problem 5.1 satisfies*

$$\begin{aligned}&\|\boldsymbol{u}\|_{L^\infty(I;H)}^2+\delta\|\boldsymbol{u}'\|_{L^2(I;H)}^2+\delta m_{\mathscr{A}}\|\boldsymbol{u}\|_{\mathcal{V}}^2\\&\qquad+(2\delta)^{-1}(m_{\mathscr{B}}-\delta m_{j_n}\|\gamma\|^2)T\|\boldsymbol{u}+\delta\boldsymbol{u}'\|_{\mathcal{V}}^2\le C(\delta,T,\boldsymbol{f}),\end{aligned} \tag{5.17}$$

*where* $C(\delta,T,\boldsymbol{f}):=\beta_\delta(e^{2\beta_\delta^{-1}T}-1)\frac{4\delta c_1^2T\|\gamma\|^2+4\delta\|\boldsymbol{f}\|_{\mathcal{V}^*}^2}{m_{\mathscr{B}}-\delta m_{j_n}\|\gamma\|^2}$ *and*

$\beta_\delta:=\max\{\delta,4\|\mathscr{B}\|^{-2}(\delta^2m_{\mathscr{A}}(m_{\mathscr{B}}-\delta m_{j_n}\|\gamma\|^2))\}.$

***Proof*** Taking $\boldsymbol{v}=-\boldsymbol{u}(t)-\delta\boldsymbol{u}'(t)$ in (5.4), we get

$$\langle \boldsymbol{u}''(t) + A\boldsymbol{u}(t) + B\boldsymbol{u}'(t), \boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\rangle \le J^\circ(\gamma(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)); -\gamma(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))) + \langle \boldsymbol{f}(t), \boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\rangle. \tag{5.18}$$

By using hypothesis $H(\mathscr{B})$,

$$\begin{aligned}\langle B\boldsymbol{u}'(t), \boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\rangle &= \frac{1}{\delta}\langle B(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)), \boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\rangle - \frac{1}{\delta}\langle B\boldsymbol{u}(t), \boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\rangle \\ &\ge \frac{m_{\mathscr{B}}}{\delta}\|\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\|_V^2 - \frac{1}{\delta}\langle B\boldsymbol{u}(t), \boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\rangle.\end{aligned}$$

In view of (5.2) and (5.3), we have

$$\begin{aligned}&J^\circ(\gamma(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)); -\gamma(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))) \\ &\quad = J^\circ(\gamma(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)); -\gamma(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))) \\ &\qquad + J^\circ(\boldsymbol{0}_X; \gamma(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))) - J^\circ(\boldsymbol{0}_X; \gamma(\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t))) \\ &\quad \le m_{j_n}\|\gamma\|^2\|\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\|_V^2 + c_1\|\gamma\|\|\boldsymbol{u}(t) + \delta\boldsymbol{u}'(t)\|_V.\end{aligned}$$

Integrating (5.18) on (0, *t*) with $t \in (0, T]$ and applying the above two inequalities, together with (3.33a)-(3.34b), we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|_H^2 - \int_0^t \|\boldsymbol{u}'(s)\|_H^2 ds + \frac{\delta}{2}\|\boldsymbol{u}'(t)\|_H^2 + m_{\mathscr{A}}\int_0^t \|\boldsymbol{u}(s)\|_V^2 ds \\ &\qquad + \frac{\delta}{2}m_{\mathscr{A}}\|\boldsymbol{u}(t)\|_V^2 + \frac{m_{\mathscr{B}} - \delta m_{j_n}\|\gamma\|^2}{4\delta}\int_0^t \|\boldsymbol{u}(s) + \delta\boldsymbol{u}'(s)\|_V^2 ds \\ &\le \int_0^t \frac{4\delta c_1^2\|\gamma\|^2 + 4\delta\|\boldsymbol{f}(s)\|_{V^*}^2}{m_{\mathscr{B}} - \delta m_{j_n}\|\gamma\|^2} ds + \frac{4\|\mathscr{B}\|^2}{\delta(m_{\mathscr{B}} - \delta m_{j_n}\|\gamma\|^2)}\int_0^t \|\boldsymbol{u}(s)\|_V^2 ds.\end{aligned}$$

Set

$$\begin{aligned}\bar{a}(t) &:= \|\boldsymbol{u}(t)\|_H^2 + \delta\int_0^t \|\boldsymbol{u}'(s)\|_H^2 ds + 2m_{\mathscr{A}}\int_0^t\int_0^s \|\boldsymbol{u}(r)\|_V^2 dr\, ds \\ &\qquad + \delta m_{\mathscr{A}}\int_0^t \|\boldsymbol{u}(s)\|_V^2 ds + \frac{m_{\mathscr{B}} - \delta m_{j_n}\|\gamma\|^2}{2\delta}\int_0^t\int_0^s \|\boldsymbol{u}(r) + \delta\boldsymbol{u}'(r)\|_V^2 dr\, ds, \\ \bar{c}(t) &:= \int_0^t \frac{4\delta c_1^2\|\gamma\|^2 + 4\delta\|\boldsymbol{f}(s)\|_{V^*}^2}{m_{\mathscr{B}} - \delta m_{j_n}\|\gamma\|^2} ds.\end{aligned}$$

From the previous inequality, we get

$$\frac{1}{2}\frac{d}{dt}\bar{a}(t) \le \beta_\delta^{-1}\bar{a}(t) + \bar{c}(t), \quad \bar{a}(0) = 0,$$

where $\beta_\delta := \min\{\delta, 4\|\mathscr{B}\|^{-2}(\delta^2 m_{\mathscr{A}}(m_{\mathscr{B}} - \delta m_{j_n}\|\gamma\|^2))\}$. Applying the Gronwall inequality, we conclude (5.17). □

**Remark 5.1** *The proof provided above does not take into account the case where $\mathscr{B}$ is symmetric. In contrast to Theorems 3.3 and 4.3, the smallness condition (5.5) is essential for the bounds on $\boldsymbol{u}$.*

## 6 Conclusion

In this paper, we establish the well-posedness of the viscoelastic contact problems under the modified Signorini, Tresca-friction, and Clarke-subdifferential type boundary conditions, respectively. Our analysis not only advances the mathematical theory of contact mechanics but also facilitates the connections and differences between conventional contact conditions concerning the displacement and the velocity. Future research will focus on developing numerical methods for solving the viscoelastic model under these modified boundary conditions.

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**Data Availability** Data sharing is not applicable as no datasets were generated or analyzed in this study.

### Declarations

**Conflict of interest** The authors declare that they have no Conflict of interest.

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