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Analysis and finite element solution of a Navier–Stokes hemivariational inequality for incompressible fluid flows with damping

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ABSTRACT

This paper provides a well-posedness analysis and a mixed finite element method for a hemivariational inequality of the stationary Navier–Stokes equations with a nonlinear damping term. The Navier–Stokes hemivariational inequality describes a steady incompressible fluid flow subject to a nonsmooth slip boundary condition of friction type. The well-posedness of the Navier–Stokes hemivariational inequality is established by constructing two auxiliary problems and applying Banach fixed point arguments twice. Mixed finite element methods are introduced to solve the problem, and error estimates for the solutions are derived. The error estimates are of optimal order for low-order mixed element pairs under suitable solution regularity assumptions. An efficient iterative algorithm is presented, and numerical results are provided to verify the theoretical analysis.

1. Introduction

To model slip or leak phenomena for real-world applications of fluid flows, Fujita proposed slip or leak boundary conditions of friction type for boundary value problems of steady motions of viscous incompressible fluids in the early 1990s [1,2]. Numerous studies have followed in this area. Further analysis of such problems, including numerical approximation and regularity of the solutions, was investigated in [3–7], and the conditions were also applied in non-Newtonian fluids [8]. Numerical methods for solving these types of problems were studied in [9–15]. In these studies, the slip or leak boundary conditions are modeled by nonsmooth monotone relations, and the corresponding weak formulations governed by the Stokes or Navier–Stokes equations are variational inequalities. When the boundary conditions involve nonsmooth non-monotone relations, the weak formulations become hemivariational inequalities, and they are studied in a number of papers, e.g., [16–21].

Hemivariational inequalities, or more generally, variational–hemivariational inequalities, were first introduced by Panagiotopoulos in early 1980s [22]. Early comprehensive references on hemivariational inequalities include [23,24], while recent studies on the mathematical theories of this subject can be found in [25–28]. In these references, abstract theory of pseudomonotone operators is applied for solution existence. An alternative approach on variational–hemivariational inequalities, more suitable for researchers in applied mathematics, sciences and engineering, is started in [29,30] and is well documented in [31]. Since there is no analytic solution formula to solve variational–hemivariational inequalities, numerical methods are required to solve the problems. The book [32] investigates finite element approximations of hemivariational inequalities, discussing the convergence

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of the numerical methods and presenting solution algorithms. Since the first optimal order error estimate for the numerical solution of a hemivariational inequality was derived in [33], many articles have been published on error analysis and optimal order error estimation for solving a variety of variational–hemivariational inequalities. For a comprehensive summary of the numerical analysis of hemivariational inequalities, the reader is referred to the survey papers [34,35].

Damping effects arise from resistance to the motion of flows. Many physical processes, such as porous media flows, drag or friction effects, and various dissipative mechanisms, involve damping phenomena (cf. [36,37]). The stationary and non-stationary Navier–Stokes equations with damping have been studied in several papers. An initial–boundary value problem of the Navier–Stokes equations with damping over the entire spatial space is studied in [38–40]. Analysis and numerical methods for a stationary Navier–Stokes variational inequality with damping can be found in [41,42]. In this paper, we study a Navier–Stokes hemivariational inequality for a viscous incompressible fluid flow with damping effect that is subject to a nonsmooth, not necessarily monotone, slip boundary condition of friction type. This represents a novel mathematical model designed to describe incompressible fluid flows influenced by damping effects, capturing a wider range of physical phenomena. Unlike most existing works that rely on abstract surjectivity results for pseudomonotone operators to prove the existence of solutions to hemivariational inequalities, our method is based on the well-posedness result from Ref. [43] on a Stokes hemivariational inequality with damping effect and follows the approach presented in [31, Chapter 8]. We construct two auxiliary problems and apply fixed-point arguments twice to prove the well-posedness of the Navier–Stokes hemivariational inequality. Only basic notions and results from functional analysis are used throughout the proof. The mixed finite element method is used to solve the Navier–Stokes hemivariational inequality and the derived error estimate is of optimal order for low-order mixed element pairs satisfying the discrete inf-sup condition under suitable solution regularity assumptions. An efficient iterative algorithm is further presented to solve the problem, and numerical results are provided to validate the theoretical analysis.

The rest of the paper is organized as follows. In Section 2, we describe the physical setting of the fluid, present the corresponding Navier–Stokes hemivariational inequality, and provide the necessary preliminaries. In Section 3, we show an existence and uniqueness result for the Navier–Stokes hemivariational inequality. In Section 4, we apply the mixed finite element method to solve the Navier–Stokes hemivariational inequality and derive error estimates for the finite element solutions. In Section 5, we introduce solution algorithms to solve the discrete Navier–Stokes hemivariational inequality and a related discrete Navier–Stokes variational inequality. In Section 6, we report numerical simulation results.

2. The Navier–Stokes hemivariational inequality

We first introduce some notation. Let $\mathbb{R}^d$ be the d-dimensional real Euclidean space, and let $\mathbb{S}^d$ be the space of second-order symmetric tensors on $\mathbb{R}^d$. In $\mathbb{R}^d$ and $\mathbb{S}^d$, the standard inner products and the induced norms are

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \qquad |\boldsymbol{v}|_{\mathbb{R}^d} = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u}=(u_i),\ \boldsymbol{v}=(v_i)\in\mathbb{R}^d,$$
$$\boldsymbol{\sigma}:\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \qquad |\boldsymbol{\sigma}|_{\mathbb{S}^d} = (\boldsymbol{\sigma}:\boldsymbol{\sigma})^{1/2} \quad \text{for all } \boldsymbol{\sigma}=(\sigma_{ij}),\ \boldsymbol{\tau}=(\tau_{ij})\in\mathbb{S}^d.$$

The summation convention over a repeated index is adopted, e.g., $u_i v_i$ stands for $u_1 v_1 + \cdots + u_d v_d$.

We consider the Navier–Stokes equations in a domain Ω in $\mathbb{R}^d$, $d = 2$ or 3. Assume the boundary $\Gamma = \partial\Omega$ is Lipschitz continuous. The stationary Navier–Stokes equations with damping for the velocity $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and pressure $p : \Omega \to \mathbb{R}$ with given external force $\boldsymbol{f} : \Omega \to \mathbb{R}^d$ are

$$-\operatorname{div}(2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \alpha\,|\boldsymbol{u}|^{r-2}\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.1}$$
$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega. \tag{2.2}$$

Here $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ denotes the deformation rate tensor, $\mu > 0$ is the kinematic viscosity coefficient. The damping effect is represented by the term $\alpha\,|\boldsymbol{u}|^{r-2}\boldsymbol{u}$ in (2.1), where $\alpha > 0$ and $r \geq 2$ are two constants. The parameter α is known as the Forchheimer coefficient, which accounts for the inertial effects of the fluid. The Forchheimer law extends Darcy's law by incorporating the additional nonlinear term to address the effects that become significant at higher flow velocities. We assume

$$2 \leq r < \infty \text{ if } d = 2, \quad 2 \leq r \leq 6 \text{ if } d = 3. \tag{2.3}$$

Then, by the Sobolev embedding theorem [44], $H^1(\Omega) \hookrightarrow L^r(\Omega)$. We split the boundary Γ to two disjoint measurable parts: $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, where Γ_0 and Γ_1 are relatively open, $|\Gamma_0| > 0$, $|\Gamma_1| > 0$, and $\Gamma_0 \cap \Gamma_1 = \emptyset$. We comment on the case $|\Gamma_1| = 0$ at the end of Section 3. Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d)^T$ represent the unit outward normal vector on the boundary Γ. The normal and tangential components of a vector field $\boldsymbol{u}$ on Γ are given by $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$, respectively. The normal and tangential components of an $\mathbb{S}^d$-valued field $\boldsymbol{\sigma}$ on the boundary are $\sigma_\nu = \boldsymbol{\nu}\cdot\boldsymbol{\sigma}\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$.

Eqs. (2.1)–(2.2) are supplemented by the following boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0, \tag{2.4}$$
$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_1. \tag{2.5}$$

Here, $\boldsymbol{\sigma}_\tau$ is the tangential component of the stress tensor $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})$, $\boldsymbol{I}$ being the identity matrix. The super-potential $\psi : \mathbb{R}^d \to \mathbb{R}$ is assumed to be locally Lipschitz continuous, and $\partial\psi$ is the subdifferential of ψ in the sense of Clarke, a concept briefly reviewed below. The relation (2.5) is known as a slip boundary condition. The first part of the condition (2.5) indicates that there is no fluid leak on Γ_1. The second part of (2.5) specifies a friction condition for the friction $\boldsymbol{\sigma}_\tau$ with respect to the tangential velocity $\boldsymbol{u}_\tau$. When the super-potential ψ is convex, the weak formulation takes the form of a variational inequality. The specific case where $\psi(\boldsymbol{v}_\tau) = g\,|\boldsymbol{v}_\tau|$, with $g > 0$, was previously studied in [42]. In this paper, we do not assume the convexity of ψ, and the weak formulation of the problem (2.1)–(2.5) is a hemivariational inequality.

We now recall the definition of the generalized directional derivative and generalized subdifferential in the sense of Clarke [45]. For a locally Lipschitz continuous functional $\Psi : V \to \mathbb{R}$ defined on a real Banach space V, the generalized (Clarke) directional derivative of Ψ at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u;v) := \limsup_{w\to u,\,\lambda\downarrow 0} \frac{\Psi(w+\lambda v) - \Psi(w)}{\lambda},$$

whereas the generalized subdifferential of Ψ at $u \in V$ is

$$\partial\Psi(u) := \left\{\eta \in V^* \mid \Psi^0(u;v) \ge \langle \eta, v\rangle\ \forall\, v \in V\right\}.$$

We refer to [45] for the basic properties of the generalized directional derivative and the generalized subdifferential, and we present some fundamental ones needed for this paper.

Proposition 2.1. *Let V be a Banach space.*

(i) *If $\Psi : V \to \mathbb{R}$ is locally Lipschitz continuous and convex, then the subdifferential $\partial\Psi(u)$ at any $u \in V$ in the sense of Clarke coincides with the convex subdifferential $\partial\Psi(u)$.*

(ii) *Let $\Psi : V \to \mathbb{R}$ be locally Lipschitz continuous. Then $\partial(\lambda\Psi)(u) = \lambda\,\partial\Psi(u)$ for all $\lambda \in \mathbb{R}$ and all $u \in V$. Moreover, Ψ^0 is positively homogeneous and subadditive, i.e.,*

$$\Psi^0(u;\lambda v) = \lambda\Psi^0(u;v) \quad \forall\,\lambda \ge 0,\ u, v \in V,$$
$$\Psi^0(u;v_1+v_2) \le \Psi^0(u;v_1) + \Psi^0(u;v_2) \quad \forall\, u, v_1, v_2 \in V.$$

(iii) *Let $\Psi_1, \Psi_2 : V \to \mathbb{R}$ be locally Lipschitz functions. Then the inclusion*

$$\partial(\Psi_1+\Psi_2)(u) \subseteq \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall\, u \in V \tag{2.6}$$

holds, or equivalently,

$$(\Psi_1+\Psi_2)^0(u;v) \le \Psi_1^0(u;v) + \Psi_2^0(u;v) \quad \forall\, u, v \in V. \tag{2.7}$$

To present the weak formulation of the problem, we introduce some function spaces. For the velocity variable, let

$$V = \left\{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0,\ v_\nu = 0 \text{ on } \Gamma_1\right\}. \tag{2.8}$$

Since $|\Gamma_0| > 0$, Korn's inequality holds (cf. [46, p. 79]): for a constant $c_e > 0$ depending only on Ω and Γ_0,

$$\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)} \le c_e\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)} \quad \forall\,\boldsymbol{v} \in V. \tag{2.9}$$

Consequently, V is a Hilbert space with the inner product $(\boldsymbol{u},\boldsymbol{v})_V := (\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^d)}$ and the induced norm $\|\cdot\|_V = \|\boldsymbol{\varepsilon}(\cdot)\|_{L^2(\Omega;\mathbb{S}^d)}$ is equivalent to the standard $H^1(\Omega;\mathbb{R}^d)$-norm over V. The following trace inequality holds

$$\|\boldsymbol{v}_\tau\|_{L^2(\Gamma_1;\mathbb{R}^d)} \le \lambda_0^{-1/2}\|\boldsymbol{v}\|_V \quad \forall\,\boldsymbol{v} \in V, \tag{2.10}$$

where $\lambda_0 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \lambda\int_{\Gamma_1} \boldsymbol{u}_\tau\cdot\boldsymbol{v}_\tau\,ds \quad \forall\,\boldsymbol{v} \in V. \tag{2.11}$$

We further introduce two subspaces of V:

$$V_{\rm div} = \{\boldsymbol{v} \in V \mid {\rm div}\boldsymbol{v} = 0 \text{ a.e. in } \Omega\},$$
$$V_0 = H_0^1\left(\Omega;\mathbb{R}^d\right).$$

For the pressure variable, we use the space

$$Q = L_0^2(\Omega) = \left\{q \in L^2(\Omega) \mid \int_\Omega q\,dx = 0\right\}. \tag{2.12}$$

Define the following forms:

$$a(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = a_0(\boldsymbol{u},\boldsymbol{v}) + a_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) \quad \forall\,\boldsymbol{w},\boldsymbol{u},\boldsymbol{v} \in V, \tag{2.13}$$

$$a_0(\boldsymbol{u},\boldsymbol{v}) = 2\mu\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx \quad \forall\,\boldsymbol{u},\boldsymbol{v} \in V, \tag{2.14}$$

$$a_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \alpha\int_\Omega |\boldsymbol{w}|^{r-2}\boldsymbol{u}\cdot\boldsymbol{v}\,dx \quad \forall\,\boldsymbol{w},\boldsymbol{u},\boldsymbol{v} \in V, \tag{2.15}$$

$$d(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (\boldsymbol{w}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\,dx \quad \forall\,\boldsymbol{w},\boldsymbol{u},\boldsymbol{v} \in V, \tag{2.16}$$

$$b(\boldsymbol{v}, q) = -\int_\Omega q \,\mathrm{div}\boldsymbol{v}\, dx \quad \forall\, \boldsymbol{v} \in V,\ q \in Q, \tag{2.17}$$

and make the assumption throughout this paper that $\boldsymbol{f} \in V^*$. Obviously, the bilinear form $a_0(\cdot,\cdot) : V \times V \to \mathbb{R}$, bilinear form $b(\cdot,\cdot) : V \times Q \to \mathbb{R}$ and trilinear form $d(\cdot;\cdot,\cdot) : V \times V \times V \to \mathbb{R}$ are bounded. Moreover, $a_0(\cdot,\cdot)$ is coercive on V:

$$a_0(\boldsymbol{v}, \boldsymbol{v}) = 2\mu \|\boldsymbol{v}\|_V^2 \quad \forall\, \boldsymbol{v} \in V. \tag{2.18}$$

The following inf-sup condition holds [47]: for a constant $\beta_0 > 0$,

$$\beta_0 \|q\|_{L^2(\Omega)} \le \sup_{\boldsymbol{v}\in V_0} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_V} \quad \forall\, q \in Q. \tag{2.19}$$

Concerning the trilinear form $d(\cdot,\cdot,\cdot)$,

$$d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -d(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \quad \text{and} \quad d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{u} \in V_{\mathrm{div}},\ \boldsymbol{v}, \boldsymbol{w} \in V. \tag{2.20}$$

We use $c_d > 0$ for the boundedness constant:

$$|d(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w})| \le c_d \|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V \|\boldsymbol{w}\|_V \quad \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V.$$

For r in the range specified in (2.3), we have [43]

$$\left|a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v})\right| \le c\, \|\boldsymbol{u}\|_V^{r-1} \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{2.21}$$

$$\left|a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{w}) - a_1(\boldsymbol{v}; \boldsymbol{v}, \boldsymbol{w})\right| \le c \left(\|\boldsymbol{u}\|_{L^r(\Omega)}^{r-2} + \|\boldsymbol{v}\|_{L^r(\Omega)}^{r-2} \right) \|\boldsymbol{u} - \boldsymbol{v}\|_V \|\boldsymbol{w}\|_V \quad \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V. \tag{2.22}$$

Inequality (2.22) can be replaced with [43]

$$\left|a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{w}) - a_1(\boldsymbol{v}; \boldsymbol{v}, \boldsymbol{w})\right| \le c \left(\|\boldsymbol{u}\|_V^{r-2} + \|\boldsymbol{v}\|_V^{r-2} \right) \|\boldsymbol{u} - \boldsymbol{v}\|_V \|\boldsymbol{w}\|_V \quad \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V. \tag{2.23}$$

The following lemma will be used later in this paper.

Lemma 2.2. *([48, Section 5.3]) For $r \ge 2$,*

$$\left(|\boldsymbol{\eta}|^{r-2}\boldsymbol{\eta} - |\boldsymbol{\xi}|^{r-2}\boldsymbol{\xi}\right) \cdot (\boldsymbol{\eta} - \boldsymbol{\xi}) \ge 0 \quad \forall\, \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d, \tag{2.24}$$

$$\left||\boldsymbol{\eta}|^{r-2}\boldsymbol{\eta} - |\boldsymbol{\xi}|^{r-2}\boldsymbol{\xi}\right| \le c\, (|\boldsymbol{\eta}| + |\boldsymbol{\xi}|)^{r-2}\, |\boldsymbol{\eta} - \boldsymbol{\xi}| \quad \forall\, \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d. \tag{2.25}$$

Regarding the super-potential $\psi : \mathbb{R}^d \to \mathbb{R}$, we assume the following hypothesis:
$H(\psi)$. $\psi : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz, and there exist constants $\alpha_\psi, c_0, c_1 \ge 0$ such that

$$\psi^0(\boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + \psi^0(\boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \le \alpha_\psi |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d, \tag{2.26}$$

$$|\boldsymbol{\eta}| \le c_0 + c_1 |\boldsymbol{\xi}| \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d, \boldsymbol{\eta} \in \partial\psi(\boldsymbol{\xi}). \tag{2.27}$$

Condition (2.26) is known as a relaxed monotonicity condition [27] and can be written equivalently as

$$(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge -\alpha_\psi |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \quad \forall\, \boldsymbol{\xi}_i \in \mathbb{R}^d,\ \boldsymbol{\eta}_i \in \partial\psi(\boldsymbol{\xi}_i),\ i = 1, 2. \tag{2.28}$$

Combining (2.10) and (2.26), we have, for $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$,

$$\begin{aligned} \int_{\Gamma_1} \left[\psi^0(\boldsymbol{v}_{1,\tau}; \boldsymbol{v}_{2,\tau} - \boldsymbol{v}_{1,\tau}) + \psi^0(\boldsymbol{v}_{2,\tau}; \boldsymbol{v}_{1,\tau} - \boldsymbol{v}_{2,\tau})\right] ds &\le \alpha_\psi \int_{\Gamma_1} |\boldsymbol{v}_{1,\tau} - \boldsymbol{v}_{2,\tau}|^2 ds \\ &\le \alpha_\psi \lambda_0^{-1} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2. \end{aligned} \tag{2.29}$$

Examples of non-convex functions ψ satisfying $H(\psi)$ can be found in [34, pp. 186—187].

By a standard procedure, we obtain the following weak formulation for the problem (2.1)–(2.5).

Problem 2.3. Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$a(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \int_{\Gamma_1} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \ge \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in V, \tag{2.30}$$

$$b(\boldsymbol{u}, q) = 0 \quad \forall\, q \in Q. \tag{2.31}$$

We comment that in the special case of a convex ψ, the weak formulation of the problem takes the form of a variational inequality.

Problem 2.4. Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} &a(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + d(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + b(\boldsymbol{v} - \boldsymbol{u}, p) + \int_{\Gamma_1} \psi(\boldsymbol{v}_\tau)\, ds - \int_{\Gamma_1} \psi(\boldsymbol{u}_\tau)\, ds \\ &\qquad \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle \quad \forall\, \boldsymbol{v} \in V, \end{aligned} \tag{2.32}$$

$$b(\boldsymbol{u}, q) = 0 \quad \forall\, q \in Q. \tag{2.33}$$

3. Well-posedness

In this section, we explore the well-posedness of Problem 2.3. As a preparation, we first introduce a boundedness result on any solution of Problem 2.3. We will assume

$$\alpha_\psi < 2\,\mu\,\lambda_0, \tag{3.1}$$

and define a subset of the space V:

$$K_f = \left\{ \boldsymbol{v} \in V_{\text{div}} \mid \|\boldsymbol{v}\|_V \le M_f \right\}, \tag{3.2}$$

where

$$M_f = \frac{c_0 \lambda_0^{-1/2} |\Gamma_1|^{1/2} + \|\boldsymbol{f}\|_{V^*}}{2\,\mu - \alpha_\psi \lambda_0^{-1}}, \tag{3.3}$$

Lemma 3.1. *If Problem 2.3 has a solution $(\boldsymbol{u}, p) \in V \times Q$ and (3.1) holds, then*

$$\|\boldsymbol{u}\|_V \le M_f. \tag{3.4}$$

Proof. Let $\boldsymbol{v} = -\boldsymbol{u}$ in (2.30) to get

$$a_0(\boldsymbol{u}, \boldsymbol{u}) + a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}) \le -d(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}) - b(\boldsymbol{u}, p) + \int_{\Gamma_1} \psi^0(\boldsymbol{u}_\tau; -\boldsymbol{u}_\tau)\, ds + \langle \boldsymbol{f}, \boldsymbol{u} \rangle.$$

Note that $\boldsymbol{u} \in V_{\text{div}}$. It follows known from (2.15), (2.20) and (2.31) that $a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}) \ge 0$, $d(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}) = 0$ and $b(\boldsymbol{u}, p) = 0$. Hence we derive from the previous inequality that

$$a_0(\boldsymbol{u}, \boldsymbol{u}) \le \int_{\Gamma_1} \psi^0(\boldsymbol{u}_\tau; -\boldsymbol{u}_\tau)\, ds + \langle \boldsymbol{f}, \boldsymbol{u} \rangle. \tag{3.5}$$

Write

$$\psi^0(\boldsymbol{u}_\tau; -\boldsymbol{u}_\tau) = \left[\psi^0(\boldsymbol{u}_\tau; -\boldsymbol{u}_\tau) + \psi^0(\boldsymbol{0}; \boldsymbol{u}_\tau) \right] - \psi^0(\boldsymbol{0}; \boldsymbol{u}_\tau).$$

Apply $H(\psi)$ to bound the two parts of the right side of the above equality to obtain

$$\psi^0(\boldsymbol{u}_\tau; -\boldsymbol{u}_\tau) \le \alpha_\psi |\boldsymbol{u}_\tau|^2 + c_0 |\boldsymbol{u}_\tau|. \tag{3.6}$$

Therefore, by the trace inequality (2.10), we derive from (3.5) that

$$\begin{aligned} 2\mu\, \|\boldsymbol{u}\|_V^2 &\le \alpha_\psi \|\boldsymbol{u}_\tau\|_{L^2(\Gamma_1)^d}^2 + c_0 |\Gamma_1|^{1/2} \|\boldsymbol{u}_\tau\|_{L^2(\Gamma_1)^d} + \|\boldsymbol{f}\|_{V^*} \|\boldsymbol{u}\|_V \\ &\le \alpha_\psi \lambda_0^{-1} \|\boldsymbol{u}\|_V^2 + c_0 |\Gamma_1|^{1/2} \lambda_0^{-1/2} \|\boldsymbol{u}\|_V + \|\boldsymbol{f}\|_{V^*} \|\boldsymbol{u}\|_V. \end{aligned}$$

Therefore, the bound (3.4) holds. □

For the well-posedness result for Problem 2.3, we consider an auxiliary problem.

Problem 3.2. For any $\boldsymbol{w}_1 \in V$, any $\boldsymbol{w}_2 \in K_f$, find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$a(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \int_{\Gamma_1} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \ge \langle \boldsymbol{f}, \boldsymbol{v} \rangle - d(\boldsymbol{w}_2; \boldsymbol{w}_1, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V, \tag{3.7}$$

$$b(\boldsymbol{u}, q) = 0 \quad \forall\, q \in Q. \tag{3.8}$$

The next result can be derived from [43, Theorem 3.6].

Proposition 3.3. *Assume $H(\psi)$ and (3.1). For any $\boldsymbol{w}_1 \in V$, any $\boldsymbol{w}_2 \in K_f$, Problem 3.2 has a unique solution $(\boldsymbol{u}, p) \in V \times Q$.*

Proposition 3.3 allows us to define an operator $P_1 : V \to V$ by

$$P_1(\boldsymbol{w}_1) = \boldsymbol{u},$$

where $\boldsymbol{u}$ is the first component of the solution to Problem 3.2. Proceeding further, we will need the condition

$$\frac{c_d\, M_f}{2\,\mu - \alpha_\psi \lambda_0^{-1}} < 1. \tag{3.9}$$

Proposition 3.4. *Assume $H(\psi)$, (3.1) and (3.9). Then for any $\boldsymbol{w}_2 \in K_f$, the operator $P_1 : V \to V$ is a contraction.*

Proof. Let $\overline{\boldsymbol{w}}_1, \boldsymbol{w}_1 \in V$, and denote by $(\overline{\boldsymbol{u}}, \overline{p}), (\boldsymbol{u}, p) \in V \times Q$ the corresponding solutions of Problem 3.2. Then for any $(\boldsymbol{v}, q) \in V \times Q$, we have the relations

$$a(\overline{\boldsymbol{u}}; \overline{\boldsymbol{u}}, \boldsymbol{v}) + b(\boldsymbol{v}, \overline{p}) + \int_{\Gamma_1} \psi^0(\overline{\boldsymbol{u}}_\tau; \boldsymbol{v}_\tau)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle - d(\boldsymbol{w}_2; \overline{\boldsymbol{w}}_1, \boldsymbol{v}), \tag{3.10}$$

$$b(\overline{\boldsymbol{u}}, q) = 0, \tag{3.11}$$

$$a(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \int_{\Gamma_1} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle - d(\boldsymbol{w}_2; \boldsymbol{w}_1, \boldsymbol{v}), \tag{3.12}$$

$$b(\boldsymbol{u}, q) = 0. \tag{3.13}$$

Take $\boldsymbol{v} = \boldsymbol{u} - \overline{\boldsymbol{u}}$ in (3.10), $\boldsymbol{v} = \overline{\boldsymbol{u}} - \boldsymbol{u}$ in (3.12), add the two resulting inequalities, and use the Eqs. (3.11) and (3.13) to obtain

$$\begin{aligned} a_0(\overline{\boldsymbol{u}} - \boldsymbol{u}, \overline{\boldsymbol{u}} - \boldsymbol{u}) \leq{} & a_1(\overline{\boldsymbol{u}}; \overline{\boldsymbol{u}}, \boldsymbol{u} - \overline{\boldsymbol{u}}) + a_1(\boldsymbol{u}; \boldsymbol{u}, \overline{\boldsymbol{u}} - \boldsymbol{u}) \\ & + \int_{\Gamma_1} \left[\psi^0(\overline{\boldsymbol{u}}_\tau; \boldsymbol{u}_\tau - \overline{\boldsymbol{u}}_\tau) + \psi^0(\boldsymbol{u}_\tau; \overline{\boldsymbol{u}}_\tau - \boldsymbol{u}_\tau)\right] ds \\ & - d(\boldsymbol{w}_2; \overline{\boldsymbol{w}}_1 - \boldsymbol{w}, \overline{\boldsymbol{u}} - \boldsymbol{u}). \end{aligned}$$

From Lemma 2.2, we know that $a_1(\overline{\boldsymbol{u}}; \overline{\boldsymbol{u}}, \boldsymbol{u} - \overline{\boldsymbol{u}}) + a_1(\boldsymbol{u}; \boldsymbol{u}, \overline{\boldsymbol{u}} - \boldsymbol{u}) \leq 0$. Then, applying (2.29), we have

$$2\,\mu\, \|\overline{\boldsymbol{u}} - \boldsymbol{u}\|_V^2 \leq \alpha_\psi \lambda_0^{-1} \|\overline{\boldsymbol{u}} - \boldsymbol{u}\|_V^2 + c_d \|\boldsymbol{w}_2\|_V \|\overline{\boldsymbol{w}}_1 - \boldsymbol{w}_1\|_V \|\overline{\boldsymbol{u}} - \boldsymbol{u}\|_V.$$

Consequently,

$$\|\overline{\boldsymbol{u}} - \boldsymbol{u}\|_V \leq \frac{c_d \|\boldsymbol{w}_2\|_V}{2\,\mu - \alpha_\psi \lambda_0^{-1}} \|\overline{\boldsymbol{w}}_1 - \boldsymbol{w}_1\|_V \leq \frac{c_d M_f}{2\,\mu - \alpha_\psi \lambda_0^{-1}} \|\overline{\boldsymbol{w}}_1 - \boldsymbol{w}_1\|_V.$$

Hence, the operator $P_1 : V \to V$ is a contraction. □

Under the conditions stated in Proposition 3.4, we can apply the Banach fixed-point theorem to conclude that for any $\boldsymbol{w}_2 \in K_f$, the operator P_1 has a unique fixed-point $\boldsymbol{u} \in V$. Then, for some element $p \in Q$, $(\boldsymbol{u}, p) \in V \times Q$ solves the next problem.

Problem 3.5. For any $\boldsymbol{w}_2 \in K_f$, find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$a(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + d(\boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v}) + \int_{\Gamma_1} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in V, \tag{3.14}$$

$$b(\boldsymbol{u}, q) = 0 \quad \forall\, q \in Q. \tag{3.15}$$

Similar to Lemma 3.1, we can show that the solution component $\boldsymbol{u}$ also belongs to the set K_f. This allows us to define an operator $P_2 : K_f \to K_f$ by

$$P_2(\boldsymbol{w}_2) = \boldsymbol{u}.$$

Proposition 3.6. *Under the assumptions $H(\psi)$, (3.1) and (3.9), the operator $P_2 : K_f \to K_f$ is a contraction.*

Proof. Let $\overline{\boldsymbol{w}}_2, \boldsymbol{w}_2 \in K_f$, and denote by $(\overline{\boldsymbol{u}}, \overline{p}), (\boldsymbol{u}, p) \in V \times Q$ the corresponding solutions of Problem 3.5. Then, for any $(\boldsymbol{v}, q) \in V \times Q$,

$$a(\overline{\boldsymbol{u}}; \overline{\boldsymbol{u}}, \boldsymbol{v}) + d(\overline{\boldsymbol{w}}_2; \overline{\boldsymbol{u}}, \boldsymbol{v}) + b(\boldsymbol{v}, \overline{p}) + \int_{\Gamma_1} \psi^0(\overline{\boldsymbol{u}}_\tau; \boldsymbol{v}_\tau)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \tag{3.16}$$

$$b(\overline{\boldsymbol{u}}, q) = 0, \tag{3.17}$$

$$a(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{w}_2; \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \int_{\Gamma_1} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \tag{3.18}$$

$$b(\boldsymbol{u}, q) = 0. \tag{3.19}$$

Take $\boldsymbol{v} = \boldsymbol{u} - \overline{\boldsymbol{u}}$ in (3.16), $\boldsymbol{v} = \overline{\boldsymbol{u}} - \boldsymbol{u}$ in (3.18), add the two resulting inequalities and use the Eqs. (3.17) and (3.19) to obtain

$$\begin{aligned} a_0(\overline{\boldsymbol{u}} - \boldsymbol{u}, \overline{\boldsymbol{u}} - \boldsymbol{u}) \leq{} & a_1(\overline{\boldsymbol{u}}; \overline{\boldsymbol{u}}, \boldsymbol{u} - \overline{\boldsymbol{u}}) + a_1(\boldsymbol{u}; \boldsymbol{u}, \overline{\boldsymbol{u}} - \boldsymbol{u}) \\ & + \int_{\Gamma_1} \left[\psi^0(\overline{\boldsymbol{u}}_\tau; \boldsymbol{u}_\tau - \overline{\boldsymbol{u}}_\tau) + \psi^0(\boldsymbol{u}_\tau; \overline{\boldsymbol{u}}_\tau - \boldsymbol{u}_\tau)\right] ds \\ & + d(\overline{\boldsymbol{w}}_2; \overline{\boldsymbol{u}}, \boldsymbol{u} - \overline{\boldsymbol{u}}) + d(\boldsymbol{w}_2; \boldsymbol{u}, \overline{\boldsymbol{u}} - \boldsymbol{u}). \end{aligned} \tag{3.20}$$

Applying (2.20), we can write

$$\begin{aligned} d(\overline{\boldsymbol{w}}_2; \overline{\boldsymbol{u}}, \boldsymbol{u} - \overline{\boldsymbol{u}}) + d(\boldsymbol{w}_2; \boldsymbol{u}, \overline{\boldsymbol{u}} - \boldsymbol{u}) &= d(\overline{\boldsymbol{w}}_2; \boldsymbol{u}, \boldsymbol{u} - \overline{\boldsymbol{u}}) + d(\boldsymbol{w}_2; \boldsymbol{u}, \overline{\boldsymbol{u}} - \boldsymbol{u}) \\ &= d(\boldsymbol{w}_2 - \overline{\boldsymbol{w}}_2; \boldsymbol{u}, \overline{\boldsymbol{u}} - \boldsymbol{u}). \end{aligned}$$

Then,

$$d(\overline{w}_2;\overline{u},u-\overline{u})+d(w_2;u,\overline{u}-u)\le c_d\|u\|_V\|\overline{w}_2-w_2\|_V\|\overline{u}-u\|_V$$
$$\le c_d M_f\|\overline{w}_2-w_2\|_V\|\overline{u}-u\|_V.$$

From (3.20), (2.24) and (2.29), we have

$$2\,\mu\,\|\overline{u}-u\|_V^2\le\alpha_\psi\lambda_0^{-1}\|\overline{u}-u\|_V^2+c_d M_f\|\overline{w}_2-w_2\|_V\|\overline{u}-u\|_V.$$

Consequently,

$$\|\overline{u}-u\|_V\le\frac{c_d M_f}{2\,\mu-\alpha_\psi\lambda_0^{-1}}\,\|\overline{w}_2-w_2\|_V.$$

Hence, the operator $P_2:K_f\to K_f$ is a contraction. □

Theorem 3.7. *Assume $H(\psi)$, (3.1) and (3.9). Then, Problem 2.3 has a unique solution $(u,p)\in V\times Q$, and for a constant $c>0$,*

$$\|u\|_V\le c\left(1+\|f\|_{V^*}\right),\tag{3.21}$$
$$\|u\|_{L^r(\Omega)^d}\le c\left(1+\|f\|_{V^*}^{2/r}\right),\tag{3.22}$$
$$\|p\|_Q\le c\left(1+\|f\|_{V^*}^2\right).\tag{3.23}$$

Moreover, $u\in V$ depends Lipschitz continuously on $f\in V^*$, and $p\in Q$ depends locally Lipschitz continuously on $f\in V^*$.

Proof. From Proposition 3.6, by the Banach fixed-point theorem, the operator P_2 has a unique fixed point $u\in K_f$. Then, for some element $p\in Q$, $(u,p)\in V\times Q$ is a solution of Problem 2.3 and the solution component u is unique. To show the uniqueness of p, assume $(u,\tilde{p})\in V\times Q$ is another solution of Problem 2.3. From (2.30),

$$a(u;u,v)+d(u;u,v)+b(v,p)=\langle f,v\rangle\quad\forall\,v\in V_0,\tag{3.24}$$
$$a(u;u,v)+d(u;u,v)+b(v,\tilde{p})=\langle f,v\rangle\quad\forall\,v\in V_0.$$

Subtract the two equalities to obtain

$$b(v,p-\tilde{p})=0\quad\forall\,v\in V_0.$$

By the inf-sup condition (2.19),

$$\beta_0\|p-\tilde{p}\|_Q\le\sup_{v\in V_0}\frac{b(v,p-\tilde{p})}{\|v\|_V}=0.$$

Hence, $\tilde{p}=p$ and the solution component p is unique.

To prove (3.21), we take $v=-u$ in (2.30) to get

$$2\mu\int_\Omega|\varepsilon(u)|^2dx+\alpha\int_\Omega|u|^r dx\le\int_{\Gamma_1}\psi^0(u_\tau;-u_\tau)\,ds+\langle f,u\rangle.$$

From the proof in [43, Theorem 3.6], we deduce that

$$\|u\|_V^2+\|u\|_{L^r(\Omega)^d}^r\le c\left(1+\|f\|_{V^*}^2\right).\tag{3.25}$$

From (2.19) and (3.24), we have

$$\beta_0\|p\|_Q\le\sup_{v\in V_0}\frac{1}{\|v\|_V}\left[\langle f,v\rangle-a_0(u,v)-a_1(u;u,v)-d(u;u,v)\right].$$

By the boundedness of the bilinear form $a_0(\cdot,\cdot)$, the trilinear form $d(\cdot;\cdot,\cdot)$ and the bound (2.21),

$$\|p\|_Q\le c\left(\|f\|_{V^*}+\|u\|_V+\|u\|_V^2+\|u\|_{L^r(\Omega)^d}^{r-1}\right).$$

Hence, (3.23) holds.

Finally, we prove the Lipschitz continuity of the solution. For any $f_1,f_2\in V^*$, let $(u_1,p_1),(u_2,p_2)\in V\times Q$ be the corresponding solutions of Problem 2.3. Then, for all $v\in V$,

$$a(u_1;u_1,v)+d(u_1;u_1,v)+b(v,p_1)+\int_{\Gamma_1}\psi^0(u_{1,\tau};v_\tau)\,ds\ge\langle f_1,v\rangle,\tag{3.26}$$
$$a(u_2;u_2,v)+d(u_2;u_2,v)+b(v,p_2)+\int_{\Gamma_1}\psi^0(u_{2,\tau};v_\tau)\,ds\ge\langle f_2,v\rangle.\tag{3.27}$$

Taking $v=u_2-u_1$ in (3.26), $v=u_1-u_2$ in (3.27), adding the two resulting inequalities and using Eq. (2.31) for $u=u_1$ and u_2, we obtain that

$$a_0(u_1-u_2,u_1-u_2)\le a_1(u_1;u_1,u_2-u_1)+a_1(u_2;u_2,u_1-u_2)$$

$$+ d(\boldsymbol{u}_1;\boldsymbol{u}_1,\boldsymbol{u}_2-\boldsymbol{u}_1) + d(\boldsymbol{u}_2;\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2) + \int_{\Gamma_1}\left[\psi^0(\boldsymbol{u}_{1,\tau};\boldsymbol{u}_{2,\tau}-\boldsymbol{u}_{1,\tau}) + \psi^0(\boldsymbol{u}_{2,\tau};\boldsymbol{u}_{1,\tau}-\boldsymbol{u}_{2,\tau})\right] ds + \langle \boldsymbol{f}_1-\boldsymbol{f}_2, \boldsymbol{u}_1-\boldsymbol{u}_2\rangle,$$

which leads to

$$2\mu\,\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2 \le \alpha_\psi \lambda_0^{-1}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2 + c_d M_{\boldsymbol{f}}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2 + \|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{V^*}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V.$$

Hence,

$$\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V \le \frac{1}{2\mu-\alpha_\psi\lambda_0^{-1}-c_d M_{\boldsymbol{f}}}\,\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{V^*}. \tag{3.28}$$

Therefore, $\boldsymbol{u}$ depends Lipschitz continuously on $\boldsymbol{f}$. Note that (2.27) is not needed for this part of the result.

Similar to (3.24), we have

$$a(\boldsymbol{u}_1;\boldsymbol{u}_1,\boldsymbol{v}) + d(\boldsymbol{u}_1;\boldsymbol{u}_1,\boldsymbol{v}) + b(\boldsymbol{v},p_1) = \langle \boldsymbol{f}_1,\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in V_0,$$
$$a(\boldsymbol{u}_2;\boldsymbol{u}_2,\boldsymbol{v}) + d(\boldsymbol{u}_2;\boldsymbol{u}_2,\boldsymbol{v}) + b(\boldsymbol{v},p_2) = \langle \boldsymbol{f}_2,\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in V_0.$$

Subtract the two equalities to obtain

$$b(\boldsymbol{v},p_1-p_2) = \langle \boldsymbol{f}_1-\boldsymbol{f}_2,\boldsymbol{v}\rangle - a_0(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{v}) - \left[a_1(\boldsymbol{u}_1;\boldsymbol{u}_1,\boldsymbol{v}) - a_1(\boldsymbol{u}_2;\boldsymbol{u}_2,\boldsymbol{v})\right] - \left[d(\boldsymbol{u}_1;\boldsymbol{u}_1,\boldsymbol{v}) - d(\boldsymbol{u}_2;\boldsymbol{u}_2,\boldsymbol{v})\right] \quad \forall\,\boldsymbol{v}\in V_0. \tag{3.29}$$

By (2.22), we have

$$-\left[a_1\left(\boldsymbol{u}_1;\boldsymbol{u}_1,\boldsymbol{v}\right) - a_1\left(\boldsymbol{u}_2;\boldsymbol{u}_2,\boldsymbol{v}\right)\right] \le c\left(\|\boldsymbol{u}_1\|_{L^r(\Omega)^d}^{r-2} + \|\boldsymbol{u}_2\|_{L^r(\Omega)^d}^{r-2}\right)\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V\,\|\boldsymbol{v}\|_V$$
$$\le c\left(\|\boldsymbol{u}_1\|_{L^r(\Omega)^d} + \|\boldsymbol{u}_2\|_{L^r(\Omega)^d}\right)^{r-2}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V\,\|\boldsymbol{v}\|_V.$$

Moreover,

$$-\left[d(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{v}) - d(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{v})\right] = -\left[d(\boldsymbol{u}_2-\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{v}) + d(\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{v})\right]$$
$$\le c_d\left(\|\boldsymbol{u}_1\|_V + \|\boldsymbol{u}_2\|_V\right)\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V\|\boldsymbol{v}\|_V.$$

We derive from (3.29) that

$$b(\boldsymbol{v},p_1-p_2) \le \|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{V^*}\|\boldsymbol{v}\|_V + 2\,\mu\,\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V\|\boldsymbol{v}\|_V + c\left(\|\boldsymbol{u}_1\|_{L^r(\Omega)^d} + \|\boldsymbol{u}_2\|_{L^r(\Omega)^d}\right)^{r-2}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V\|\boldsymbol{v}\|_V + c\left(\|\boldsymbol{u}_1\|_V + \|\boldsymbol{u}_2\|_V\right)\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V\|\boldsymbol{v}\|_V.$$

By (3.22),

$$\|\boldsymbol{u}_1\|_{L^r(\Omega)^d} + \|\boldsymbol{u}_2\|_{L^r(\Omega)^d} \le c\left(1+\|\boldsymbol{f}_1\|_{V^*}+\|\boldsymbol{f}_2\|_{V^*}\right)^{2/r}.$$

Hence,

$$b(\boldsymbol{v},p_1-p_2) \le \|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{V^*}\|\boldsymbol{v}\|_V + 2\,\mu\,\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V\|\boldsymbol{v}\|_V + c\left(1+\|\boldsymbol{f}_1\|_{V^*}+\|\boldsymbol{f}_2\|_{V^*}\right)^{\max\{1,2(r-2)/r\}}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V\|\boldsymbol{v}\|_V.$$

Applying the inf-sup condition (2.19) and (3.28), we find

$$\|p_1-p_2\|_Q \le \frac{1}{\beta_0}\sup_{\boldsymbol{v}\in V_0}\frac{b(\boldsymbol{v},p_1-p_2)}{\|\boldsymbol{v}\|_V} \le c\left(1+\|\boldsymbol{f}_1\|_{V^*}+\|\boldsymbol{f}_2\|_{V^*}\right)^{\max\{1,2(r-2)/r\}}\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{V^*}.$$

Thus, $p\in Q$ depends locally Lipschitz continuously on $\boldsymbol{f}\in V^*$. □

In the special case of a convex function ψ, $\alpha_\psi = 0$, the condition (3.1) is automatically satisfied, and the condition (3.9) reduces to $c_d M_{\boldsymbol{f}} < 2\,\mu$. Then we have the next result on Problems 2.4.

Theorem 3.8. *Assume ψ is convex. Then for any $\boldsymbol{f}\in V^*$ with $c_d M_{\boldsymbol{f}} < 2\,\mu$, Problem 2.4 has a unique solution $(\boldsymbol{u},p)\in V\times Q$. Moreover, (3.21)–(3.23) hold, $\boldsymbol{u}\in V$ depends Lipschitz continuously on $\boldsymbol{f}\in V^*$ and $p\in Q$ depends locally Lipschitz continuously on $\boldsymbol{f}\in V^*$.*

In the case where $|\Gamma_1| = 0$, the weak formulation of the original problem reduces to an equality, and both Problems 2.3 and 2.4 are replaced by Problem 3.9 below.

Problem 3.9. Find $(\boldsymbol{u},p)\in V\times Q$ such that

$$a(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + d(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in V, \tag{3.30}$$

$$b(\boldsymbol{u},q) = 0 \quad \forall\,q\in Q. \tag{3.31}$$

The argument presented earlier in this section, in a simpler form, also applies to establishing the well-posedness result for Problem 3.9. Here, for any $\boldsymbol{f} \in V^*$ satisfying $c_d \|\boldsymbol{f}\|_{V^*} < 4\mu^2$, Problem 3.9 has a unique solution $(\boldsymbol{u}, p) \in V \times Q$. Moreover, (3.21)–(3.23) hold, $\boldsymbol{u} \in V$ depends Lipschitz continuously on $\boldsymbol{f} \in V^*$ and $p \in Q$ depends locally Lipschitz continuously on $\boldsymbol{f} \in V^*$.

4. Mixed finite element methods

We study the mixed finite element method for Problem 2.3 in this section. For simplicity, we assume Ω is a polygonal domain ($d = 2$) or a polyhedral domain ($d = 3$). Let $\{\mathcal{T}^h\}_h$ be a regular family of finite element partitions of the domain $\bar{\Omega}$ into triangular/tetrahedral elements that are compatible with the boundary decomposition to Γ_0 and Γ_1, i.e., if a side or face of an element has non-trivial intersection with Γ_0 or Γ_1, then the side or face lies entirely on $\overline{\Gamma_0}$ or $\overline{\Gamma_1}$. Let h represent the discretization parameter. We use finite element spaces $V^h \subset V$ and $Q^h \subset Q$ that satisfy the discrete inf-sup condition holds: for a constant $\beta > 0$ independent of h,

$$\beta \|q^h\|_Q \le \sup_{\boldsymbol{v}^h \in V_0^h} \frac{b(\boldsymbol{v}^h, q^h)}{\|\boldsymbol{v}^h\|_V} \quad \forall\, q^h \in Q^h, \tag{4.1}$$

where $V_0^h = V^h \cap V_0$.

As examples, we may use the P1b/P1 finite elements [49],

$$V^h = \left\{ \boldsymbol{v}^h \in V \cap C^0(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in [P_1(T) \oplus B(T)]^d \ \forall\, T \in \mathcal{T}^h \right\},$$
$$Q^h = \left\{ q^h \in Q \cap C^0(\overline{\Omega}) \mid q|_T \in P_1(T) \ \forall\, T \in \mathcal{T}^h \right\},$$

or P2/P1 finite elements ([50, Chapter II, Corollary 4.1]),

$$V^h = \left\{ \boldsymbol{v}^h \in V \cap C^0(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in [P_2(T)]^d \ \forall\, T \in \mathcal{T}^h \right\},$$
$$Q^h = \left\{ q^h \in Q \cap C^0(\overline{\Omega}) \mid q|_T \in P_1(T) \ \forall\, T \in \mathcal{T}^h \right\},$$

where $P_k(T)$ denotes the space of polynomials of a degree less than or equal to k on T, and $B(T)$ represents the space of bubble functions on T.

The mixed finite element method for Problem 2.3 is as follows.

Problem 4.1. Find $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ such that

$$a(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h) + d(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^h) + \int_{\Gamma_1} \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h)\, ds \ge \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall\, \boldsymbol{v}^h \in V^h, \tag{4.2}$$

$$b(\boldsymbol{u}^h, q^h) = 0 \quad \forall\, q^h \in Q^h. \tag{4.3}$$

Similar to the result for Problem 2.3, we obtain the following Theorem.

Theorem 4.2. *Assume $H(\psi)$, (3.1), (3.9), and the discrete inf-sup condition (4.1). Then Problem 4.1 has a unique solution $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$. The solution component $\boldsymbol{u}^h$ belongs to the set K_f and depends Lipschitz continuously on $\boldsymbol{f}$. Moreover, for a constant $c > 0$,*

$$\|\boldsymbol{u}^h\|_V^2 + \|\boldsymbol{u}^h\|_{L^r(\Omega)^d}^r \le c \left(1 + \|\boldsymbol{f}\|_{V^*}^2\right), \tag{4.4}$$

and p^h depends locally Lipschitz continuously on $\boldsymbol{f}$.

Next we will present a Céa-type error estimate result. The following modified Cauchy–Schwarz inequality will be applied several times:

$$x\, y \le \epsilon\, x^2 + \frac{1}{4\,\epsilon}\, y^2 \quad \forall\, x, y \in \mathbb{R}, \tag{4.5}$$

where $\epsilon > 0$ is arbitrarily small.

Theorem 4.3. *Keep the assumptions of Theorem 4.2. Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}^h, p^h)$ be solutions of Problems 2.3 and 4.1, respectively. Then there exists a positive constant c independent of h such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 + \|p - p^h\|_Q^2 \le c \left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{L^2(\Gamma_1)^d} + \|p - q^h\|_Q^2 \right) \quad \forall\, \boldsymbol{v}^h \in V^h,\ q^h \in Q^h. \tag{4.6}$$

Proof. For an arbitrary $\boldsymbol{v}^h \in V^h$, we have

$$\begin{aligned} 2\,\mu\, \|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 &\le a_0(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{u}^h) = a_0(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{v}^h) + a_0(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h) \\ &\le a_0(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{v}^h) + a_0(\boldsymbol{u}, \boldsymbol{v}^h - \boldsymbol{u}^h) + a_0(\boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h). \end{aligned} \tag{4.7}$$

Take $\boldsymbol{v} = \boldsymbol{u}^h - \boldsymbol{v}^h$ in (2.30) to obtain

$$a_0(\boldsymbol{u}, \boldsymbol{v}^h - \boldsymbol{u}^h) \le a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}^h - \boldsymbol{v}^h) + d(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}^h - \boldsymbol{v}^h) + b(\boldsymbol{u}^h - \boldsymbol{v}^h, p)$$

$$+ \int_{\Gamma_1} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{v}_\tau^h)\, ds - \langle \boldsymbol{f}, \boldsymbol{u}^h - \boldsymbol{v}^h \rangle.$$

Substitute $\boldsymbol{v}^h$ with $\boldsymbol{v}^h - \boldsymbol{u}^h$ in (4.2),

$$\begin{aligned} a_0(\boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h) \le{}& a_1(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h) + d(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h) + b(\boldsymbol{v}^h - \boldsymbol{u}^h, p^h) \\ &+ \int_{\Gamma_1} \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h)\, ds - \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}^h \rangle. \end{aligned}$$

Apply these two inequalities in (4.7), we have

$$2\,\mu\, \|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 \le a_0(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{v}^h) + I_{a_1} + I_d + I_\psi + I_b, \tag{4.8}$$

where

$$\begin{aligned} I_{a_1} &= a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}^h - \boldsymbol{v}^h) + a_1(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h), \\ I_d &= d(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}^h - \boldsymbol{v}^h) + d(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h), \\ I_\psi &= \int_{\Gamma_1} \left[\psi^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{v}_\tau^h) + \psi^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h)\right] ds, \\ I_b &= b(\boldsymbol{u}^h - \boldsymbol{v}^h, p - p^h). \end{aligned}$$

From [43, Theorem 4.2], we have the estimation

$$a_0(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{v}^h) \le 2\mu\, \|\boldsymbol{u} - \boldsymbol{u}^h\|_V \|\boldsymbol{u} - \boldsymbol{v}^h\|_V \le \epsilon\, \|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 + c\, \|\boldsymbol{u} - \boldsymbol{v}^h\|_V^2, \tag{4.9}$$

$$I_{a_1} \le c\, \|\boldsymbol{u} - \boldsymbol{u}^h\|_V \|\boldsymbol{u} - \boldsymbol{v}^h\|_V \le \epsilon\, \|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 + c\, \|\boldsymbol{u} - \boldsymbol{v}^h\|_V^2, \tag{4.10}$$

$$I_\psi \le \alpha_\psi \lambda_0^{-1} \|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 + c\left(1 + \|\boldsymbol{u}_\tau\|_{L^2(\Gamma_1)^d} + \|\boldsymbol{u}_\tau^h\|_{L^2(\Gamma_1)^d}\right) \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{L^2(\Gamma_1)^d}, \tag{4.11}$$

$$I_b \le \epsilon\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 + \|p - p^h\|_Q^2\right) + c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V^2 + \|p - q^h\|_Q^2\right). \tag{4.12}$$

We now bound the term I_d. Write

$$\begin{aligned} I_d &= d\left(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h\right) + d\left(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h\right) \\ &= d\left(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u} - \boldsymbol{v}^h\right) + d\left(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{v}^h\right), \end{aligned}$$

where we used the equality $d\left(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{u}^h - \boldsymbol{u}\right) = 0$. Then

$$\begin{aligned} I_d &\le c_d \|\boldsymbol{u}\|_V \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_V \left\|\boldsymbol{u} - \boldsymbol{v}^h\right\|_V + c_d \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_V \left\|\boldsymbol{u}^h\right\|_V \left\|\boldsymbol{u}^h - \boldsymbol{v}^h\right\|_V \\ &\le c_d M_f \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_V \left\|\boldsymbol{u} - \boldsymbol{v}^h\right\|_V + c_d M_f \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_V \left\|\boldsymbol{u}^h - \boldsymbol{v}^h\right\|_V. \end{aligned}$$

Applying the triangle inequality

$$\left\|\boldsymbol{u}^h - \boldsymbol{v}^h\right\|_V \le \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_V + \left\|\boldsymbol{u} - \boldsymbol{v}^h\right\|_V$$

and the modified Cauchy–Schwarz inequality (4.5), we have a constant $c > 0$ depending on ϵ such that

$$I_d \le \left(c_d M_f + 2\epsilon\right) \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_V^2 + c\left\|\boldsymbol{u} - \boldsymbol{v}^h\right\|_V^2. \tag{4.13}$$

Applying (4.9)–(4.13) to (4.8), and recalling the assumption (3.9), we find that for any sufficiently small $\epsilon > 0$, there exists a constant c depending on ϵ such that

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_V^2 \le c\left(\|\boldsymbol{u} - \boldsymbol{v}^h\|_V^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{L^2(\Gamma_1)^d} + \|p - q^h\|_Q^2\right) + \epsilon\, \|p - p^h\|_Q^2. \tag{4.14}$$

By the triangle inequality,

$$\|p - p^h\|_Q \le \|p - q^h\|_Q + \|p^h - q^h\|_Q. \tag{4.15}$$

From the discrete inf-sup condition (4.1),

$$\beta\, \|p^h - q^h\|_Q \le \sup_{\boldsymbol{v}^h \in V_0^h} \frac{b(\boldsymbol{v}^h, p^h - q^h)}{\|\boldsymbol{v}^h\|_V}.$$

Write

$$b(\boldsymbol{v}^h, p^h - q^h) = b(\boldsymbol{v}^h, p^h) - b(\boldsymbol{v}^h, p) + b(\boldsymbol{v}^h, p - q^h).$$

From (2.30) and (4.2), it follows that

$$\begin{aligned} &a_0(\boldsymbol{u}, \boldsymbol{v}^h) + a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}^h) + d(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p) = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall\, \boldsymbol{v}^h \in V_0, \\ &a_0(\boldsymbol{u}^h, \boldsymbol{v}^h) + a_1(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h) + d(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^h) = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall\, \boldsymbol{v}^h \in V_0^h. \end{aligned}$$

Hence,

$$\begin{aligned} b(\boldsymbol{v}^h, p^h) - b(\boldsymbol{v}^h, p) &= -\left[a_0(\boldsymbol{u}^h, \boldsymbol{v}^h) + a_1(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h) + d(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h)\right] \\ &\quad + \left[a_0(\boldsymbol{u}, \boldsymbol{v}^h) + a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}^h) + d(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}^h)\right] \\ &= a_0(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) + a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}^h) - a_1(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h) \\ &\quad + d(\boldsymbol{u} - \boldsymbol{u}^h; \boldsymbol{u}, \boldsymbol{v}^h) + d(\boldsymbol{u}^h; \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h). \end{aligned}$$

From (2.22), we obtain

$$a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}^h) - a_1(\boldsymbol{u}^h; \boldsymbol{u}^h, \boldsymbol{v}^h) \le c\left(\|\boldsymbol{u}\|_{L^r(\Omega)^d} + \|\boldsymbol{u}^h\|_{L^r(\Omega)^d}\right)^{r-2} \|\boldsymbol{u} - \boldsymbol{u}^h\|_V \|\boldsymbol{v}^h\|_V.$$

By the boundedness of the form d, we get

$$\begin{aligned} d(\boldsymbol{u} - \boldsymbol{u}^h; \boldsymbol{u}, \boldsymbol{v}^h) + d(\boldsymbol{u}^h; \boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) &\le c_d\left(\|\boldsymbol{u}\|_V + \|\boldsymbol{u}^h\|_V\right) \|\boldsymbol{u} - \boldsymbol{u}^h\|_V \|\boldsymbol{v}^h\|_V \\ &\le c\|\boldsymbol{u} - \boldsymbol{u}^h\|_V \|\boldsymbol{v}^h\|_V. \end{aligned}$$

Hence, we deduce the following bound from (4.15)

$$\|p - p^h\|_Q \le c\left(\|\boldsymbol{u} - \boldsymbol{u}^h\|_V + \|p - q^h\|_Q\right). \tag{4.16}$$

Finally, (4.6) follows from (4.14) with a sufficiently small ϵ and (4.16). □

As sample error estimates for applications of the Céa's inequality (4.6) and the standard finite element interpolation error estimates [48,51,52], we consider the numerical methods with the use of the P1b/P1 elements and the P2/P1 elements. We express $\overline{\Gamma_1}$ as a union of a finite number of flat components:

$$\overline{\Gamma_1} = \cup_{l=1}^{l_0} \Gamma_{1,l}.$$

Theorem 4.4. *Keep the assumptions of Theorem 4.2. Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}^h, p^h)$ be solutions of Problems 2.3 and 4.1 with the P1b/P1 elements. Assume the following solution regularities:*

$$\boldsymbol{u} \in H^2(\Omega)^d, \quad \boldsymbol{u}_\tau|_{\Gamma_{1,l}} \in H^2(\Gamma_{1,l})^d,\ 1 \le l \le l_0, \quad p \in H^1(\Omega).$$

Then,

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_V + \|p - p^h\|_Q \le c\, h.$$

Theorem 4.5. *Keep the assumptions of Theorem 4.2. Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}^h, p^h)$ be solutions of Problems 2.3 and 4.1 with the P2/P1 elements. Assume the following solution regularities:*

$$\boldsymbol{u} \in H^3(\Omega)^d, \quad \boldsymbol{u}_\tau|_{\Gamma_{1,l}} \in H^3(\Gamma_{1,l})^d,\ 1 \le l \le l_0, \quad p \in H^2(\Omega).$$

Then,

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_V + \|p - p^h\|_Q \le c\, h^{3/2}.$$

5. Solution algorithms

We introduce a numerical algorithm to solve Problem 4.1 following the idea presented in [43] in the context of a Stokes hemivariational inequality. To handle the two nonlinear terms in the problem, we apply a Newton-type linearization. Let $\boldsymbol{u}_{n+1}^h$ be represented as $\boldsymbol{u}_n^h + \boldsymbol{\delta}_n^h$, where $\boldsymbol{\delta}_n^h = \boldsymbol{u}_{n+1}^h - \boldsymbol{u}_n^h$ is assumed to be small. Consider the following real-valued functions of a real variable,

$$\begin{aligned} \boldsymbol{\phi}_1(t) &= \left|\boldsymbol{u}_n^h + t\boldsymbol{\delta}_n^h\right|^{r-2} \left(\boldsymbol{u}_n^h + t\boldsymbol{\delta}_n^h\right), \quad t \in \mathbb{R}, \\ \boldsymbol{\phi}_2(t) &= \left(\boldsymbol{u}_n^h + t\boldsymbol{\delta}_n^h\right) \cdot \nabla\left(\boldsymbol{u}_n^h + t\boldsymbol{\delta}_n^h\right), \quad t \in \mathbb{R}. \end{aligned}$$

We use the approximations

$$\boldsymbol{\phi}_1(1) \approx \boldsymbol{\phi}_1(0) + \boldsymbol{\phi}_1'(0), \quad \boldsymbol{\phi}_2(1) \approx \boldsymbol{\phi}_2(0) + \boldsymbol{\phi}_2'(0)$$

to obtain

$$\begin{aligned} &\left|\boldsymbol{u}_{n+1}^h\right|^{r-2} \boldsymbol{u}_{n+1}^h \approx \left|\boldsymbol{u}_n^h\right|^{r-2} \boldsymbol{u}_{n+1}^h + (r-2)\left|\boldsymbol{u}_n^h\right|^{r-4} (\boldsymbol{u}_n^h \cdot \boldsymbol{u}_{n+1}^h)\boldsymbol{u}_n^h - (r-2)\left|\boldsymbol{u}_n^h\right|^{r-2} \boldsymbol{u}_n^h, \\ &\boldsymbol{u}_{n+1}^h \cdot \nabla \boldsymbol{u}_{n+1}^h \approx \boldsymbol{u}_{n+1}^h \cdot \nabla \boldsymbol{u}_n^h + \boldsymbol{u}_n^h \cdot \nabla \boldsymbol{u}_{n+1}^h - \boldsymbol{u}_n^h \cdot \nabla \boldsymbol{u}_n^h. \end{aligned}$$

Thus, we are led to the following linearized iterative algorithm.

Algorithm 5.1. Choose an initial guess $\boldsymbol{u}_0^h \in V^h$. Then for $n \geq 0$ until a stopping criterion is satisfied, find $\left(\boldsymbol{u}_{n+1}^h, p_{n+1}^h\right) \in V^h \times Q^h$ such that

$$\begin{aligned}
&a_0(\boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) + a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p_{n+1}^h) + d(\boldsymbol{u}_{n+1}^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h) + d(\boldsymbol{u}_n^h, \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h)\\
&- d(\boldsymbol{u}_n^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h) + \int_{\Gamma_1} \psi^0(\boldsymbol{u}_{\tau,n+1}^h; \boldsymbol{v}_\tau^h)\, ds + \alpha(r-2)\left(\left|\boldsymbol{u}_n^h\right|^{r-4}\left(\boldsymbol{u}_n^h \cdot \boldsymbol{u}_{n+1}^h\right)\boldsymbol{u}_n^h, \boldsymbol{v}^h\right)\\
&\quad \geq \left\langle \boldsymbol{f}, \boldsymbol{v}^h \right\rangle + (r-2) a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_n^h, \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in V^h,\\
&b(\boldsymbol{u}_{n+1}^h, q^h) = 0 \quad \forall q^h \in Q^h.
\end{aligned}$$

For the special case of a convex function ψ, the algorithm is as follows:

Algorithm 5.2. Choose an initial guess $\boldsymbol{u}_0^h \in V^h$. Then for $n \geq 0$ until a stopping criterion is satisfied, find $\left(\boldsymbol{u}_{n+1}^h, p_{n+1}^h\right) \in V^h \times Q^h$ such that

$$\begin{aligned}
&a_0(\boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h - \boldsymbol{u}_{n+1}^h) + a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h - \boldsymbol{u}_{n+1}^h) + b(\boldsymbol{v}^h - \boldsymbol{u}_{n+1}^h, p_{n+1}^h)\\
&+ d(\boldsymbol{u}_{n+1}^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h) + d(\boldsymbol{u}_n^h, \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) - d(\boldsymbol{u}_n^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h)\\
&+ \int_{\Gamma_1} \psi(\boldsymbol{v}_\tau^h)\, ds - \int_{\Gamma_1} \psi(\boldsymbol{u}_{\tau,n+1}^h)\, ds + \alpha(r-2)\left(\left|\boldsymbol{u}_n^h\right|^{r-4}\left(\boldsymbol{u}_n^h \cdot \boldsymbol{u}_{n+1}^h\right)\boldsymbol{u}_n^h, \boldsymbol{v}^h - \boldsymbol{u}_{n+1}^h\right)\\
&\quad \geq \left\langle \boldsymbol{f}, \boldsymbol{v}^h \right\rangle + (r-2) a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_n^h, \boldsymbol{v}^h - \boldsymbol{u}_{n+1}^h) \quad \forall \boldsymbol{v}^h \in V^h,\\
&b(\boldsymbol{u}_{n+1}^h, q^h) = 0 \quad \forall q^h \in Q^h.
\end{aligned}$$

6. Numerical results

Since the exact solution of the hemivariational inequality is unknown, we demonstrate the performance of the algorithms in two parts, based on the selection of the nonlinear slip boundary condition. First, we consider the special case where the function ψ is convex. In this instance, the exact solution of the variational inequality is known, and the performance of the algorithm can be clearly demonstrated by the experimental results. Then, we consider the general case where ψ is not a convex function, and we showcase the convergence of the algorithm through numerical examples. The P1b/P1 elements are employed as the finite element spaces.

6.1. An example with a convex ψ

Let $\psi(\boldsymbol{v}_\tau) = g|\boldsymbol{v}_\tau|$, $g > 0$. For the implementation, we use the Uzawa iterative algorithm [53] to convert the inequality to an equality. We consider a two-dimensional domain Ω and choose the boundary Γ_1 to be piecewise parallel to the axes for the remainder of this section in describing the Uzawa algorithm. The iteration step in Algorithm 5.2 is reformulated as: find $\left(\boldsymbol{u}_{n+1}^h, p_{n+1}^h, \lambda_{n+1}^h\right) \in V^h \times Q^h \times \Lambda$ such that

$$\begin{aligned}
&a_0(\boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) + a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p_{n+1}^h) + d(\boldsymbol{u}_{n+1}^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h) + d(\boldsymbol{u}_n^h, \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h)\\
&- d(\boldsymbol{u}_n^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h) + \int_{\Gamma_1} g \lambda_{n+1}^h \cdot \boldsymbol{v}_\tau^h ds + \alpha(r-2)\left(\left|\boldsymbol{u}_n^h\right|^{r-4}\left(\boldsymbol{u}_n^h \cdot \boldsymbol{u}_{n+1}^h\right)\boldsymbol{u}_n^h, \boldsymbol{v}^h\right)\\
&\quad = \left\langle \boldsymbol{f}, \boldsymbol{v}^h \right\rangle + (r-2) a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_n^h, \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in V^h,\\
&b(\boldsymbol{u}_{n+1}^h, q^h) = 0 \quad \forall q^h \in Q^h,\\
&\lambda_{n+1}^h \cdot \boldsymbol{u}_{\tau,n+1}^h = \left|\boldsymbol{u}_{\tau,n+1}^h\right| \quad \text{a.e. on } \Gamma_1,
\end{aligned}$$

where

$$\Lambda = \left\{ \boldsymbol{\mu} \in L^2\left(\Gamma_1\right)^2 \mid |\boldsymbol{\mu}| \leq 1 \text{ a.e. on } \Gamma_1 \right\}.$$

Hence we will implement the following Uzawa iteration scheme: Begin with an initial guess $\boldsymbol{u}_0^h \in V^h$ and $\lambda_0^h \in \Lambda$. Then for $n \geq 0$, find $\left(\boldsymbol{u}_{n+1}^h, p_{n+1}^h, \lambda_{n+1}^h\right) \in V^h \times Q^h \times \Lambda$,

$$\begin{aligned}
&a_0(\boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) + a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) + d(\boldsymbol{u}_{n+1}^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h) + d(\boldsymbol{u}_n^h, \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) - d(\boldsymbol{u}_n^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h)\\
&+ b(\boldsymbol{v}^h, p_{n+1}^h) + \alpha(r-2)\left(\left|\boldsymbol{u}_n^h\right|^{r-4}\left(\boldsymbol{u}_n^h \cdot \boldsymbol{u}_{n+1}^h\right)\boldsymbol{u}_n^h, \boldsymbol{v}^h\right)\\
&\quad = \left\langle \boldsymbol{f}, \boldsymbol{v}^h \right\rangle - \int_{\Gamma_1} g \lambda_{n+1}^h \cdot \boldsymbol{v}_\tau^h ds + (r-2) a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_n^h, \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in V^h,\\
&b(\boldsymbol{u}_{n+1}^h, q^h) = 0 \quad \forall q^h \in Q^h,
\end{aligned}$$

and

$$\lambda_{n+1}^h = P_\Lambda\left(\lambda_n^h + \rho g \boldsymbol{u}_{\tau,n}^h\right), \quad \rho > 0,$$

Table 1
Numerical convergence orders, Example 6.1.

h	$\frac{\|u-u^h\|_V}{\|u\|_V}$	Order	$\frac{\|u-u^h\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}}$	Order	$\frac{\|p-p^h\|_Q}{\|p\|_Q}$	Order	Iteration
2^{-3}	3.819e−1	\	1.116e−1	\	2.181e−2	\	3
2^{-4}	1.946e−1	0.972	2.761e−2	2.015	8.248e−3	1.403	4
2^{-5}	9.811e−2	0.988	6.709e−3	2.041	2.953e−3	1.482	4
2^{-6}	4.923e−2	0.995	1.675e−3	2.002	1.041e−3	1.504	4

where

$$P_\Lambda(\boldsymbol{\mu}) = \frac{\inf(1, |\boldsymbol{\mu}|)}{|\boldsymbol{\mu}|}\boldsymbol{\mu} \quad \forall \boldsymbol{\mu} \in L^2(\Gamma_1)^2.$$

Here P_Λ represents the projection operator onto Λ. The stopping criterion for both Algorithms 5.1 and 5.2 is

$$\frac{\left\|\boldsymbol{u}_{n+1}^h - \boldsymbol{u}_n^h\right\|_V}{\left\|\boldsymbol{u}_{n+1}^h\right\|_V} \le 10^{-8}.$$

For the initial guess in the iteration in the following examples, we set $\boldsymbol{u}_0^h = \boldsymbol{0}$. For λ_0^h, on the boundary Γ_1 parallel to the x-axis, we set it as $(1,0)^T$, and on the boundary Γ_1 parallel to the y-axis, we set it as $(0,-1)^T$.

Example 6.1. Let $\Omega = [0,1]^2$, $\Gamma_1 = [0,1]\times\{1\} \cup \{1\}\times[0,1]$ and $\Gamma_0 = \Gamma\backslash\Gamma_1$. We solve the problem with the exact solution

$$\boldsymbol{u}(x,y) = \begin{pmatrix} -x^2 y(x-1)(3y-2) \\ xy^2(y-1)(3x-2) \end{pmatrix}, \quad p(x,y) = (2x-1)(2y-1).$$

It is easy to verify that $\boldsymbol{u} = \boldsymbol{0}$ on Γ_0 and we have

$$\boldsymbol{\sigma}_\tau = \begin{pmatrix} 0 \\ 4\mu y^2(y-1) \end{pmatrix} \text{ on } \{1\}\times[0,1], \quad \boldsymbol{\sigma}_\tau = \begin{pmatrix} -4\mu x^2(x-1) \\ 0 \end{pmatrix} \text{ on } [0,1]\times\{1\}.$$

From the boundary condition (2.5), we know that

$$|\boldsymbol{\sigma}_\tau| \le g.$$

Thus, we can set

$$g(x,y) = \begin{cases} -4\mu y^2(y-1) & \text{on } \{1\}\times[0,1], \\ -4\mu x^2(x-1) & \text{on } [0,1]\times\{1\}. \end{cases}$$

We choose the parameter $\rho = 0.5\mu$ and the meshes are uniform triangle meshes with the unit interval $[0,1]$ divided into $1/h$ equal parts. We set $r = 4$, $\mu = 0.05$ and $\alpha = 0.1$. Table 1 shows the numerical convergence orders as the mesh is refined. The numerical results match the theoretical prediction on the H^1-norm error of the velocity and appear to be half order higher for the L^2-norm error of the pressure.

6.2. Examples with a non-convex ψ

Let

$$\psi\left(\boldsymbol{u}_\tau\right) = \int_0^{|\boldsymbol{u}_\tau|} \omega(t)dt, \quad \omega(t) = (a-b)\exp(-\gamma t) + b,$$

where a, b and γ are constants with $a > b > 0$ and $\gamma > 0$. The slip boundary condition $-\boldsymbol{\sigma}_\tau \in \partial\psi\left(\boldsymbol{u}_\tau\right)$ can be expressed in the following form:

$$|\boldsymbol{\sigma}_\tau| \le \omega\left(|\boldsymbol{u}_\tau|\right), \quad -\boldsymbol{\sigma}_\tau = \omega\left(|\boldsymbol{u}_\tau|\right)\frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0}.$$

Again, we employ the Uzawa algorithm for implementation. We introduce a Lagrange multiplier $\lambda = -\boldsymbol{\sigma}_\tau/\omega\left(|\boldsymbol{u}_\tau|\right)$, an element of the set

$$\Lambda = \left\{\boldsymbol{\mu} \in L^2\left(\Gamma_1\right)^2 \mid |\boldsymbol{\mu}| \le 1 \text{ a.e. on } \Gamma_1\right\}.$$

Then, the iteration step in Algorithm 5.1 is reformulated as: find $\left(\boldsymbol{u}_{n+1}^h, p_{n+1}^h, \lambda_{n+1}^h\right) \in V^h \times Q^h \times \Lambda$ such that

$$a_0(\boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) + a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p_{n+1}^h) + d(\boldsymbol{u}_{n+1}^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h) + d(\boldsymbol{u}_n^h, \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h)$$
$$- d(\boldsymbol{u}_n^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h) + \int_{\Gamma_1} \omega\left(\left|\boldsymbol{u}_{\tau,n+1}^h\right|\right)\lambda_{n+1}^h \cdot \boldsymbol{v}_\tau^h ds + \alpha(r-2)\left(\left|\boldsymbol{u}_n^h\right|^{r-4}\left(\boldsymbol{u}_n^h \cdot \boldsymbol{u}_{n+1}^h\right)\boldsymbol{u}_n^h, \boldsymbol{v}^h\right)$$

Table 2
Numerical convergence orders, Example 6.2.

h	$\frac{\Vert u-u^h\Vert_V}{\Vert u\Vert_V}$	Order	$\frac{\Vert u-u^h\Vert_{L^2(\Omega)}}{\Vert u\Vert_{L^2(\Omega)}}$	Order	$\frac{\Vert p-p^h\Vert_Q}{\Vert p\Vert_Q}$	Order	Iteration
2^{-3}	4.563e−1	\	1.293e−1	\	2.633e−1	\	6
2^{-4}	2.566e−1	0.830	3.480e−2	1.893	1.447e−1	0.863	7
2^{-5}	1.387e−1	0.887	9.110e−3	1.934	7.429e−2	0.962	7
2^{-6}	6.983e−2	0.990	2.315e−3	1.976	3.358e−2	1.146	7

Table 3
Numerical convergence orders, Example 6.3.

h	$\frac{\Vert u-u^h\Vert_V}{\Vert u\Vert_V}$	Order	$\frac{\Vert u-u^h\Vert_{L^2(\Omega)}}{\Vert u\Vert_{L^2(\Omega)}}$	Order	$\frac{\Vert p-p^h\Vert_Q}{\Vert p\Vert_Q}$	Order	Iteration
2^{-3}	5.799e−1	\	3.537e−1	\	9.547e−1	\	6
2^{-4}	3.345e−1	0.794	1.240e−1	1.512	3.865e−1	1.305	6
2^{-5}	1.741e−1	0.942	3.420e−2	1.858	1.352e−1	1.515	6
2^{-6}	8.684e−2	1.004	8.415e−3	2.023	4.658e−2	1.538	6

$$= \left\langle \boldsymbol{f}, \boldsymbol{v}^h \right\rangle + (r-2)a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_n^h, \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in V^h,$$
$$b\left(\boldsymbol{u}_{n+1}^h, q^h\right) = 0 \quad \forall q^h \in Q^h,$$
$$\boldsymbol{\lambda}_{n+1}^h \cdot \boldsymbol{u}_{\tau,n+1}^h = \left|\boldsymbol{u}_{\tau,n+1}^h\right| \qquad \text{a.e. on } \Gamma_1.$$

The corresponding Uzawa iteration scheme is: Begin with an initial guess $\boldsymbol{u}_0^h \in V^h$, $\lambda_0^h \in \Lambda$. Then for $n \geq 0$, find $\left(\boldsymbol{u}_{n+1}^h, p_{n+1}^h, \lambda_{n+1}^h\right) \in V^h \times Q^h \times \Lambda$,

$$a_0(\boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) + a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h) + d(\boldsymbol{u}_{n+1}^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h) + d(\boldsymbol{u}_n^h, \boldsymbol{u}_{n+1}^h, \boldsymbol{v}^h)$$
$$- d(\boldsymbol{u}_n^h, \boldsymbol{u}_n^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p_{n+1}^h) + \alpha(r-2)\left(\left|\boldsymbol{u}_n^h\right|^{r-4}\left(\boldsymbol{u}_n^h \cdot \boldsymbol{u}_{n+1}^h\right)\boldsymbol{u}_n^h, \boldsymbol{v}^h\right)$$
$$= \left\langle \boldsymbol{f}, \boldsymbol{v}^h \right\rangle - \int_{\Gamma_1} \omega\left(\left|\boldsymbol{u}_{\tau,n}^h\right|\right) \boldsymbol{\lambda}_{n+1}^h \cdot \boldsymbol{v}_\tau^h ds + (r-2)a_1(\boldsymbol{u}_n^h; \boldsymbol{u}_n^h, \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in V^h,$$
$$b(\boldsymbol{u}_{n+1}^h, q^h) = 0 \quad \forall q^h \in Q^h,$$

and

$$\boldsymbol{\lambda}_{n+1}^h = P_\Lambda\left(\boldsymbol{\lambda}_n^h + \rho \boldsymbol{u}_{\tau,n}^h\right), \quad \rho > 0,$$

where

$$P_\Lambda(\boldsymbol{\mu}) = \frac{\inf(1, |\boldsymbol{\mu}|)}{|\boldsymbol{\mu}|}\boldsymbol{\mu} \quad \forall \boldsymbol{\mu} \in L^2(\Gamma_1)^2.$$

Example 6.2. Let $\Omega = [0,1]^2$, $\Gamma_1 = [0,1] \times \{1\} \cup \{1\} \times [0,1]$ and $\Gamma_0 = \Gamma \backslash \Gamma_1$. We choose $\mu = 0.5$, $\alpha = 0.1$, $r = 4$, $a = 0.255$, $b = 0.25$, $\gamma = 10.0$ and $\rho = 0.5\mu$. The source term is defined by

$$\boldsymbol{f}_0 = -\mu \Delta \boldsymbol{u}_0 + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \alpha \left|\boldsymbol{u}_0\right|^{r-2} \boldsymbol{u}_0 + \nabla p_0$$

with

$$\boldsymbol{u}_0(x,y) = \begin{pmatrix} -x^2 y(x-1)(3y-2) \\ xy^2(y-1)(3x-2) \end{pmatrix}, \quad p_0(x,y) = (2x-1)(2y-1).$$

We use uniform triangular meshes, dividing the unit interval $[0,1]$ into $1/h$ equal parts. In the absence of a known true solution, we employ the numerical solution obtained on a sufficiently fine mesh ($h = 1/256$) as the reference solution $(\boldsymbol{u}, p)$ to assess errors in the numerical solutions $(\boldsymbol{u}_n^h, p_n^h)$ on coarser meshes at $h = 1/2^n$ for $n \leq 6$. From Table 2, we observe that the convergence orders in the velocity space and pressure space align well with theoretical predictions.

Example 6.3. Let $\Omega = [0,1]^2$, $\Gamma_1 = [0,1] \times \{1\}$ and $\Gamma_0 = \Gamma \backslash \Gamma_1$. We choose $\mu = 1$, $\alpha = 1$, $r = 3$, $a = 0.255$, $b = 0.25$, $\gamma = 10.0$ and $\rho = 1$. The source term is defined by

$$\boldsymbol{f}_0 = -\mu \Delta \boldsymbol{u}_0 + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \alpha \left|\boldsymbol{u}_0\right|^{r-2} \boldsymbol{u}_0 + \nabla p_0$$

with

$$\boldsymbol{u}_0(x,y) = \begin{pmatrix} -\sin^2(2\pi x)\sin(4\pi y) \\ \sin^2(2\pi y)\sin(4\pi x) \end{pmatrix}, \quad p_0(x,y) = \cos(x)\cos(y) - \sin^2(1).$$

We use uniform triangular meshes as in Example 6.2, and employ the numerical solution calculated on a sufficiently fine mesh ($h = 1/256$) as the reference solution $(\boldsymbol{u}, p)$. The results are presented in Table 3. We notice that the convergence orders align well with theoretical predictions and the L^2-norm error estimate of the pressure is half order higher than our theoretical analysis in this example.

CRediT authorship contribution statement

Wensi Wang: Formal analysis, Writing – original draft, Software. **Xiaoliang Cheng:** Validation, Writing – review & editing, Supervision. **Weimin Han:** Conceptualization, Methodology, Writing – review & editing.

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