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OPTIMAL CONTROL OF A STATIONARY NAVIER-STOKES HEMIVARIATIONAL INEQUALITY WITH NUMERICAL APPROXIMATION

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Abstract. In this paper, we study an optimal control problem for a stationary Navier–Stokes hemivariational inequality with control constraints and its numerical approximation. The hemivariational inequality is the weak formulation of a stationary incompressible fluid flow problem, modeled by the Navier-Stokes equations subject to a nonleak boundary condition and a subdifferential condition of friction type. We investigate the stability of the solution of the hemivariational inequality with respect to perturbations in the density of external force and superpotential, and demonstrate the existence of a solution for the optimal control problem with the external force density as the control. Moreover, we consider the numerical solution of the optimal control problem and show its convergence. As an example and for definiteness, the numerical solution is defined through the finite element method.

1. **Introduction.** Optimal control of a stationary Stokes hemivariational inequality and its numerical approximation is discussed in several publications ([6, 9, 10]). In this paper, we take one step further and discuss the optimal control problem of a stationary Navier-Stokes hemivariational inequality and its numerical approximation. The hemivariational inequality models a steady flow of incompressible viscous fluid subject to a nonleak boundary condition and a subdifferential condition of friction type.

The notion of hemivariational inequalities was proposed by Panagiotopoulos ([30]), and it provides a mathematical framework for understanding and addressing complex interactions and constraints in various systems involving nonsmooth and nonmonotone relations among physical quantities. Mathematical theory of hemivariational inequalities has been developed in a variety of publications, e.g., the comprehensive references [26, 27]. Since there is no analytic solution formula, hemivariational inequalities must be solved by numerical methods. In this regard, one is referred to [21] for early development of numerical methods and algorithms for solving hemivariational inequalities and to [20] for a survey of recent development and mathematical theories of the numerical solution of hemivariational inequalities.

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For the Navier-Stokes equations, when a boundary condition involves non-smooth monotone relations, the weak formulation is a variational inequality, which has been studied in various publications, e.g., [5, 11]. When the boundary condition involves non-smooth nonmonotone relations, the weak formulation becomes a hemivariational inequality. Well-posedness of the Navier-Stokes hemivariational inequality was first addressed in [23, 25] by employing an abstract surjectivity result for pseudomonotone operators. The optimal control problem and the existence of its solutions were subsequently introduced in [24]. Unlike [25, 23], well-posedness of the Navier-Stokes hemivariational inequality is established using only basic concepts and results from functional analysis in [15, 22], following the approach developed in [13, 14] for elliptic hemivariational inequalities and in [17, 18] for mixed hemivariational inequalities.

The control of fluid flow has garnered sustained attention from both engineers and scientists prompted by the demand for intricate technological applications. Optimal control problems subject to the regular boundary value problems of the Navier-Stokes equations are studied in a variety of publications, e.g., [1, 4]. Optimal control problems related to variational-hemivariational inequalities have also been studied in various papers, e.g., [9, 24, 33]. Here, we study the optimal control problem of the Navier-Stokes hemivariational inequality with force density as control and its numerical approximation by the finite element method. We comment that other numerical methods, such as the discontinuous Galerkin method, the virtual element method, can be also applied and analyzed for solving the optimal control problem of the Navier-Stokes hemivariational inequality.

To prepare for a study of the optimal control problem, we first analyze the stability for the solution of the Navier-Stokes hemivariational inequality with respect to the density of the external force and superpotential. The stability result is of interest by itself. We comment that some results on the stability of elliptic hemivariational inequalities are shown in [16, 33].

The rest of the paper is structured as follows. We review some basic notation and present preliminary material needed later in this paper in Section 2. In Section 3, we first provide a stability analysis for the solution of the stationary Navier-Stokes hemivariational inequality subject to perturbations of external force density and superpotential. Subsequently, we study the optimal control of the stationary Navier-Stokes hemivariational inequality with control constraints. In Section 4, we investigate the numerical approximation of the optimal control problem and show the convergence of the numerical method in a fairly broad context.

2. Notation and preliminaries.

We consider a viscous incompressible fluid flow in $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, where Ω is a bounded simply connected Lipschitz domain with a smooth boundary $\Gamma = \partial\Omega$ partitioned into two measurable and disjoint parts Γ_D and Γ_S with $\mathrm{meas}(\Gamma_D) \neq 0$. Denote the fluid velocity and the pressure by $\boldsymbol{u} = (u_1, \ldots, u_d)^T$ and p respectively, which are the unknown variables of the problem. Let $\nu > 0$ be the constant coefficient of kinematic viscosity and $\boldsymbol{f}$ the external force density. Denote by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ the deformation rate tensor. The stress tensor is $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})$, where $\boldsymbol{I}$ is the identity matrix.

Denote by $\boldsymbol{n}$ the unit outward normal to Γ. Since the boundary is Lipschitz continuous, $\boldsymbol{n}$ exists a.e. on Γ. We decompose the velocity and the stress tensor on the boundary into their normal and tangential components, respectively:

$$u_n = \boldsymbol{u} \cdot \boldsymbol{n}, \quad \boldsymbol{u}_\tau = \boldsymbol{u} - u_n \boldsymbol{n},$$
$$\sigma_n(\boldsymbol{u}) = (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}) \cdot \boldsymbol{n}, \quad \boldsymbol{\sigma}_\tau(\boldsymbol{u}) = \boldsymbol{\sigma}(\boldsymbol{u}) - \sigma_n(\boldsymbol{u})\boldsymbol{n}.$$

Let $\mathbb{S}^d$ stand for the space of second-order symmetric tensors on $\mathbb{R}^d$. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{u} = (u_i), \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\tau\|_{\mathbb{S}^d} = (\boldsymbol{\tau} : \boldsymbol{\tau})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

The definitions and some properties of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function are recalled bellow for later use.

Definition 2.1. ([8, p. 10]) Let X be a Banach space, and let $\psi : X \to \mathbb{R}$ be a locally Lipschitz continuous function. The generalized directional derivative of ψ at $x \in X$ in the direction $z \in X$, denoted by $\psi^0(x; z)$, is defined by

$$\psi^0(x; z) = \limsup_{y \to x, \lambda \to 0+} \frac{\psi(y + \lambda z) - \psi(y)}{\lambda}.$$

The generalized gradient or subdifferential of ψ at x, denoted by $\partial\psi(x)$, is a subset of the dual space X^* given by

$$\partial\psi(x) = \left\{ \zeta \in X^* \mid \psi^0(x; z) \geq \langle \zeta, z \rangle_{X^* \times X} \ \forall\, z \in X \right\}.$$

Proposition 2.2. ([26, Propsition 3.23]) *Let X be a Banach space, and let $\psi : X \to \mathbb{R}$ be a locally Lipschitz continuous function. Then the following statements hold:*

(i) For every $x \in X$, the function $X \ni z \mapsto \psi^0(x; z) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $\psi^0(x; \lambda z) = \lambda \psi^0(x; z)$ for all $\lambda \geq 0, z \in X$ and $\psi^0(x; z_1 + z_2) \leq \psi^0(x; z_1) + \psi^0(x; z_2)$ for all $z_1, z_2 \in X$, respectively.

(ii) The function $X \times H \ni (x, z) \mapsto \psi^0(x; z) \in \mathbb{R}$ is upper semicontinuous, i.e., for all $x \in X$, $z \in X$, $\{x_n\} \subset X$, $\{z_n\} \subset H$ such that $x_n \to x$ in X and $z_n \to z$ in X, we have $\limsup \psi^0(x_n; z_n) \leq \psi^0(x; z)$.

(iii) For all $z \in X$, $\psi^0(x; z) = \max\{\langle \zeta, z \rangle_{X^* \times X} \mid \zeta \in \partial\psi(x)\}$.

3. Navier–Stokes hemivariational inequality and optimal control.

Consider the incompressible stationary Navier-Stokes problem with the Dirichlet boundary condition on Γ_D and a nonleak and nonsmooth slip boundary condition of friction type on Γ_S. The equations are as follows

$$-\nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1}$$
$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \tag{2}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D, \tag{3}$$
$$u_n = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi(\boldsymbol{u}_\tau) \quad \text{on} \quad \Gamma_S. \tag{4}$$

Here $\boldsymbol{f}$ and $\psi : \Gamma_S \times \mathbb{R}^d \to \mathbb{R}$ are given functions and $\partial\psi$ is the subdifferential of $\psi(\boldsymbol{x}, \cdot)$ in the sense of Clarke. The divergence free condition (2) reflects the incompressibility of the fluid. As for boundary conditions, the first part of (4) signifies the nonleak property, indicating that the flow cannot penetrate beyond Γ_S outside the domain and the second part shows the connection between the frictional force $\boldsymbol{\sigma}_\tau$ and the tangential velocity $\boldsymbol{u}_\tau$. We use $\psi(\boldsymbol{u}_\tau)$ to represent $\psi(\boldsymbol{x}, \boldsymbol{u}_\tau)$ for simplicity and assume ψ satisfies the assumptions.

$H(\psi)$. $\psi : \Gamma_S \times \mathbb{R}^d \to \mathbb{R}$ is such that
(i) $\psi(\cdot, \boldsymbol{\xi})$ is measurable on Γ_S for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and $\psi(\cdot, \mathbf{0}) \in L^1(\Gamma_S)$;
(ii) $\psi(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Gamma_S$;
(iii) $\|\boldsymbol{\eta}\|_{\mathbb{R}^d} \le c_0 + c_1 \|\boldsymbol{\xi}\|_{\mathbb{R}^d}$ $\forall \boldsymbol{\xi} \in \mathbb{R}^d$, $\boldsymbol{\eta} \in \partial\psi(\boldsymbol{x}, \boldsymbol{\xi})$ a.e. $\boldsymbol{x} \in \Gamma_S$ with $c_0, c_1 \ge 0$;
(iv) $\psi^0(\boldsymbol{x}, \boldsymbol{\xi_1}; \boldsymbol{\xi_2} - \boldsymbol{\xi_1}) + \psi^0(\boldsymbol{x}, \boldsymbol{\xi_2}; \boldsymbol{\xi_1} - \boldsymbol{\xi_2}) \le m \|\boldsymbol{\xi_1} - \boldsymbol{\xi_2}\|^2_{\mathbb{R}^d}$ $\forall \boldsymbol{\xi_1}, \boldsymbol{\xi_2} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_S$ with $m \ge 0$.

By Prop. 2.2, $H(\psi)$ (iii) implies

$$\left\|\psi^0(\boldsymbol{x}, \boldsymbol{\xi_1}; \boldsymbol{\xi_2})\right\|_{\mathbb{R}^d} \le (c_0 + c_1 \|\boldsymbol{\xi_1}\|_{\mathbb{R}^d}) \|\boldsymbol{\xi_2}\|_{\mathbb{R}^d} \quad \forall \boldsymbol{\xi_1}, \boldsymbol{\xi_2} \in \mathbb{R}^d, \text{a.e. } \boldsymbol{x} \in \Gamma_S. \tag{5}$$

The problem (1)–(4) is studied through its weak formulation. To derive the weak formulation, assume the problem has a smooth solution $(\boldsymbol{u}, p)$. First we notice that the incompressibility condition (2) implies that

$$\Delta \boldsymbol{u} = 2 \,\mathrm{Div}\, \boldsymbol{\varepsilon}(\boldsymbol{u}).$$

Thus, (1) can be equivalently written as

$$-2\nu \,\mathrm{Div}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega,$$

or, thanks to the definition of the stress tensor $\boldsymbol{\sigma}$,

$$-\,\mathrm{Div}\, \boldsymbol{\sigma} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega.$$

Multiply the above equation by an arbitrary smooth test function $\boldsymbol{v}$ with $\boldsymbol{v} = \mathbf{0}$ on Γ_D and $v_n = 0$ on Γ_S, and integrate over Ω to obtain

$$-\int_\Omega \mathrm{Div}\, \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx + \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx.$$

Integrate by parts,

$$-\int_{\partial\Omega} \boldsymbol{\sigma n} \cdot \boldsymbol{v}\, ds + \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx. \tag{6}$$

By making use of the conditions $\boldsymbol{v} = \mathbf{0}$ on Γ_D and $v_n = 0$ on Γ_S, as well as the identity

$$\boldsymbol{\sigma n} \cdot \boldsymbol{v} = \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau + \sigma_n v_n,$$

we have

$$-\int_{\partial\Omega} \boldsymbol{\sigma n} \cdot \boldsymbol{v}\, ds = -\int_{\Gamma_S} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau ds.$$

We further have, thanks to the second boundary condition in (4),

$$-\int_{\partial\Omega} \boldsymbol{\sigma n} \cdot \boldsymbol{v}\, ds \le \int_{\Gamma_S} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds.$$

Moreover, from the definition of $\boldsymbol{\sigma}$,

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}) = 2\nu\, \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) - p \,\mathrm{div} \boldsymbol{v}.$$

Hence, from (6), we obtain the inequality

$$2\nu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{v}\, dx - \int_\Omega p \,\mathrm{div}\, \boldsymbol{v}\, dx + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds$$
$$\ge \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \tag{7}$$

for any smooth function $\boldsymbol{v}$ with $\boldsymbol{v} = \mathbf{0}$ on Γ_D and $v_n = 0$ on Γ_S. We multiply (2) by any $L^2(\Omega)$ function q with a vanishing mean in Ω and integrate over Ω to get

$$\int_\Omega q \,\mathrm{div}\, \boldsymbol{u}\, dx = 0. \tag{8}$$

An inspection of the smoothness requirements on the functions $\boldsymbol{u}$, $\boldsymbol{v}$, p and q in (7)–(8) suggests that we use the function spaces

$$\boldsymbol{V} := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D, v_n = 0 \text{ on } \Gamma_S \right\},$$
$$Q := L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \mid \int_\Omega q\, dx = 0 \right\}$$

for the velocity and pressure variables. Here, the space $\boldsymbol{H}^1(\Omega) := (H^1(\Omega))^d$.Since $\text{meas}(\Gamma_D) > 0$, Korn's inequality ([29, p. 79]) implies that over the space $\boldsymbol{V}$,

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega} = \left(\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) dx \right)^{\frac{1}{2}}$$

defines a norm and is equivalent to the standard $\boldsymbol{H}^1(\Omega)$-norm. We use $\|\cdot\|_{\boldsymbol{V}} = \|\boldsymbol{\varepsilon}(\cdot)\|_{0,\Omega}$ for the norm on $\boldsymbol{V}$ and $(\boldsymbol{V}, \|\cdot\|_{\boldsymbol{V}})$ is a Hilbert space. The space Q is equipped with the standard $L^2(\Omega)$-norm. Denote by $\boldsymbol{V}_0 := H_0^1(\Omega;\mathbb{R}^d)$ a subspace of $\boldsymbol{V}$, and denote $\boldsymbol{H} := L^2(\Omega;\mathbb{R}^d)$. We identify $\boldsymbol{H}$ with its dual $\boldsymbol{H}^*$, and denote the duality pairing between $\boldsymbol{V}$ and $\boldsymbol{V}^*$ by $\langle\cdot,\cdot\rangle$. Note that $\boldsymbol{V} \subset \boldsymbol{H} = \boldsymbol{H}^* \subset \boldsymbol{V}^*$ and the embeddings are dense, continuous and compact.

Define

$$a(\boldsymbol{u},\boldsymbol{v}) = 2\nu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \forall\, \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V},$$
$$d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\, dx \quad \forall\, \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V},$$
$$b(\boldsymbol{v},q) = \int_\Omega q \operatorname{div} \boldsymbol{v}\, dx \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, q \in Q.$$

and assume $\boldsymbol{f} \in \boldsymbol{V}^*$. Then (7)–(8) lead to the following weak formulation of the problem (1)–(4).

Problem (3.1)**.** *Find $(\boldsymbol{u},p) \in \boldsymbol{V}\times Q$ such that*

$$a(\boldsymbol{u},\boldsymbol{v}) + d(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - b(\boldsymbol{v},p) + \int_{\Gamma_S} \psi^0\left(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau\right) ds \ge \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{9}$$
$$b(\boldsymbol{u},q) = 0 \quad \forall\, q \in Q. \tag{10}$$

Obviously, the bilinear form $a(\cdot,\cdot)$ is continuous and coercive on $\boldsymbol{V}\times\boldsymbol{V}$:

$$a(\boldsymbol{u},\boldsymbol{v}) \le 2\nu\|\boldsymbol{u}\|_{\boldsymbol{V}}\|\boldsymbol{v}\|_{\boldsymbol{V}} \quad \text{and} \quad a(\boldsymbol{v},\boldsymbol{v}) = 2\nu\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 \quad \forall\, \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}.$$

The trilinear form $d(\cdot,\cdot,\cdot)$ is continuous on $\boldsymbol{V}\times\boldsymbol{V}\times\boldsymbol{V}$:

$$d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le c_d\|\boldsymbol{u}\|_{\boldsymbol{V}}\|\boldsymbol{v}\|_{\boldsymbol{V}}\|\boldsymbol{w}\|_{\boldsymbol{V}} \quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}, \tag{11}$$

and for $\boldsymbol{u} \in \boldsymbol{V}$ satisfies (2), the following estimates hold:

$$d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -d(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) \quad \text{and} \quad d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0 \quad \forall \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}. \tag{12}$$

The bilinear form $b(\cdot,\cdot)$ is bounded on $\boldsymbol{V}\times Q$:

$$b(\boldsymbol{v},q) \le c_b\|\boldsymbol{v}\|_{\boldsymbol{V}}\|q\|_Q \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\, q \in Q.$$

The following inf-sup condition holds ([32]): for a constant $\beta > 0$,

$$\beta\|q\|_{L^2(\Omega)} \le \sup_{\boldsymbol{v}\in\boldsymbol{V}_0} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \quad \forall q \in Q. \tag{13}$$

By the Sobolev trace theorem ([28]), we have the inequality

$$\|\boldsymbol{v}_\tau\|_{0,\Gamma_S} \le \lambda_0^{-1/2}\|\boldsymbol{v}\|_{\boldsymbol{V}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{14}$$

where $\lambda_0 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in \boldsymbol{V}, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v})dx = \lambda \int_{\Gamma_S} \boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau ds \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}.$$

Combining $H(\psi)$ (iv) and (14), we have, for $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{V}$,

$$\begin{aligned}\int_{\Gamma_S} \left[\psi^0\left(\boldsymbol{v}_{1\tau}; \boldsymbol{v}_{2\tau} - \boldsymbol{v}_{1\tau}\right) + \psi^0\left(\boldsymbol{v}_{2\tau}; \boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau}\right)\right] ds &\le \int_{\Gamma_S} m\left\|\boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau}\right\|^2 ds \\ &\le m\lambda_0^{-1}\left\|\boldsymbol{v}_1 - \boldsymbol{v}_2\right\|_{\boldsymbol{V}}^2.\end{aligned}$$

Let

$$\alpha_{\boldsymbol{f}} = \frac{c_0\lambda_0^{-1/2}|\Gamma_S|^{1/2} + \|\boldsymbol{f}\|_{\boldsymbol{V}^*}}{2\nu - m\lambda_0^{-1}}. \tag{15}$$

The existence and uniqueness result for Problem 3.1 and the boundedness of the solution are stated below, which can be found in [15, Theorem 3.1].

Theorem 3.1. *Assume $H(\psi), \boldsymbol{f} \in \boldsymbol{V}^*$, and the smallness condition*

$$0 < c_d\alpha_{\boldsymbol{f}} < 2\nu - m\lambda_0^{-1}.$$

Then Problem 3.1 has a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$. Moreover,

$$\|\boldsymbol{u}\|_{\boldsymbol{V}} \le \alpha_{\boldsymbol{f}}.$$

3.1. **A stability result.** For further analysis, we first study a perturbed stationary Navier-Stokes hemivariational inequality in which the external force density $\boldsymbol{f}$ and the superpotential ψ are substituted with their respective perturbations $\boldsymbol{f}_n \in \boldsymbol{V}^*$ and ψ_n for $n \ge 1$.

Similar to $H(\psi)$, we make assumptions regarding ψ_n.

$H(\psi_n)$. $\psi_n : \Gamma_S \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $\psi_n(\cdot, \boldsymbol{\xi})$ is measurable on Γ_S for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and $\psi(\cdot, \boldsymbol{0}) \in L^1(\Gamma_S)$;

(ii) $\psi_n(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Gamma_S$;

(iii) $\|\boldsymbol{\eta}\|_{\mathbb{R}^d} \le c_{0n} + c_{1n}\|\boldsymbol{\xi}\|_{\mathbb{R}^d}$ $\forall\, \boldsymbol{\xi} \in \mathbb{R}^d$, $\boldsymbol{\eta} \in \partial\psi_n(\boldsymbol{x}, \boldsymbol{\xi})$ a.e. $\boldsymbol{x} \in \Gamma_S$ with $0 \le c_{0n} \le \bar{c}_0, 0 \le c_{1n} \le \bar{c}_1$;

(iv) $\psi_n^0\left(\boldsymbol{x}, \boldsymbol{\xi_1}; \boldsymbol{\xi_2} - \boldsymbol{\xi_1}\right) + \psi_n^0\left(\boldsymbol{x}, \boldsymbol{\xi_2}; \boldsymbol{\xi_1} - \boldsymbol{\xi_2}\right) \le m_{\mu n}\left\|\boldsymbol{\xi_1} - \boldsymbol{\xi_2}\right\|_{\mathbb{R}^d}^2$ $\forall\, \boldsymbol{\xi_1}, \boldsymbol{\xi_2} \in \mathbb{R}^d$ a.e. $\boldsymbol{x} \in \Gamma_S$ with $m_{\mu n} \ge 0$, and there exists a constant $m_\mu > 0$ such that $m_{\mu n} \le m_\mu$ for each $n \in \mathbb{N}$.

The perturbed problem is then constructed as follows.

Problem (3.2)**.** *Find $(\boldsymbol{u}_n, p_n) \in \boldsymbol{V} \times Q$ such that*

$$a(\boldsymbol{u}_n, \boldsymbol{v}) + d(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{v}) - b(\boldsymbol{v}, p_n) + \int_{\Gamma_S} \psi_n^0\left(\boldsymbol{u}_{n\tau}; \boldsymbol{v}_\tau\right) ds \ge \langle \boldsymbol{f}_n, \boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{16}$$

$$b(\boldsymbol{u}_n, q) = 0 \quad \forall q \in Q. \tag{17}$$

To assess the proximity of the problem data, we introduce the following assumption related to the superpotential.

$H(\psi_n')$. If $\boldsymbol{\xi}_n \to \boldsymbol{\xi}$ and $\boldsymbol{\eta}_n \to \boldsymbol{\eta}$ in $\mathbb{R}^d$, then

$$\limsup_{n\to\infty} \psi_n^0\left(\boldsymbol{\xi}_n; \boldsymbol{\eta}_n\right) \le \psi^0(\boldsymbol{\xi}; \boldsymbol{\eta}).$$

In [10], an example is presented where $H(\psi)$, $H(\psi_n)$ and $H(\psi_n')$ are satisfied.

To guarantee the existence result of the perturbed problem, an additional constraint on the force density is required. For a given constant $m_0 > 0$, define a subset $\boldsymbol{V}^*_{m_0} \subset \boldsymbol{V}^*$ by

$$\boldsymbol{V}^*_{m_0} = \{\boldsymbol{f} \in \boldsymbol{V}^* : \|\boldsymbol{f}\|_{\boldsymbol{V}^*} \le m_0\}.$$

Similar to (15), denote

$$\alpha_{m_0} = \frac{\bar{c}_0 \lambda_0^{-1/2} |\Gamma_S|^{1/2} + m_0}{2\nu - m_\mu \lambda_0^{-1}}.$$

We have the following result.

Theorem 3.2. *Assume $H(\psi)$, $H(\psi_n)$, $\boldsymbol{f}, \boldsymbol{f}_n \in \boldsymbol{V}^*_{m_0}$ and the smallness condition*

$$0 < c_d \alpha_{m_0} < 2\nu - m_\mu \lambda_0^{-1}. \tag{18}$$

Then Problem 3.1 has a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$, Problem 3.2 has a unique solution $(\boldsymbol{u}_n, p_n) \in \boldsymbol{V} \times Q$, and

$$\|\boldsymbol{u}\|_{\boldsymbol{V}} \le \alpha_{m_0}, \quad \|\boldsymbol{u}_n\|_{\boldsymbol{V}} \le \alpha_{m_0}.$$

Moreover, assume $\|\boldsymbol{f}_n - \boldsymbol{f}\|_{\boldsymbol{V}^*} \to 0$ and $H(\psi'_n)$, then

$$\boldsymbol{u}_n \to \boldsymbol{u} \text{ in } \boldsymbol{V}, \quad p_n \to p \text{ in } Q, \quad \text{as } n \to \infty.$$

Proof. It follows from Theorem 3.1 that under the stated assumptions, Problem 3.1 and Problem 3.2 have a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ and $(\boldsymbol{u}_n, p_n) \in \boldsymbol{V} \times Q$, respectively. Now assume $\boldsymbol{f}_n \to \boldsymbol{f}$ in $\boldsymbol{V}^*$ and $H(\psi'_n)$.

Step 1. First, we prove that the sequences $\{\|\boldsymbol{u}_n\|_V\}$ and $\left\{\|p_n\|_{L^2(\Omega)}\right\}$ are bounded. For $\boldsymbol{v} \in \boldsymbol{V}_0$, following a standard procedure similar to the derivation of Problem 3.1, equations (1)-(4) lead to

$$a(\boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0. \tag{19}$$

Likewise, from the perturbed problem, we have

$$a(\boldsymbol{u}_n, \boldsymbol{v}) + d(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{v}) - b(\boldsymbol{v}, p_n) = \langle \boldsymbol{f}_n, \boldsymbol{v} \rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V}_0. \tag{20}$$

Subtracting (19) from (20) yields

$$b(\boldsymbol{v}, p_n - p) = a(\boldsymbol{u}_n - \boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{v}) - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \langle \boldsymbol{f} - \boldsymbol{f}_n, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0. \tag{21}$$

With $\boldsymbol{v} \in \boldsymbol{V}_0$,

$$\begin{aligned} d(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{v}) - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) &= -d(\boldsymbol{u}_n, \boldsymbol{v}, \boldsymbol{u}_n) + d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}) \\ &= d(\boldsymbol{u} - \boldsymbol{u}_n, \boldsymbol{v}, \boldsymbol{u}) + d(\boldsymbol{u}_n, \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{u}_n) \\ &\le c_d (\|\boldsymbol{u}\|_{\boldsymbol{V}} + \|\boldsymbol{u}_n\|_{\boldsymbol{V}}) \|\boldsymbol{v}\|_{\boldsymbol{V}} \|\boldsymbol{u} - \boldsymbol{u}_n\|_{\boldsymbol{V}}. \end{aligned} \tag{22}$$

In view of (21), (22) and the inf -sup condition (13), we have

$$\begin{aligned} \beta \|p_n - p\|_{L^2(\Omega)} &\le \sup_{\boldsymbol{v} \in \boldsymbol{V}_0} \frac{b(\boldsymbol{v}, p_n - p)}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \\ &= \sup_{\boldsymbol{v} \in \boldsymbol{V}_0} \frac{a(\boldsymbol{u}_n - \boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{v}) - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \langle \boldsymbol{f} - \boldsymbol{f}_n, \boldsymbol{v} \rangle}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \\ &\le 2\nu \|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{V}} + c_d \left(\|\boldsymbol{u}\|_{\boldsymbol{V}} + \|\boldsymbol{u}_n\|_{\boldsymbol{V}}\right) \|\boldsymbol{u} - \boldsymbol{u}_n\|_{\boldsymbol{V}} + \|\boldsymbol{f}_n - \boldsymbol{f}\|_{\boldsymbol{V}^*}. \end{aligned} \tag{23}$$

As $\{\|\boldsymbol{u}_n\|_{\boldsymbol{V}}\}$ and $\{\|\boldsymbol{f}_n\|_{\boldsymbol{V}^*}\}$ are bounded, it follows that $\left\{\|p_n\|_{L^2(\Omega)}\right\}$ is also bounded.

Step 2. We now prove the weak convergence:

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \text{ in } \boldsymbol{V}, \quad p_n \rightharpoonup p \text{ in } Q, \qquad \text{as } n \to \infty.$$

Since $\{\|\boldsymbol{u}_n\|_V\}$ and $\left\{\|p_n\|_{L^2(\Omega)}\right\}$ are bounded, there exist subsequences of $\{\boldsymbol{u}_n\}$ and $\{p_n\}$, still denoted by $\{\boldsymbol{u}_n\}$ and $\{p_n\}$, and two elements $\overline{\boldsymbol{u}} \in \boldsymbol{V}$ and $\bar{p} \in Q$ such that

$$\boldsymbol{u}_n \rightharpoonup \overline{\boldsymbol{u}} \text{ in } \boldsymbol{V} \text{ and } p_n \rightharpoonup \bar{p} \text{ in } Q \text{ as } n \to \infty.$$

By resorting to a subsequence if necessary, we have $\boldsymbol{u}_n \to \overline{\boldsymbol{u}}$ a.e. on Γ_S. Taking upper limit in (16), from $H(\psi_n)$ we obtain that for any $\boldsymbol{v} \in \boldsymbol{V}$,

$$\begin{aligned}\langle \boldsymbol{f}, \boldsymbol{v}\rangle &\le a(\overline{\boldsymbol{u}}, \boldsymbol{v}) + d(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}, \boldsymbol{v}) - b(\boldsymbol{v}, \bar{p}) + \limsup_{n\to\infty} \int_{\Gamma_S} \psi_n^0 (\boldsymbol{u}_{n\tau}; \boldsymbol{v}_\tau)\, ds \\ &\le a(\overline{\boldsymbol{u}}, \boldsymbol{v}) + d(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}, \boldsymbol{v}) - b(\boldsymbol{v}, \bar{p}) + \int_{\Gamma_S} \limsup_{n\to\infty} \psi_n^0 (\boldsymbol{u}_{n\tau}; \boldsymbol{v}_\tau)\, ds.\end{aligned} \tag{24}$$

Since $\boldsymbol{u}_{n\tau} \to \overline{\boldsymbol{u}}_\tau$ a.e. on Γ_S, by $H(\psi_n')$,

$$\limsup_{n\to\infty} \psi_n^0 (\boldsymbol{u}_{n\tau}; \boldsymbol{v}_\tau) \le \psi^0 (\overline{\boldsymbol{u}}_\tau; \boldsymbol{v}_\tau).$$

Therefore, we can derive from (24) that for any $\boldsymbol{v} \in \boldsymbol{V}$,

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle \le a(\overline{\boldsymbol{u}}, \boldsymbol{v}) + d(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}, \boldsymbol{v}) - b(\boldsymbol{v}, \bar{p}) + \int_{\Gamma_S} \psi^0 (\overline{\boldsymbol{u}}_\tau; \boldsymbol{v}_\tau)\, ds. \tag{25}$$

Letting $n \to \infty$ in (17), we have

$$b(\overline{\boldsymbol{u}}, q) = 0 \quad \forall q \in Q. \tag{26}$$

Hence, from (25)-(26), we conclude that $(\overline{\boldsymbol{u}}, \bar{p}) \in \boldsymbol{V} \times Q$ is a solution of Problem 3.1. And it follows from Theorem 3.1, the unique solvability of Problem 3.1, that $\boldsymbol{u} = \overline{\boldsymbol{u}}$ and $p = \bar{p}$. This implies that every subsequence of $\{(\boldsymbol{u}_n, p_n)\}$ which converges weakly in $\boldsymbol{V} \times Q$ has the same limit and hence the entire sequence $\{(\boldsymbol{u}_n, p_n)\}$ converges weakly in $\boldsymbol{V} \times Q$ to $(\boldsymbol{u}, p)$, as $n \to \infty$.

Step 3. We are left to prove the strong convergence of the sequences. Again, without loss of generality, we can assume $\boldsymbol{u}_n \to \boldsymbol{u}$ a.e. on Γ_S for the solution sequence $\{\boldsymbol{u}_n\}$. By (9) and (16), we have

$$\begin{aligned}&a(\boldsymbol{u}_n - \boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{v}) - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p_n) + b(\boldsymbol{v}, p) \\ &\quad + \int_{\Gamma_S} \left[\psi_n^0 (\boldsymbol{u}_{n\tau}; \boldsymbol{v}_\tau) + \psi^0 (\boldsymbol{u}_\tau; -\boldsymbol{v}_\tau)\right] ds \ge \langle \boldsymbol{f}_n - \boldsymbol{f}, \boldsymbol{v}\rangle.\end{aligned} \tag{27}$$

Take $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_n$ in (27), we obtain

$$b(\boldsymbol{u} - \boldsymbol{u}_n, p) - b(\boldsymbol{u} - \boldsymbol{u}_n, p_n) = 0,$$

this is a direct result from (10) and (17). Thus

$$\begin{aligned}2\nu \|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{V}}^2 =& a(\boldsymbol{u}_n - \boldsymbol{u}, \boldsymbol{u}_n - \boldsymbol{u}) \\ &\le \int_{\Gamma_S} \left[\psi_n^0 (\boldsymbol{u}_{n\tau}; \boldsymbol{u}_\tau - \boldsymbol{u}_{n\tau}) + \psi^0 (\boldsymbol{u}_\tau; \boldsymbol{u}_{n\tau} - \boldsymbol{u}_\tau)\right] ds \\ &\quad + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_n - \boldsymbol{u}) - d(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) + \langle \boldsymbol{f}_n - \boldsymbol{f}, \boldsymbol{u}_n - \boldsymbol{u}\rangle.\end{aligned} \tag{28}$$

By (11)-(12), we have

$$\begin{aligned} d(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}_n-\boldsymbol{u})-d(\boldsymbol{u}_n,\boldsymbol{u}_n,\boldsymbol{u}_n-\boldsymbol{u}) &\le d(\boldsymbol{u}-\boldsymbol{u}_n,\boldsymbol{u},\boldsymbol{u}_n-\boldsymbol{u})\\ &\le c_d\|\boldsymbol{u}\|_{\boldsymbol{V}}\,\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{V}}^2\\ &\le c_d\alpha_{m_0}\,\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{V}}^2\,. \end{aligned}$$

Hence, by (18), we can derive from (28) that

$$\begin{aligned} (2\nu-c_d\alpha_{m_0})\,\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{V}}^2 \le \int_{\Gamma_S}&\left[\psi_n^0(\boldsymbol{u}_{n\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{n\tau})+\psi^0(\boldsymbol{u}_\tau;\boldsymbol{u}_{n\tau}-\boldsymbol{u}_\tau)\right]ds\\ &+\langle\boldsymbol{f}_n-\boldsymbol{f},\boldsymbol{u}_n-\boldsymbol{u}\rangle. \end{aligned} \tag{29}$$

Applying $H(\psi_n')$, $H(\psi)$, and Proposition 2.2(i),

$$\begin{aligned} \limsup_{n\to\infty}\int_{\Gamma_S}\psi_n^0(\boldsymbol{u}_{n\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{n\tau})\,ds &\le \int_{\Gamma_S}\limsup_{n\to\infty}\psi_n^0(\boldsymbol{u}_{n\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{n\tau})\,ds\\ &\le \int_{\Gamma_S}\psi^0(\boldsymbol{u}_\tau;\boldsymbol{u}_\tau-\boldsymbol{u}_\tau)\,ds\\ &= \int_{\Gamma_S}\psi^0(\boldsymbol{u}_\tau;\boldsymbol{0})\,ds\\ &=0. \end{aligned}$$

Similarly, applying $H(\psi)$ and Proposition 2.2(ii),

$$\begin{aligned} \limsup_{n\to\infty}\int_{\Gamma_S}\psi^0(\boldsymbol{u}_{n\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{n\tau})\,ds &\le \int_{\Gamma_S}\limsup_{n\to\infty}\psi^0(\boldsymbol{u}_{n\tau};\boldsymbol{u}_\tau-\boldsymbol{u}_{n\tau})\,ds\\ &\le \int_{\Gamma_S}\psi^0(\boldsymbol{u}_\tau;\boldsymbol{0})\,ds\\ &=0. \end{aligned}$$

Since $\boldsymbol{f}_n\to\boldsymbol{f}$ in $\boldsymbol{V}^*$, $\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}$ in $\boldsymbol{V}$, taking the upper limit in (29), we have

$$\limsup_{n\to\infty}\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{V}}\le 0.$$

Hence,

$$\lim_{n\to\infty}\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{V}}=0,$$

i.e., $\boldsymbol{u}_n$ converges to $\boldsymbol{u}$ in $\boldsymbol{V}$. By (23), p_n converges to p in Q. □

If we only consider perturbations in the external force density, we have a similar result for the following problem.

Problem (3.3)**.** *Find* $(\boldsymbol{u}_n,p_n)\in\boldsymbol{V}\times Q$ *such that*

$$a(\boldsymbol{u}_n,\boldsymbol{v})+d(\boldsymbol{u}_n,\boldsymbol{u}_n,\boldsymbol{v})-b(\boldsymbol{v},p_n)+\int_{\Gamma_S}\psi^0(\boldsymbol{u}_{n\tau};\boldsymbol{v}_\tau)\,ds\ge\langle\boldsymbol{f}_n,\boldsymbol{v}\rangle\quad\forall\boldsymbol{v}\in\boldsymbol{V},$$

$$b(\boldsymbol{u}_n,q)=0\quad\forall q\in Q.$$

Corollary 3.3. *Assume* $H(\psi)$, $\boldsymbol{f},\boldsymbol{f}_n\in\boldsymbol{V}_{m_0}^*$ *and the smallness condition*

$$0<c_d\alpha_{m_0^*}<2\nu-m\lambda_0^{-1}. \tag{30}$$

where

$$\alpha_{m_0}^*=\frac{c_0\lambda_0^{-1/2}|\Gamma_S|^{1/2}+m_0}{2\nu-m\lambda_0^{-1}}.$$

Then Problem 3.1 has a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$*, Problem 3.3 has a unique solution* $(\boldsymbol{u}_n, p_n) \in \boldsymbol{V} \times Q$*, and*

$$\|\boldsymbol{u}\|_{\boldsymbol{V}} \leq \alpha^*_{m_0}, \quad \|\boldsymbol{u}_n\|_{\boldsymbol{V}} \leq \alpha^*_{m_0}.$$

If $\|\boldsymbol{f}_n - \boldsymbol{f}\|_{\boldsymbol{V}^*} \to 0$*, then*

$$\boldsymbol{u}_n \to \boldsymbol{u} \text{ in } \boldsymbol{V}, \quad p_n \to p \text{ in } Q, \quad \text{as } n \to \infty.$$

3.2. **An optimal control.** Consider the optimal control of Navier-Stokes hemivariational inequality (9)–(10) with the external force density $\boldsymbol{f} \in \boldsymbol{V}^*$ as the control space. Let $\boldsymbol{V}^*_{ad} \subset \boldsymbol{V}^*_{m_0}$ be the set of admissible controls and $\mathcal{S} : \boldsymbol{V} \times Q \times \boldsymbol{V}^* \to \mathbb{R} \cup \{+\infty\}$ be the objective functional, which is of the form

$$\mathcal{S}(\boldsymbol{f}) = S(\boldsymbol{u}(\boldsymbol{f}), p(\boldsymbol{f}), \boldsymbol{f}),$$

where $(\boldsymbol{u}(\boldsymbol{f}), p(\boldsymbol{f})) \in \boldsymbol{V} \times Q$ is the solution of Problem 3.1 corresponding to $\boldsymbol{f}$. To simplify the notation, we denote the cost function by

$$\mathcal{S}(\boldsymbol{f}) = S(\boldsymbol{u}, p, \boldsymbol{f}),$$

where $(\boldsymbol{u}, p) = (\boldsymbol{u}(\boldsymbol{f}), p(\boldsymbol{f}))$. The optimal control problem can then be derived in the form

$$\inf \left\{\mathcal{S}(\boldsymbol{f}) \mid \boldsymbol{f} \in \boldsymbol{V}^*_{ad}\right\}. \tag{31}$$

Concerning the problem (31), we assume the following hypotheses for the control space and the objective functional.

$H(\boldsymbol{V}^*_{ad})$: $\boldsymbol{V}^*_{ad} \subset \boldsymbol{V}^*_{m_0}$ is a nonempty and compact subset of $\boldsymbol{V}^*$.

$H(S)$: $S : \boldsymbol{V} \times Q \times \boldsymbol{V}^* \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, i.e., if $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\boldsymbol{V}$, $p_n \to p$ in Q and $\boldsymbol{f}_n \to \boldsymbol{f}$ in $\boldsymbol{V}^*$, then

$$S(\boldsymbol{u}, p, \boldsymbol{f}) \leq \liminf_{n\to\infty} S\left(\boldsymbol{u}_n, p_n, \boldsymbol{f}_n\right).$$

An example of the objective functional is

$$\begin{aligned}\mathcal{S}(\boldsymbol{f}) = \frac{\alpha_1}{2} \int_\Omega \|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_d(\boldsymbol{x})\|^2_{\mathbb{R}^d}\, dx + \frac{\alpha_2}{2} \int_\Omega |p(\boldsymbol{x}) - p_d(\boldsymbol{x})|^2\, dx \\ + \frac{\alpha_3}{2} \int_\Omega \|\boldsymbol{f}(\boldsymbol{x})\|^2_{\mathbb{R}^d} dx,\end{aligned} \tag{32}$$

where the external force density $\boldsymbol{f} \in \boldsymbol{H} \subset \boldsymbol{V}^*$ is the control, $(\boldsymbol{u}, p)$ is the solution of Problem 3.1 corresponding to $\boldsymbol{f}$. The targets $\boldsymbol{u}_d$ and p_d are the ideal velocity distribution and pressure distribution, respectively. The cost parameters are fixed with $\alpha_1, \alpha_2 \geq 0$ satisfying $\alpha_1 + \alpha_2 = 1$, and $\alpha_3 > 0$. These constants determine the relative weights of the three integral terms in $\mathcal{S}(\boldsymbol{f})$. When $\alpha_1 = 0$, the control focuses solely on matching the desired pressure field. In contrast, when $\alpha_2 = 0$, the control shifts to matching the desired velocity field exclusively. The inclusion of the stabilization term with $\alpha_3 > 0$ ensures the force density to be within a reasonable range so that the control can still be physically realized. The goal is to control the force density of the flow field so that the velocity and pressure are close to the desired velocity and pressure. The choice (32) satisfies $H(S)$.

We are now in a position to deliver an existence result for the optimal control problem (31).

Theorem 3.4. *Assume* $H(\psi)$*,* $H(\boldsymbol{V}^*_{ad})$*,* $H(S)$ *and* (30)*. Then the optimal control problem* (31) *has a solution.*

Proof. Let $\{\boldsymbol{f}_n\}_{n\geq 1}$ be a minimizing sequence for the problem (31), i.e., $\boldsymbol{f}_n \in \boldsymbol{V}^*_{ad}$ and

$$\lim \mathcal{S}(\boldsymbol{f}_n) = \inf\{\mathcal{S}(\boldsymbol{f}) \mid \boldsymbol{f} \in \boldsymbol{V}^*_{ad}\} =: M.$$

We write $(\boldsymbol{u}_n, p_n) \in \boldsymbol{V} \times Q$ for the solution of Problem 3.3. Since $\{\boldsymbol{f}_n\} \subset \boldsymbol{V}^*_{ad}$, which is a compact subset of $\boldsymbol{V}^*$, by considering a subsequence if necessary, we have an element $\boldsymbol{f}^* \in \boldsymbol{V}^*_{ad}$ such that $\boldsymbol{f}_n \to \boldsymbol{f}^*$ in $\boldsymbol{V}^*$. We apply Corollary 3.3 to conclude that $\boldsymbol{u}_n \to \boldsymbol{u}^*$ in $\boldsymbol{V}$ and $p_n \to p^*$ in Q, where $(\boldsymbol{u}^*, p^*)$ is the solution of Problem 3.1 corresponding to $\boldsymbol{f}^*$. Due to hypothesis $H(\mathcal{S})$, we have

$$M \leq \mathcal{S}(\boldsymbol{u}^*, p^*, \boldsymbol{f}^*) \leq \liminf \mathcal{S}(\boldsymbol{u}_n, p_n, \boldsymbol{f}_n) = M.$$

Therefore, $\boldsymbol{f}^* \in \boldsymbol{V}^*_{ad}$ is a solution of the optimal control problem (31). □

4. Numerical approximation of the optimal control problem.

We study the numerical approximation of the optimal control problem (31) in this section. We keep all the assumptions stated in Theorem 3.4. Let $h > 0$ denote the discretization parameter and let $\left\{\left(\boldsymbol{V}_h, Q_h, \boldsymbol{V}^*_{ad,h}\right)\right\}_h$ be finite dimensional approximations of $(\boldsymbol{V}, Q, \boldsymbol{V}^*_{ad})$ as $h \to 0$.

We make the following assumptions for the discrete spaces.

$(H_{\boldsymbol{V}_h})$: $\boldsymbol{V}_h \subset \boldsymbol{V}$, and for any $\boldsymbol{v} \in \boldsymbol{V}$, there exists $\boldsymbol{v}_h \in \boldsymbol{V}_h$ with $\|\boldsymbol{v}_h - \boldsymbol{v}\|_{\boldsymbol{V}} \to 0$ as $h \to 0$.

(H_{Q_h}): $Q_h \subset Q$, and for any $q \in Q$, there exists $q_h \in Q_h$ with $\|q_h - q\|_Q \to 0$ as $h \to 0$.

$(H_{\boldsymbol{V}^*_{ad,h}})$: $\boldsymbol{V}^*_{ad,h} \subset \boldsymbol{V}^*_{m_0}$ is a nonempty and compact subset of $\boldsymbol{V}^*$. For any $\boldsymbol{f} \in \boldsymbol{V}^*_{ad}$, there exists $\boldsymbol{f}_h \in \boldsymbol{V}^*_{ad,h}$ with $\|\boldsymbol{f}_h - \boldsymbol{f}\|_{\boldsymbol{V}^*} \to 0$ as $h \to 0$ and for any $\{\boldsymbol{f}_h\}_h$, $\boldsymbol{f}_h \in \boldsymbol{V}^*_{ad,h}$, there exists a sequence $\left\{\boldsymbol{f}'_h\right\}_h \subset \boldsymbol{V}^*_{ad}$ with $\left\|\boldsymbol{f}_h - \boldsymbol{f}'_h\right\|_{\boldsymbol{V}^*} \to 0$ as $h \to 0$.

Assumption $(H_{\boldsymbol{V}^*_{ad,h}})$ is weaker than assuming $\boldsymbol{V}^*_{ad,h} \subset \boldsymbol{V}^*_{ad}$.

$(H_{V,Q,h})$: There exists a constant $\beta_1 > 0$ such that

$$\beta_1 \|q_h\|_Q \leq \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{0h}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}} \quad \forall q_h \in Q_h, \tag{33}$$

where $\boldsymbol{V}_{0h} = \boldsymbol{V}_h \cap \boldsymbol{V}_0$.

Note that $(H_{V,Q,h})$ is known as the Babuška-Brezzi condition. We list some well-known finite element spaces which satisfy the condition. We assume Ω is a polygonal domain ($d = 2$) or a polyhedral domain ($d = 3$). Let $\left\{\mathcal{T}^h\right\}_h$ be a regular family of finite element partitions of the domain $\bar{\Omega}$ into triangular/tetrahedral elements. For an integer $k \geq 0$, $P_k(T)$ is the space of polynomials of a total degree less than or equal to k in T, and $B(T)$ is the space of bubble functions on T. In this context, one option is to employ the Mini element ([2, Section 2])

$$\boldsymbol{V}_h = \left\{\boldsymbol{v}_h \in \boldsymbol{V} \cap C^0(\bar{\Omega})^d \mid \boldsymbol{v}_h|_T \in [P_1(T)]^d \oplus [B(T)]^d \ \forall T \in \mathcal{T}^h\right\},$$
$$Q_h = \left\{q_h \in Q \cap C^0(\bar{\Omega}) \mid q_h|_T \in P_1(T) \ \forall T \in \mathcal{T}^h\right\},$$

or P_2/P_1 finite element pair ([12, Section IV.4.2])

$$\boldsymbol{V}_h = \left\{\boldsymbol{v}_h \in \boldsymbol{V} \cap C^0(\bar{\Omega})^d \mid \boldsymbol{v}_h|_T \in [P_2(T)]^d \ \forall T \in \mathcal{T}^h\right\},$$
$$Q_h = \left\{q_h \in Q \cap C^0(\bar{\Omega}) \mid q_h|_T \in P_1(T) \ \forall T \in \mathcal{T}^h\right\}.$$

For commonly used finite element spaces, $(H_{\boldsymbol{V}_h})$ and (H_{Q_h}) are verified with a standard technique well-known in the finite element research community (cf. e.g., the proof of Theorem 10.4.1 in [3]), through a combination of the density of smooth functions in a function space under concern, $\boldsymbol{V}$ or Q in the current context, and standard error bounds for finite element interpolations of smooth functions found in virtually any textbook on the finite element method. One is referred to [19, Section 7.1] and references therein on the density of $\boldsymbol{V} \cap C^{\infty}(\overline{\Omega}; \mathbb{R}^d)$ in $\boldsymbol{V}$. The density of $C^{\infty}(\overline{\Omega})$ functions in $L^2(\Omega)$ is well-known in the Sobolev space theory, and it is then trivial to deduce the density of $Q \cap C^{\infty}(\overline{\Omega})$ in Q.

Regarding the approximation of the control space, as the admissible set does not have the structure of a vector space, the set $\boldsymbol{V}^*_{ad,h}$ is not in general contained in the set $\boldsymbol{V}^*_{ad}$. We follow the approach developed for the obstacle problem and make assumption $(H_{\boldsymbol{V}^*_{ad,h}})$. For $(H_{\boldsymbol{V}^*_{ad,h}})$, the specific approximation space is determined by the given form of $\boldsymbol{V}^*_{ad}$ and the final property in it is automatically satisfied if $\boldsymbol{V}^*_{ad,h} \subset \boldsymbol{V}^*_{ad}$ $\forall\, h \in (0,1)$. We will present an example to illustrate the validity of this assumption. Let $\boldsymbol{g}_1, \boldsymbol{g}_2 \in \boldsymbol{H}$ and define $\boldsymbol{G}(\boldsymbol{x}) = G := \max\left(\|\boldsymbol{g}_1\|_{L^\infty(\Omega;\mathbb{R}^d)}, \|\boldsymbol{g}_2\|_{L^\infty(\Omega;\mathbb{R}^d)}\right)$, $\forall\, \boldsymbol{x} \in \Omega$. Assume $\|\boldsymbol{G}\|_{\boldsymbol{V}^*} \le m_0$ and consider the admissible control set described by

$$\boldsymbol{V}^*_{ad} = \{\boldsymbol{f} \in \boldsymbol{H} \mid \boldsymbol{g}_1 \le \boldsymbol{f} \le \boldsymbol{g}_2 \text{ a.e. in } \Omega\}.$$

From the above definition and the properties of $\boldsymbol{g}_1, \boldsymbol{g}_2$, it follows that $\boldsymbol{V}^*_{ad} \subset \boldsymbol{V}^*_{m_0}$. Since $\boldsymbol{V}^*_{ad}$ is a closed nonempty convex subset of $\boldsymbol{H}$, it is a nonempty and weakly closed subset of $\boldsymbol{H}$ ([3, Section 3.3]). Given that $\boldsymbol{H}$ compactly embeds into $\boldsymbol{V}^*$, we see that $H(\boldsymbol{V}^*_{ad})$ is satisfied.

Define

$$\boldsymbol{U}_h = \left\{\boldsymbol{f}_h \in \boldsymbol{H} \cap C^0(\bar{\Omega}) \mid \boldsymbol{f}_h|_T \in P_1(T)\ \forall T \in \mathcal{T}^h\right\}.$$

Letting $\mathcal{N}_h$ denote the set of the nodes of the spaces $\boldsymbol{U}_h$ and define

$$\boldsymbol{V}^*_{ad,h} = \{\boldsymbol{f}_h \in \boldsymbol{U}_h \mid \forall\, \boldsymbol{b} \in \mathcal{N}_h, \boldsymbol{g}_1(\boldsymbol{b}) \le \boldsymbol{f}_h(\boldsymbol{b}) \le \boldsymbol{g}_2(\boldsymbol{b})\}.$$

Thus, $\boldsymbol{V}^*_{ad,h} \subset \boldsymbol{V}^*_{m_0}$ is a nonempty and compact subset of $\boldsymbol{V}^*$, and by ([7, Theorem 5.1.2]), we know that $(H_{\boldsymbol{V}^*_{ad,h}})$ is satisfied.

We then give the discrete form of Problem 3.1.

Problem (4.1)**.** *Find* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ *such that*

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_{h,\tau}; \boldsymbol{v}_{h,\tau})\, ds \ge \langle \boldsymbol{f}_h, \boldsymbol{v}_h\rangle \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{34}$$

$$b(\boldsymbol{u}_h, q_h) = 0 \quad \forall q_h \in Q_h. \tag{35}$$

We have an existence, uniqueness and boundedness result for Problem 4.1, similar to Theorem 3.1. The numerical approximation of (31) is then

$$\inf\left\{\mathcal{S}_h(\boldsymbol{f}_h) \mid \boldsymbol{f}_h \in \boldsymbol{V}^*_{ad,h}\right\}, \tag{36}$$

where

$$\mathcal{S}_h(\boldsymbol{f}_h) = S(\boldsymbol{u}_h, p_h, \boldsymbol{f}_h),$$

and $(\boldsymbol{u}_h, p_h) = (\boldsymbol{u}_h(\boldsymbol{f}_h), p_h(\boldsymbol{f}_h))$ is the solution of Problem 4.1.

Similar to the optimal control problem (31), under the assumptions $H(\psi)$, $H(S)$, (30), and $(H_{\boldsymbol{V}_h})$, (H_{Q_h}), $(H_{\boldsymbol{V}^*_{ad,h}})$, $(H_{\boldsymbol{V},Q,h})$, the discrete problem (36) has a solution. To obtain the convergence analysis of (36), we first prove a convergence result for Problem 4.1 as $\boldsymbol{f}_h \to \boldsymbol{f}$ in $\boldsymbol{V}^*$.

Theorem 4.1. *Assume $H(\psi)$, (30), and $(H_{\boldsymbol{V}_h})$, (H_{Q_h}), and $(H_{\boldsymbol{V},Q,h})$. Let $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ and $(\boldsymbol{u}_h, p_h)$ be the solutions of Problem 3.1 and Problem 4.1, respectively. Then as $\boldsymbol{f}_h \to \boldsymbol{f}$ in $\boldsymbol{V}^*$, we have*

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\boldsymbol{V}} + \|p_h - p\|_Q \to 0 \quad \text{as } h \to 0.$$

Proof. We split the proof into several steps.

Step 1. We prove the uniform boundedness of the numerical solutions. Since $\boldsymbol{V}^*_{ad,h} \subset \boldsymbol{V}^*_{m_0}$, we have $\{\|\boldsymbol{f}_h\|_{\boldsymbol{V}^*}\}$ is bounded. Thus from the discrete analog of Theorem 3.1, $\{\|\boldsymbol{u}_h\|_{\boldsymbol{V}}\}$ is bounded. Similar to (19),

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) = \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_{0h}, \tag{37}$$

so

$$b(\boldsymbol{v}_h, p_h) = a(\boldsymbol{u}_h, \boldsymbol{v}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_{0h}.$$

By the Babuška-Brezzi condition (33)

$$\beta_1 \|p_h\|_Q \le \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{0h}} \frac{b(\boldsymbol{v}_h, p_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}},$$

we thus have

$$\begin{aligned}\beta_1 \|p_h\|_Q &\le \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{0h}} \frac{a(\boldsymbol{u}_h, \boldsymbol{v}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}} \\ &\le c\left(\|\boldsymbol{u}_h\|_{\boldsymbol{V}} + \|\boldsymbol{u}_h\|^2_{\boldsymbol{V}} + \|\boldsymbol{f}_h\|_{\boldsymbol{V}^*}\right).\end{aligned}$$

Therefore, since $\{\|\boldsymbol{u}_h\|_{\boldsymbol{V}}\}_h$ and $\{\|\boldsymbol{f}_h\|_{\boldsymbol{V}^*}\}_h$ are bounded, so is $\left\{\|p_h\|_Q\right\}_h$.

Step 2. Since $\{\|\boldsymbol{u}_h\|_{\boldsymbol{V}}\}_h$ and $\left\{\|p_h\|_Q\right\}_h$ are bounded, there exist a subsequence, still denoted by $\{(\boldsymbol{u}_h, p_h)\}_h$, and elements $\overline{\boldsymbol{u}} \in \boldsymbol{V}, \overline{p} \in Q$ such that

$$\boldsymbol{u}_h \rightharpoonup \overline{\boldsymbol{u}} \text{ in } \boldsymbol{V}, \quad \boldsymbol{u}_h \to \overline{\boldsymbol{u}} \text{ in } \boldsymbol{H}, \quad p_h \rightharpoonup \overline{p} \text{ in } Q \quad \text{as } h \to 0. \tag{38}$$

Step 3. Let us prove that

$$b(\overline{\boldsymbol{u}}, q) = 0 \quad \forall\, q \in Q. \tag{39}$$

For an arbitrarily fixed $q \in Q$, by (H_{Q_h}), there exists $q_h \in Q_h$ such that

$$q_h \to q \text{ in } Q. \tag{40}$$

By (35),

$$b(\boldsymbol{u}_h, q_h) = 0.$$

Write

$$\begin{aligned} b(\boldsymbol{u}_h, q_h) &= b(\boldsymbol{u}_h - \overline{\boldsymbol{u}}, q_h) + b(\overline{\boldsymbol{u}}, q_h) \\ &= b(\boldsymbol{u}_h - \overline{\boldsymbol{u}}, q_h - q) + b(\boldsymbol{u}_h - \overline{\boldsymbol{u}}, q) + b(\overline{\boldsymbol{u}}, q_h). \end{aligned} \tag{41}$$

Since $\|\boldsymbol{u}_h - \overline{\boldsymbol{u}}\|_{\boldsymbol{V}}$ is bounded independent of h and $\|q_h - q\|_Q \to 0$ as $h \to 0$, we have

$$|b(\boldsymbol{u}_h - \overline{\boldsymbol{u}}, q_h - q)| \le c\,\|\boldsymbol{u}_h - \overline{\boldsymbol{u}}\|_{\boldsymbol{V}} \|q_h - q\|_Q \to 0 \quad \text{as } h \to 0.$$

Due to (38), as $h \to 0$,

$$b(\boldsymbol{u}_h - \overline{\boldsymbol{u}}, q) \to 0.$$

By (40), as $h \to 0$,

$$b(\overline{\boldsymbol{u}}, q_h) \to b(\overline{\boldsymbol{u}}, q).$$

Thus, taking the limit $h \to 0$ in (41), we prove (39).

Step 4. We prove the strong convergence

$$\boldsymbol{u}_h \to \overline{\boldsymbol{u}} \text{ in } \boldsymbol{V}, \quad \text{as } h \to 0. \tag{42}$$

By $(H_{\boldsymbol{V}_h})$ and (H_{Q_h}), there exist $\overline{\boldsymbol{u}}_h \in \boldsymbol{V}_h$ and $\overline{p}_h \in Q_h$ such that

$$\bar{\boldsymbol{u}}_h \to \overline{\boldsymbol{u}} \text{ in } \boldsymbol{V}, \quad \overline{p}_h \to \overline{p} \text{ in } Q, \quad \text{as } h \to 0. \tag{43}$$

We start with

$$\begin{aligned} 2\nu \|\overline{\boldsymbol{u}} - \boldsymbol{u}_h\|_{\boldsymbol{V}}^2 &\le a(\overline{\boldsymbol{u}} - \boldsymbol{u}_h, \overline{\boldsymbol{u}} - \boldsymbol{u}_h) \\ &= a(\overline{\boldsymbol{u}}, \overline{\boldsymbol{u}} - \boldsymbol{u}_h) - a(\boldsymbol{u}_h, \overline{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h) - a(\boldsymbol{u}_h, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_h). \end{aligned} \tag{44}$$

By (34) with $\boldsymbol{v}_h = \bar{\boldsymbol{u}}_h - \boldsymbol{u}_h$,

$$\begin{aligned} -a(\boldsymbol{u}_h, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_h) &\le d(\boldsymbol{u}_h, \boldsymbol{u}_h, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_h) - b(\bar{\boldsymbol{u}}_h - \boldsymbol{u}_h, p_h) \\ &\quad + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_{h,\tau}; \bar{\boldsymbol{u}}_{h,\tau} - \boldsymbol{u}_{h,\tau})\, ds - \langle \boldsymbol{f}_h, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_h \rangle. \end{aligned}$$

Then from (44),

$$\begin{aligned} 2\nu \|\overline{\boldsymbol{u}} - \boldsymbol{u}_h\|_{\boldsymbol{V}}^2 \le{}& a(\overline{\boldsymbol{u}}, \overline{\boldsymbol{u}} - \boldsymbol{u}_h) - a(\boldsymbol{u}_h, \overline{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_h) \\ & - b(\bar{\boldsymbol{u}}_h - \boldsymbol{u}_h, p_h) + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_{h,\tau}; \overline{\boldsymbol{u}}_{h,\tau} - \boldsymbol{u}_{h,\tau})\, ds \\ & - \langle \boldsymbol{f}_h, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_h \rangle. \end{aligned} \tag{45}$$

By (12), we have that:

$$d(\boldsymbol{u}_h, \boldsymbol{u}_h, \bar{\boldsymbol{u}}_h - \boldsymbol{u}_h) = d(\boldsymbol{u}_h, \boldsymbol{u}_h, \bar{\boldsymbol{u}}_h - \overline{\boldsymbol{u}}) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \overline{\boldsymbol{u}}).$$

We now consider each term on the right side of (45). Since $\boldsymbol{u}_h \rightharpoonup \overline{\boldsymbol{u}}$ in $\boldsymbol{V}$,

$$\begin{aligned} a(\overline{\boldsymbol{u}}, \overline{\boldsymbol{u}} - \boldsymbol{u}_h) &\to 0, \\ d(\boldsymbol{u}_h, \boldsymbol{u}_h, \overline{\boldsymbol{u}}) &\to 0. \end{aligned}$$

Since $\|\overline{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h\|_{\boldsymbol{V}} \to 0$ and $\|\boldsymbol{u}_h\|_{\boldsymbol{V}}$ is bounded independent of h,

$$\begin{aligned} |a(\boldsymbol{u}_h, \overline{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h)| &\le 2\nu \|\boldsymbol{u}_h\|_{\boldsymbol{V}} \|\overline{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h\|_{\boldsymbol{V}} \to 0, \\ |d(\boldsymbol{u}_h, \boldsymbol{u}_h, \bar{\boldsymbol{u}}_h - \overline{\boldsymbol{u}})| &\le c_d \|\boldsymbol{u}_h\|_{\boldsymbol{V}}^2 \|\overline{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h\|_{\boldsymbol{V}} \to 0. \end{aligned}$$

Write

$$\begin{aligned} -b(\bar{\boldsymbol{u}}_h - \boldsymbol{u}_h, p_h) &= -b(\bar{\boldsymbol{u}}_h, p_h) + b(\boldsymbol{u}_h, p_h) \\ &= -b(\bar{\boldsymbol{u}}_h, p_h) \\ &= -b(\bar{\boldsymbol{u}}_h - \overline{\boldsymbol{u}}, p_h) \\ &\le c_b \|p_h\|_Q \|\bar{\boldsymbol{u}}_h - \overline{\boldsymbol{u}}\|_{\boldsymbol{V}}, \end{aligned}$$

where for the second equality, we used (35), and for the third equality, we used (39). Since $\|\overline{\boldsymbol{u}} - \bar{\boldsymbol{u}}_h\|_{\boldsymbol{V}} \to 0$ and $\|p_h\|_Q$ is bounded independent of h, we obtain

$$b(\bar{\boldsymbol{u}}_h - \boldsymbol{u}_h, p_h) \to 0.$$

Using $H(\psi)$ and for a subsequence, due to the compact embedding $H^1(\Omega) \subset L^2(\Gamma_S)$ ([28, p. 7]),

$$\|\bar{\boldsymbol{u}}_{h,\tau} - \boldsymbol{u}_{h,\tau}\|_{L^2(\Gamma_S;\mathbb{R}^d)} \le \|\bar{\boldsymbol{u}}_{h,\tau} - \overline{\boldsymbol{u}}_\tau\|_{L^2(\Gamma_S;\mathbb{R}^d)} + \|\overline{\boldsymbol{u}}_\tau - \boldsymbol{u}_{h,\tau}\|_{L^2(\Gamma_S;\mathbb{R}^d)} \to 0,$$

we have

$$\begin{aligned}\int_{\Gamma_S} \psi^0\left(\boldsymbol{u}_{h,\tau};\bar{\boldsymbol{u}}_{h,\tau}-\boldsymbol{u}_{h,\tau}\right)ds &\le \int_{\Gamma_S}\left(c_0+c_1\left|\boldsymbol{u}_{h,\tau}\right|\right)\left|\bar{\boldsymbol{u}}_{h,\tau}-\boldsymbol{u}_{h,\tau}\right|ds\\ &\le c\left(1+\|\boldsymbol{u}_h\|_{\boldsymbol{V}}\right)\|\bar{\boldsymbol{u}}_{h,\tau}-\boldsymbol{u}_{h,\tau}\|_{L^2(\Gamma_S;\mathbb{R}^d)}\\ &\to 0.\end{aligned}$$

Note that $\overline{\boldsymbol{u}}_h-\boldsymbol{u}_h\to\boldsymbol{0}$ in $\boldsymbol{V}$ and $\boldsymbol{f}_h\to\boldsymbol{f}$ in $\boldsymbol{V}^*$, the latter implies the uniform boundedness of $\boldsymbol{f}_h$ in $\boldsymbol{V}^*$. Then,

$$|\langle\boldsymbol{f}_h,\overline{\boldsymbol{u}}_h-\boldsymbol{u}_h\rangle|\le\|\boldsymbol{f}_h\|_{\boldsymbol{V}^*}\|\overline{\boldsymbol{u}}_h-\boldsymbol{u}_h\|_{\boldsymbol{V}}\to 0.$$

Hence, we obtain the strong convergence (42) from (45).

Step 5. We prove that the limit $(\overline{\boldsymbol{u}},\overline{p})$ is the unique solution of Problem 3.1. For any $\boldsymbol{v}\in\boldsymbol{V}$ and $q\in Q$, by assumptions $(H_{\boldsymbol{V}_h})$ and (H_{Q_h}), we have a sequence $\{(\boldsymbol{v}_h,q_h)\}\subset\boldsymbol{V}\times Q$ with $\boldsymbol{v}_h\in\boldsymbol{V}_h, q_h\in Q_h$ such that

$$\boldsymbol{v}_h\to\boldsymbol{v}\text{ in }\boldsymbol{V},\quad q_h\to q\text{ in }Q.$$

From (34),

$$a\left(\boldsymbol{u}_h,\boldsymbol{v}_h\right)+d\left(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h\right)-b\left(\boldsymbol{v}_h,p_h\right)+\int_{\Gamma_S}\psi^0\left(\boldsymbol{u}_{h,\tau};\boldsymbol{v}_{h,\tau}\right)ds\ge\langle\boldsymbol{f}_h,\boldsymbol{v}_h\rangle. \quad (46)$$

Due to (4), (38) and $\boldsymbol{f}_h\to\boldsymbol{f}$ in $\boldsymbol{V}^*$, we have, as $h\to 0$,

$$\begin{aligned}&a\left(\boldsymbol{u}_h,\boldsymbol{v}_h\right)\to a(\overline{\boldsymbol{u}},\boldsymbol{v}),\\ &d\left(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h\right)\to d\left(\overline{\boldsymbol{u}},\overline{\boldsymbol{u}},\boldsymbol{v}\right),\\ &b\left(\boldsymbol{v}_h,p_h\right)\to b(\boldsymbol{v},\overline{p}),\\ &\langle\boldsymbol{f}_h,\boldsymbol{v}_h\rangle\to\langle\boldsymbol{f},\boldsymbol{v}\rangle.\end{aligned}$$

Also, for subsequences, still denoted by $\{\boldsymbol{u}_{h,\tau}\}_h$ and $\{\boldsymbol{v}_{h,\tau}\}_h$, we have

$$\boldsymbol{u}_{h,\tau}\to\overline{\boldsymbol{u}}_\tau\text{ and }\boldsymbol{v}_{h,\tau}\to\boldsymbol{v}_\tau\qquad\text{a.e. on }\Gamma_S,$$

and then

$$\limsup_{h\to 0}\int_{\Gamma_S}\psi^0\left(\boldsymbol{u}_{h,\tau};\boldsymbol{v}_{h,\tau}\right)ds\le\int_{\Gamma_S}\limsup_{h\to 0}\psi^0\left(\boldsymbol{u}_{h,\tau};\boldsymbol{v}_{h,\tau}\right)ds\le\int_{\Gamma_S}\psi^0\left(\overline{\boldsymbol{u}}_\tau;\boldsymbol{v}_\tau\right)ds.$$

Thus, we derive from (46) that

$$a(\overline{\boldsymbol{u}},\boldsymbol{v})+d\left(\overline{\boldsymbol{u}},\overline{\boldsymbol{u}},\boldsymbol{v}\right)-b(\boldsymbol{v},\overline{p})+\int_{\Gamma_S}\psi^0\left(\overline{\boldsymbol{u}}_\tau;\boldsymbol{v}_\tau\right)ds\ge\langle\boldsymbol{f},\boldsymbol{v}\rangle.$$

This relation and (39) show that $(\overline{\boldsymbol{u}},\overline{p})=(\boldsymbol{u},p)$ is the unique solution of Problem 3.1. Since the limit $(\boldsymbol{u},p)$ is unique, the entire sequence converges: as $h\to 0$,

$$\boldsymbol{u}_h\to\boldsymbol{u}\text{ in }\boldsymbol{V},\quad q_h\to q\text{ in }Q.$$

Step 6. We prove the strong convergence:

$$p_h\to p\quad\text{ in }Q. \quad (47)$$

By the Babuška-Brezzi condition (33),

$$\beta_1\|p_h-\overline{p}_h\|_Q\le\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{0h}}\frac{b\left(\boldsymbol{v}_h,p_h-\overline{p}_h\right)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}}.$$

Write

$$b\left(\boldsymbol{v}_h,p_h-\overline{p}_h\right)=b\left(\boldsymbol{v}_h,p_h-p\right)+b\left(\boldsymbol{v}_h,p-\overline{p}_h\right).$$

Recall (37),

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h) = \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{0h}.$$

From (19), we have

$$a(\boldsymbol{u}, \boldsymbol{v}_h) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{0h}.$$

Thus, it follows that

$$b(\boldsymbol{v}_h, p_h - p) = a(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) - \langle \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h \rangle,$$

here

$$\begin{aligned} d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) &= -d(\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{u}_h) + d(\boldsymbol{u}, \boldsymbol{v}_h, \boldsymbol{u}) \\ &= d(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{u}) + d(\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{u} - \boldsymbol{u}_h) \\ &\le c_d \left(\|\boldsymbol{u}\|_{\boldsymbol{V}} + \|\boldsymbol{u}_h\|_{\boldsymbol{V}}\right) \|\boldsymbol{v}_h\|_{\boldsymbol{V}} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{V}}. \end{aligned}$$

Then,

$$\begin{aligned} \beta_1 \|p_h - \overline{p}_h\|_Q \le & \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{0h}} \frac{1}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}} [a(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) + d(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) \\ & - \langle \boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h \rangle + b(\boldsymbol{v}_h, p - \overline{p}_h)] \\ \le & \, c\left[\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\boldsymbol{V}} + \|\boldsymbol{f}_h - \boldsymbol{f}\|_{\boldsymbol{V}^*} + \|p - \overline{p}_h\|_Q\right]. \end{aligned}$$

By the triangle inequality

$$\|p_h - p\|_Q \le \|p_h - \overline{p}_h\|_Q + \|p - \overline{p}_h\|_Q,$$

we then obtain

$$\|p_h - p\|_Q \le c\left[\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\boldsymbol{V}} + \|\boldsymbol{f}_h - \boldsymbol{f}\|_{\boldsymbol{V}^*} + \|p - \overline{p}_h\|_Q\right].$$

From this inequality and (42), (43), noting that $\overline{\boldsymbol{u}} = \boldsymbol{u}$, $\overline{p} = p$ and $\boldsymbol{f}_h \to \boldsymbol{f}$ in $\boldsymbol{V}^*$, we conclude that (47) holds. □

Let us make a further assumption on the cost function S in the convergence analysis of the numerical approximation of the optimal control problem (36).

$H(S')$: If $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\boldsymbol{V}$, $p_n \to p$ in Q, and $\boldsymbol{f}_n \to \boldsymbol{f}$ in $\boldsymbol{V}^*$, then

$$S(\boldsymbol{u}, p, \boldsymbol{f}) = \lim_{n\to\infty} S(\boldsymbol{u}_n, p_n, \boldsymbol{f}_n).$$

Note that the function (32) has the property $H(S')$.

Theorem 4.2. *Assume $H(\psi)$, $H(\boldsymbol{V}^*_{ad})$, $H(S)$, (30), $(H_{\boldsymbol{V}_h})$, (H_{Q_h}), $(H_{\boldsymbol{V}^*_{ad,h}})$, $(H_{\boldsymbol{V},Q,h})$, and in addition $H(S')$. For each $h > 0$, let $\boldsymbol{f}_h$ be a solution of the problem (36). Then there exist a subsequence, again denoted as $\{\boldsymbol{f}_h\}$, and an element $\boldsymbol{f} \in \boldsymbol{V}^*_{ad}$ such that*

$$\boldsymbol{f}_h \to \boldsymbol{f} \text{ in } \boldsymbol{V}^*, \quad \boldsymbol{u}_h(\boldsymbol{f}_h) \to \boldsymbol{u}(\boldsymbol{f}) \text{ in } \boldsymbol{V}, \quad p_h(\boldsymbol{f}_h) \to p(\boldsymbol{f}) \text{ in } Q,$$

and $\boldsymbol{f} \in \boldsymbol{V}^*_{ad}$ is a solution of the optimal control problem (31).

Proof. By $H(\boldsymbol{V}^*_{ad,h})$, we have a sequence $\{\boldsymbol{f}'_h\}_h \subset \boldsymbol{V}^*_{ad}$ with $\|\boldsymbol{f}_h - \boldsymbol{f}'_h\|_{\boldsymbol{V}^*} \to 0$ as $h \to 0$. Since $\boldsymbol{V}^*_{ad}$ is compact, by considering a subsequence if necessary, we can obtain that $\boldsymbol{f}'_h \to \boldsymbol{f}$ in $\boldsymbol{V}^*$ for some element $\boldsymbol{f} \in \boldsymbol{V}^*_{ad}$. It immediately follows from the triangle inequality that $\boldsymbol{f}_h \to \boldsymbol{f}$ in $\boldsymbol{V}^*$. By Theorem 4.1, we obtain

$$\boldsymbol{u}_h \to \boldsymbol{u} \text{ in } \boldsymbol{V}, \quad p_h \to p \text{ in } Q.$$

So by $H(S')$,

$$\mathcal{S}(\boldsymbol{f}) = \lim_{h\to 0} \mathcal{S}_h\left(\boldsymbol{f}_h\right).$$

We need to show that $\boldsymbol{f}$ is a solution of (31). Let $\overline{\boldsymbol{f}}$ be a solution of (31). By $\left(H_{\boldsymbol{V}^*_{ad,h}}\right)$, there exists $\bar{\boldsymbol{f}}_h \in \boldsymbol{V}^*_{ad,h}$ with

$$\bar{\boldsymbol{f}}_h \to \overline{\boldsymbol{f}} \quad \text{in } \boldsymbol{V}^*.$$

Then by Theorem 4.1,

$$\boldsymbol{u}_h\left(\bar{\boldsymbol{f}}_h\right) \to \boldsymbol{u}(\overline{\boldsymbol{f}}) \text{ in } \boldsymbol{V}, \quad p_h\left(\bar{\boldsymbol{f}}_h\right) \to p(\overline{\boldsymbol{f}}) \text{ in } Q.$$

By $H\left(S'\right)$,

$$\lim_{h\to 0} \mathcal{S}_h\left(\bar{\boldsymbol{f}}_h\right) = \mathcal{S}(\overline{\boldsymbol{f}}).$$

From the definition of $\boldsymbol{f}_h$,

$$\mathcal{S}_h\left(\boldsymbol{f}_h\right) \le \mathcal{S}_h\left(\bar{\boldsymbol{f}}_h\right).$$

Taking the limit of both sides of the above inequality as $h \to 0$, we obtain

$$\mathcal{S}(\boldsymbol{f}) \le \mathcal{S}(\overline{\boldsymbol{f}}).$$

Thus, $\boldsymbol{f}$ is a solution of (31). □

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