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Numerical analysis of a class of hemivariational inequalities governed by fluid–fluid coupled flow

Feifei Jing [a,1], Weimin Han [b,2], Guanyu Zhou [c,*,3]

[a] School of Mathematics and Statistics and Xi'an Key Laboratory of Scientific Computation and Applied Statistics, Northwestern Polytechnical University, Xi'an 710129, Shanxi, China
[b] Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA
[c] Institute of Fundamental and Frontier Sciences & School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, 610051, Sichuan China

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ABSTRACT

We explore the well-posedness and conduct a numerical analysis of hemivariational inequalities for the coupled stationary Navier–Stokes/Navier–Stokes system. The interface condition involves the Clark subgradient and serves as a generalization of various interface interaction relations, including nonlinear transmission conditions and friction-type conditions. We present an existence and uniqueness result for a solution of the continuous model. We propose a domain decomposition approach to solve the coupled system and examine the convergence of iterations. Moreover, we use the finite element approximation to discretize the hemivariational inequality of the coupled system and derive error estimates, which lead to an optimal order for the $P1b/P1$ pair under appropriate solution regularity assumptions. Numerical results are reported that illustrate the optimal convergence order predicted by theoretical analysis.

1. Introduction

Various geophysical flows, including atmosphere–ocean interaction [1–4], two layers of a stratified fluid [5,6], and coupled turbulent fluids [7–10], rely on fluid–fluid interaction models. These models couple fluids through linear or nonlinear transmission conditions on a shared interface. In this work, we introduce a generalized interface condition to stationary Navier–Stokes/Navier–Stokes (NS/NS) coupled flows, incorporating the Clark subgradient and producing a hemivariational inequality. The mathematical model is described as follows.

Let Ω be a Lipschitz domain in $\mathbb{R}^d$ $(d = 2, 3)$, which is decomposed into two subdomains Ω_1 and Ω_2. The common boundary of Ω_1 and Ω_2 is the interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. For $i = 1, 2$, the remaining part of the boundary of Ω_i is denoted by $\Gamma_i := \partial\Omega_i \backslash \Gamma$, and we assume $|\Gamma_i| > 0$. We denote by (u_i, p_i) the velocity and pressure in the subregion Ω_i, and by ν_i the unit outer normal vector to $\partial\Omega_i$ (see Fig. 1). The deformation rate tensor and stress tensor are represented by $\mathbb{D}(u_i) := (\nabla u_i + \nabla^\top u_i)/2$ and $\sigma_i(u_i, p_i) := 2\mu_i \mathbb{D}(u_i) - p_i \mathbb{I}$,

* Corresponding author.
E-mail addresses: ffjing@nwpu.edu.cn (F. Jing), weimin-han@uiowa.edu (W. Han), wind_geno@live.com, zhoug@uestc.edu.cn (G. Zhou).
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respectively, with the constant viscosity $\mu_i > 0$ and unit tensor $\mathbb{I}$. In the subregion Ω_i, the fluid flow is governed by the stationary incompressible Navier–Stokes equations supplemented by a Dirichlet boundary condition on Γ_i:

$$-\nabla \cdot \sigma(u_i, p_i) + (u_i \cdot \nabla)u_i = f_i \quad \text{in } \Omega_i, \tag{1.1a}$$

$$\nabla \cdot u_i = 0 \quad \text{in } \Omega_i, \tag{1.1b}$$

$$u_i = 0 \quad \text{on } \Gamma_i. \tag{1.1c}$$

To describe the transmission condition on the interface Γ, we adopt the rigid lid hypothesis [5]: Γ is a mean interface and the values of u_i and p_i are mean values of the velocity and the pressure; and Γ is unmoving. We now introduce the notation of the normal and tangential components of the velocity and the stress vector:

$$u_{i,\nu} := u_i \cdot \nu_i, \qquad u_{i,\tau} := (\mathbb{I} - \nu \otimes \nu)u_i,$$
$$\sigma_{i,\nu} := \sigma_i \nu \cdot \nu, \qquad \sigma_{i,\tau} := (\mathbb{I} - \nu \otimes \nu)2\mu_i \mathbb{D}(u_i)\nu_i.$$

where $\nu := \nu_1 = -\nu_2$. We use $[s]$ to represent the jump of s on Γ, for example,

$$[u] := u_1 - u_2, \qquad [u_\tau] := u_{1,\tau} - u_{2,\tau}.$$

Different transmission conditions have been introduced to simulate diverse physical phenomena on the interface, for instance, adhesion [11,12], friction [3,4,7–9], coupling with the porous medium flows [13]. An example of modeling the atmosphere–ocean interaction is through the introduction of the linear coupling condition [14]:

$$\sigma_i \tau_i = (-1)^i \kappa [u] \quad \text{on } \Gamma \qquad (i = 1, 2), \tag{1.2}$$

where $\kappa > 0$ is the friction coefficient. It reflects the bulk fluids across the boundary layers slide past each other with a horizontal frictional force linearly depending on the jump of velocities. Meanwhile, the nonlinear transmission condition

$$\sigma_i \tau_i = (-1)^i \kappa |[u]|[u] \quad \text{on } \Gamma \qquad (i = 1, 2), \tag{1.3}$$

has also been utilized to model geophysical flows [5,7–9,15–17]. As is proposed in [11,12,18], the relationship between the horizontal frictional drag force and the jump of velocities can be represented by the general power-law type condition:

$$\sigma_i \tau_i = (-1)^i \kappa |[u]|^\alpha [u] \quad \text{on } \Gamma \qquad (\alpha \ge 0,\ i = 1, 2). \tag{1.4}$$

Note that (1.4) can be equivalently expressed as

$$\sigma_i \tau_i = \frac{(-1)^i \kappa}{(\alpha + 2)} \nabla_{[u]}(|[u]|^{\alpha+2}) \qquad (\alpha \ge 0,\ i = 1, 2).$$

Instead of using the gradient $\nabla_{[u]}(|[u]|^{\alpha+2})$, a subgradient of a convex function can be used as a general type condition. The Tresca-friction type slip boundary condition, as explained in [19,20], is suggested for modeling snowslides and mud-rock flows. The interface version of this condition is stated below:

$$u_{1,\nu} = u_{2,\nu} = 0, \tag{1.5a}$$

$$\sigma_{1,\tau} = -\sigma_{2,\tau}, \quad -\sigma_{1,\tau} \in g\partial|[u_\tau]|, \tag{1.5b}$$

where $g \in C(\Gamma)$ is a given threshold with $g > 0$. This friction-type slip condition expresses a *monotone* relationship between the traction force and the jump of tangential velocity. When integrated into the incompressible Navier–Stokes equations [19,20], this condition leads to a second-kind variational inequality (VI) for weak formulation. Fujita and his collaborators [19–24] and other researchers [25–32] have explored the variational inequalities of the second kind that are governed by the Stokes and Navier–Stokes equations.

Our paper presents an extension of the interface condition to include a broader range of friction types. Specifically, we consider nonsmooth boundary conditions that involve *non-monotone* relationships between physical quantities. This leads to hemivariational inequalities (HVIs) in the weak formulation.

We propose the following friction-type slip interface condition:

$$u_{i,\nu} = 0, \quad i = 1, 2, \tag{1.6a}$$

$$\sigma_{1,\tau} = -\sigma_{2,\tau}, \quad -\sigma_{1,\tau} \in \partial\psi([u_\tau]), \tag{1.6b}$$

where $\psi(u_\tau)$ is a short-hand notation for $\psi(x, u_\tau)$. The function $\psi : \Gamma \times \mathbb{R}^d \to \mathbb{R}$ is called a superpotential. It is assumed to be locally Lipschitz with respect to its last argument. The subdifferential of $\psi(x, \cdot)$ in the Clarke sense is denoted by $\partial\psi$. Note that (1.5) is a special case of (1.6) when the function $\psi(x, u_\tau) = g|u_\tau|$, where g is again a positive threshold function.

The notion of hemivariational inequality was first introduced by Panagiotopoulos in early 1980s [33], which is closely related to the development of the concept of the generalized directional derivative and subdifferential of a locally Lipschitz functional in the sense of Clarke [34,35]. Hemivariational inequalities provide mathematical formulations to treat successfully problems involving non-monotone, nonsmooth, and multivalued constitutive laws, forces, and boundary conditions. The mixed finite element methods for stationary hemivariational inequalities of the hydrodynamic equations have been studied in [36–40], optimal control problems related to the stationary hemivariational inequality are considered in [41–43], nonstationary hemivariational inequality in [44,45]. One is referred to [46–48] for comprehensive coverages of mathematical theories of hemivariational inequalities, and [49–52] for representative papers on numerical analysis of hemivariational inequalities.

Our research delves into the coupling of two viscous fluids with the non-monotone and nonsmooth slip interface condition (1.6). This paper makes a threefold contribution. Firstly, we investigate the existence and uniqueness of the NS/NS hemivariational inequality (HVI) solution. Then, we propose and analyze a domain decomposition method, which decouples the NS/NS HVI system into two NS problems in the subregions with the Dirichlet boundary condition and the slip boundary condition, respectively. Moreover, we study the mixed finite element method for the NS/NS hemivariational inequality using the P1b/P1-element and conduct error analysis. Numerical analysis of NS hemivariational inequalities has been explored in the literature, e.g., [37,38]. However, in this paper, we investigate a novel NS/NS hemivariational inequality for a coupled system of two fluids with an interface transmission condition. Solving the NS/NS hemivariational inequality numerically is more complex than solving a standard NS hemivariational inequality. Specifically, we propose and analyze a domain decomposition method to tackle this new problem.

The rest of the paper is organized as follows. In Section 2, we review some definitions and auxiliary material. In Section 3, we introduce the NS/NS HVI and establish its well-posedness. The domain decomposition approach for the solution of the NS/NS HVI is studied in Section 4. We apply the finite element method to NS/NS HVI in Section 5 and establish the convergence analysis. Section 6 is devoted to the numerical experiments.

2. Preliminaries

All the function spaces in this paper are real. For a normed space X, we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual, and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing between X^* and X. For simplicity in writing, in the following we always assume X is a Banach space, unless stated otherwise.

We first recall the definition of generalized directional derivative and subdifferential in the sense of Clarke for a locally Lipschitz function [35].

Definition 2.1. Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of ψ at $x \in X$ in the direction $v \in X$, denoted by $\psi^0(x; v)$, is defined by

$$\psi^0(x;v) = \limsup_{y\to x, h\downarrow 0} \frac{\psi(y+hv)-\psi(y)}{h}.$$

The generalized gradient or subdifferential of ψ at x, denoted by $\partial\psi(x)$, is a subset of the dual space X^* given by

$$\partial\psi(x) = \{\zeta \in X^* \ :\ \psi^0(x;v) \ge \langle \zeta, v\rangle_{X^*\times X} \ \forall\, v \in X\}.$$

We list the basic properties of the generalized directional derivative and the generalized gradient in the next two propositions [35, 53].

Proposition 2.1. *Assume that $\psi : X \to \mathbb{R}$ is a locally Lipschitz function. The following statements are valid.*
(i) $\psi^0(u, kv) = k\psi^0(u, v) \quad \forall\, k > 0, u, v \in X$.
(ii) $\psi^0(u; v) = \max\{\langle\zeta, v\rangle \ :\ \zeta \in \partial\psi(u)\} \quad \forall\, u, v \in X$.
(iii) $\psi^0(u; v_1 + v_2) \le \psi^0(u; v_1) + \psi^0(u; v_2) \quad \forall\, u, v_1, v_2 \in X$.
(iv) *If $u_n \to u$ and $v_n \to v$ in X, then* $\lim \psi^0(u_n; v_n) \le \psi^0(u; v)$.
(v) *For every $u \in X$, $\partial\psi(u)$ is nonempty, convex and weakly* compact in X^*.*
(vi) *If $u_n \to u$ in X, $\zeta_n \in \partial\psi(u_n)$, and $\zeta_n \to \zeta$ weakly* in X^*, then $\zeta \in \partial\psi(u)$.*
(vii) *If $\psi : X \to \mathbb{R}$ is convex, then the Clark subdifferential $\partial\psi(u)$ at any $u \in X$ coincides with the convex subdifferential $\partial\psi(u)$.*

Proposition 2.2. *Let $\psi, \psi_1, \psi_2 : X \to \mathbb{R}$ be locally Lipschitz functions. Then*
(i) $\partial(k\psi)(u) = k\partial\psi(u) \quad \forall\, k > 0, u \in X$.
(ii) $\partial(\psi_1 + \psi_2)(u) \subset \partial\psi_1(u) + \partial\psi_2(u) \quad \forall\, u \in X$,
equivalently,

$$(\psi_1 + \psi_2)^0(u; v) \le \psi_1^0(u; v) + \psi_2^0(u; v) \quad \forall\, u, v \in X.$$

3. Hemivariational inequalities

To present the weak formulation of the problem defined by (1.1) and (1.6), we use the following function spaces:

$$V_i := \{v_i \in H^1(\Omega_i)^d \ :\ v_i|_{\Gamma_i} = 0,\ v_{i,\nu}|_{\Gamma} = 0\}, \quad V := V_1 \times V_2,$$
$$Q_i := L^2(\Omega_i), \quad \mathring{Q}_i := L_0^2(\Omega_i), \quad Q := Q_1 \times Q_2, \quad \mathring{Q} := \mathring{Q}_1 \times \mathring{Q}_2.$$

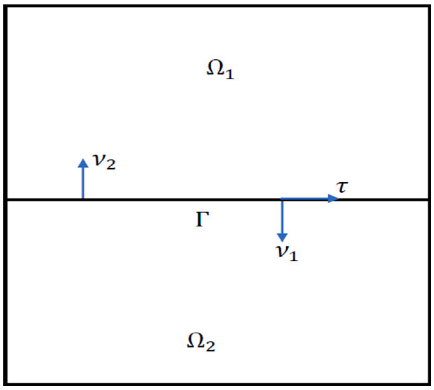


Fig. 1. A 2D domain.

Here, $L_0^2(\Omega_i)$ denotes the subspace of $L^2(\Omega)$ of functions with a vanishing integral over Ω_i. For all $u := (u_1, u_2), v := (v_1, v_2), w := (w_1, w_2) \in V$ and $p := (p_1, p_2) \in Q$, we define

$$a_i(u_i, v_i) := 2\mu_i \int_{\Omega_i} \mathbb{D}(u_i) : \mathbb{D}(v_i)\, dx, \qquad a(u,v) := a_1(u_1,v_1) + a_2(u_2,v_2),$$

$$b_i(v_i, p_i) := -\int_{\Omega_i} p_i \nabla \cdot v_i\, dx, \qquad b(v,p) := b_1(v_1,p_1) + b_2(v_2,p_2),$$

$$C_i(u_i; v_i, w_i) := \int_{\Omega_i} (u_i \cdot \nabla) v_i \ w_i\, dx, \qquad C(u;v,w) := C_1(u_1;v_1,w_1) + C_2(u_2;v_2,w_2),$$

$$\langle f_i, v_i \rangle := \int_{\Omega_i} f_i \cdot v_i\, dx, \qquad \langle f, v \rangle := \langle f_1, v_1 \rangle + \langle f_2, v_2 \rangle.$$

By Korn's inequality [54], we can equip V_i and V with norms defined by the following relations:

$$\|v_i\|_{V_i}^2 := \int_{\Omega_i} \mathbb{D}(v_i) : \mathbb{D}(v_i)\, dx \quad \forall\, v_i \in V_i, \qquad \|v\|_V^2 := \|v_1\|_{V_1}^2 + \|v_2\|_{V_2}^2 \quad \forall\, v = (v_1, v_2) \in V.$$

For $b_i(\cdot,\cdot)$ and $b(\cdot,\cdot)$, the following inf–sup conditions hold [55]: for a constant $C > 0$,

$$\sup_{v_i \in V_i} \frac{b_i(v_i, p_i)}{\|v_i\|_{V_i}} \geq C \|p_i\|_{Q_i} \quad \forall\, p_i \in \mathring{Q}_i, \qquad \sup_{v \in V} \frac{b(v,p)}{\|v\|_V} \geq C \|p\|_Q \quad \forall\, p \in \mathring{Q}. \tag{3.1}$$

Evidently, for $i = 1, 2$,

$$a_i(v_i, v_i) = 2\mu_i \|v_i\|_{V_i}^2, \quad |a_i(u_i, v_i)| \leq 2\mu_i \|u_i\|_{V_i} \|v_i\|_{V_i} \quad \forall\, u_i, v_i \in V_i,$$

and there exist constants $C, N > 0$ such that

$$b_i(v_i, q_i) \leq C \|v_i\|_{V_i} \|q_i\|_{Q_i}, \qquad \forall\, v_i \in V_i, q_i \in Q_i,$$
$$C_i(u_i; v_i, w_i) \leq N \|u_i\|_{V_i} \|v_i\|_{V_i} \|w_i\|_{V_i}, \qquad \forall\, u_i, v_i, w_i \in V_i.$$

With the bilinear forms b_1 and b_2 at our disposal, we further define the spaces

$$V_i^{\mathrm{div}} := \{ v_i \in V_i : b_i(v_i, q_i) = 0 \ \forall\, q_i \in \mathring{Q}_i \}, \quad V^{\mathrm{div}} := V_1^{\mathrm{div}} \times V_2^{\mathrm{div}},$$
$$\mathring{V}_i := H_0^1(\Omega_i)^d, \quad \mathring{V}_i^{\mathrm{div}} := \mathring{V}_i \cap V_i^{\mathrm{div}}, \quad \mathring{V}^{\mathrm{div}} := \mathring{V}_1^{\mathrm{div}} \times \mathring{V}_2^{\mathrm{div}}.$$

Concerning the superpotential ψ, we assume the following properties.

$H(\psi)$: $\psi : \Gamma \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $\psi(\cdot, \xi)$ is measurable on Γ for all $\xi \in \mathbb{R}^d$ and $\psi(\cdot, 0) \in L^1(\Gamma)$.

(ii) $\psi(x, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $x \in \Gamma$.

(iii) $|\eta| \leq c_0 + c_1 |\xi|$ for all $\xi \in \mathbb{R}^d$, $\eta \in \partial\psi(x, \xi)$, a.e. $x \in \Gamma$ with $c_0, c_1 \geq 0$.

(iv) $(\eta_1 - \eta_2)\cdot(\xi_1 - \xi_2) \geq -c_\psi\, |\xi_1 - \xi_2|^2$ for all $\xi_i \in \mathbb{R}^d$, $\eta_i \in \partial\psi(x, \xi_i)$, $i = 1, 2$, a.e. $x \in \Gamma$ with $c_\psi \geq 0$.

The condition $H(\psi)$(iv) is known as a relaxed monotonicity condition in the literature [56], and it is equivalent to

$$\psi^0(\xi_1; \xi_2 - \xi_1) + \psi^0(\xi_2; \xi_1 - \xi_2) \leq c_\psi\, |\xi_1 - \xi_2|^2 \quad \forall\, \xi_1, \xi_2 \in \mathbb{R}^d. \tag{3.2}$$

Define a functional $\Psi : V \to \mathbb{R}$ by

$$\Psi(v) = \int_\Gamma \psi(x, [v_\tau](x))\, ds, \quad v \in V. \tag{3.3}$$

The next result is similar to [57, Theorem 3.47].

Lemma 3.1. *Assume that $\psi : \Gamma\times\mathbb{R} \to \mathbb{R}$ satisfies the hypothesis $H(\psi)$. Then the functional Ψ defined by (3.3) has the following properties.*
(i) *$\Psi(\cdot)$ is locally Lipschitz in V.*
(ii) *$\|z\|_{V^*} \le c\left(1 + \|v\|_V\right) \;\forall\, v \in V,\; z \in \partial\Psi(v)$.*
(iii) *$\Psi^0(u; v) \le \int_\Gamma \psi^0([u_\tau](x); [v_\tau](x))\, ds \;\forall\, u, v \in V$.*

Let us derive a weak formulation of the problem defined by (1.1) and (1.6). For this purpose, we temporarily assume that $f_1 \in L^2(\Omega_1;\mathbb{R}^d)$ and $f_2 \in L^2(\Omega_2;\mathbb{R}^d)$, and that the problem admits a smooth solution (u_i, p_i), $i = 1, 2$. Multiplying (1.1a) by an arbitrary $v_i \in V_i$ and integrating by parts, we get

$$\int_{\Omega_i} \left(\sigma_i \cdot \mathbb{D}(v_i) + (u_i \cdot \nabla)u_i \cdot v_i\right) dx - \int_{\partial\Omega_i} (\sigma_i \nu_i) \cdot v_i\, ds = \int_{\Omega_i} f_i \cdot v_i\, dx. \tag{3.4}$$

Since $v_i = 0$ on Γ_i and $v_{i,\nu} = 0$ on Γ,

$$-\int_{\partial\Omega_i} (\sigma_i \nu_i) \cdot v_i\, ds = -\int_\Gamma \sigma_{i,\tau} \cdot v_{i,\tau} ds.$$

Since $\sigma_i = 2\mu_i \mathbb{D}(u_i) - p_i\mathbb{I}$, we obtain from (3.4) that

$$\int_{\Omega_i} \left(2\,\mu_i\mathbb{D}(u_i) \cdot \mathbb{D}(v_i) + (u_i \cdot \nabla)u_i \cdot v_i - p_i\nabla \cdot v_i\right) dx - \int_\Gamma \sigma_{i,\tau} \cdot v_{i,\tau} ds = \int_{\Omega_i} f_i \cdot v_i\, dx.$$

We add the above equality for $i = 1$ and that for $i = 2$ to get

$$a(u, v) + b(v, p) + C(u; u, v) + \int_\Gamma (-\sigma_{1,\tau} \cdot v_{1,\tau} - \sigma_{2,\tau} \cdot v_{2,\tau})\, ds = \langle f, v\rangle. \tag{3.5}$$

It follows from (1.6b) that

$$\int_\Gamma (-\sigma_{1,\tau} \cdot v_{1,\tau} - \sigma_{2,\tau} \cdot v_{2,\tau})\, ds = \int_\Gamma (-\sigma_{1,\tau}) \cdot [v_\tau]\, ds \le \int_\Gamma \psi^0([u_\tau]; [v_\tau])\, ds.$$

Hence, we derive from (3.5) that

$$a(u, v) + b(v, p) + C(u; u, v) + \int_\Gamma \psi^0([u_\tau]; [v_\tau])\, ds \ge \langle f, v\rangle \quad \forall\, v \in V.$$

Then we multiply (1.1b) by an arbitrary $q_i \in \mathring{Q}_i$, integrate over Ω_i, and add the two equalities for $i = 1$ and 2 to get

$$b(u, q) = 0.$$

Summarizing, we have derived the following hemivariational inequality for the problem defined by (1.1) and (1.6).

Problem 3.1 (*HVI-NS*)**.** Find $(u, p) \in V \times \mathring{Q}$ such that

$$a(u, v) + b(v, p) + C(u; u, v) + \int_\Gamma \psi^0([u_\tau]; [v_\tau])\, ds \ge \langle f, v\rangle \quad \forall\, v \in V, \tag{3.6}$$

$$b(u, q) = 0 \quad \forall\, q \in \mathring{Q}. \tag{3.7}$$

Restricting u and v to the subspace V^{div}, we can eliminate $b(\cdot, \cdot)$ and get the following reduced problem:

Problem 3.2 (*HVI-NS*)**.**$^{\mathrm{div}}$ Find $u \in V^{\mathrm{div}}$ such that

$$a(u, v) + C(u; u, v) + \int_\Gamma \psi^0([u_\tau]; [v_\tau])\, ds \ge \langle f, v\rangle \quad \forall\, v \in V^{\mathrm{div}}. \tag{3.8}$$

By removing the trilinear term $C(\cdot; \cdot, \cdot)$, we get the hemivariational inequalities for the Stokes/Stokes coupling problem.

Problem 3.3 (*HVI-S*)**.** Find $(u, p) \in V \times \mathring{Q}$ such that

$$a(u, v) + b(v, p) + \int_\Gamma \psi^0([u_\tau]; [v_\tau])\, ds \ge \langle f, v\rangle \quad \forall\, v \in V, \tag{3.9}$$

$$b(u, q) = 0 \quad \forall\, q \in \mathring{Q}. \tag{3.10}$$

Problem 3.4 (*HVI-S*)**.**$^{\mathrm{div}}$ Find $u \in V^{\mathrm{div}}$ such that

$$a(u, v) + \int_\Gamma \psi^0([u_\tau]; [v_\tau])\, ds \ge \langle f, v\rangle \quad \forall\, v \in V^{\mathrm{div}}. \tag{3.11}$$

For a well-posedness analysis of the above hemivariational inequalities, we introduce a smallness condition:

$$c_\psi < 2\mu\lambda_0, \quad \mu := \min(\mu_1, \mu_2), \tag{3.12}$$

where $\lambda_0 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$u \in V, \quad \int_\Omega \mathbb{D}(u) : \mathbb{D}(v)\,dx = \lambda \int_\Gamma [u_\tau]\cdot[v_\tau]\,ds \quad \forall\, v \in V.$$

In other words, $\lambda_0^{-1/2}$ is the best constant of trace's inequality

$$\|[v_\tau]\|_{L^2(\Gamma;\mathbb{R}^d)} \le c\,\|v\|_V \quad \forall\, v \in V. \tag{3.13}$$

Note that the condition (3.12) means the viscosity term dominates, or equivalently, the degree of non-convexity of ψ is relatively not strong. Combining (3.2), Lemma 3.1(iii) and (3.13), we have

$$\begin{aligned}\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) &\le \int_\Gamma (\psi^0([v_{1\tau}]; [v_{2\tau} - v_{1\tau}]) + \psi^0([v_{2\tau}]; [v_{1\tau} - v_{2\tau}]))\,ds \\ &\le c_\psi \int_\Gamma |[v_{1\tau} - v_{2\tau}]|^2\,ds \\ &\le c_\psi \lambda_0^{-1} \|v_1 - v_2\|_V^2 \quad \forall\, v_1 \in V_1,\ v_2 \in V_2. \end{aligned} \tag{3.14}$$

3.1. The Stokes hemivariational inequalities

Let us fist discuss the well-posedness of Problems 3.3 and 3.4.

Theorem 3.1. (i) *If* (HVI-S) *has a solution* (u, p)*, then* u *is a solution of* (HVI-S)^{div}.

(ii) *Suppose that* $f \in V^*$*,* $H(\psi)$ *and* (3.12) *hold. Then,* (HVI-S)^{div} *admits a unique solution* u*, and there exists a unique* $p \in \mathring{Q}$ *such that* (u, p) *is the unique solution of* (HVI-S)*. In addition, we have the following bound:*

$$\|u\|_V \le c_f := \frac{c_0 \lambda_0^{-1/2} |\Gamma|^{1/2} + \|f\|_{V^*}}{2\mu - c_\psi \lambda_0^{-1}}. \tag{3.15}$$

The solution (u, p) *depends Lipschitz-continuously on* f*, i.e., there exists a constant* $\tilde{c} > 0$ *such that for solutions* (u^1, p^1) *and* (u^2, p^2) *of* (HVI-S) *corresponding to* $f = f^1$ *and* f^2*,*

$$\|u^1 - u^2\|_V + \|p^1 - p^2\|_Q \le \tilde{c}\,\|f^1 - f^2\|_{V^*}. \tag{3.16}$$

Proof. (i) It is trivial. (ii) Since $a(\cdot,\cdot)$ is coercive, applying [58, Theorem 10] with $\Psi(v)$ there replaced by $\Psi([v])$, we can see that (HVI-S)$^{\rm div}$ admits a unique solution $u \in V^{\rm div}$. In view of

$$a(u, v) = \langle f, v\rangle \quad \forall\, v \in \mathring{V}^{\rm div},$$

and by using the inf–sup condition (3.1), we know that there exists a unique $p_i \in \mathring{Q}_i$ such that

$$a_i(u_i, v_i) + b_i(v_i, p_i) = \langle f_i, v_i\rangle \quad \forall\, v_i \in \mathring{V}_i, \quad i = 1, 2,$$

or equivalently,

$$a(u, v) + b(v, p) = \langle f, v\rangle \quad \forall\, v \in \mathring{V}. \tag{3.17}$$

Let $v = (v_1, v_2) \in V$ be arbitrary but fixed. According to the inf–sup condition (3.1), there is a $v^1 = (v_1^1, v_2^1) \in \mathring{V}$ such that

$$b(v^1, q) = b(v, q) \quad \forall\, q \in \mathring{Q}. \tag{3.18}$$

Set $v^2 := v - v^1$. Then $v^2 \in V^{\rm div}$. It follows from (3.11) that

$$a(u, v^2) + \int_\Gamma \psi^0([u_\tau]; [v_\tau^2])\,ds \ge \langle f, v^2\rangle. \tag{3.19}$$

In view of (3.17) we have

$$a(u, v^1) + b(v^1, p) = \langle f, v^1\rangle.$$

By (3.18) and (3.19), this equality leads to

$$\begin{aligned} a(u, v) + b(v, p) + \int_\Gamma \psi^0([u_\tau]; [v_\tau])\,ds &= a(u, v^1) + b(v^1, p) + a(u, v^2) + \int_\Gamma \psi^0([u_\tau]; [v_\tau^2])\,ds \\ &\ge \langle f, v^1\rangle + \langle f, v^2\rangle = \langle f, v\rangle. \end{aligned}$$

Thus, we have shown that (u, p) is the unique solution of (HVI-S). The bound (3.15) and the Lipschitz continuity (3.16) can be derived by standard arguments as in [38, Theorem 3.2]. □

Remark 3.1. The well-posedness of (HVI-S) is proved through consideration of a related convex minimization following the idea in [58,59]. This extends to well-posedness analysis of general HVIs without related convex minimization problems through a fixed-point argument [60]. Such an approach eliminates the need for the notion of pseudomonotonicity, as well as an abstract surjectivity result of pseudomonotone operators commonly used in the literature on hemivariational inequalities (cf. [36,56]).

3.2. The Navier–Stokes hemivariational inequalities

We now give the well-posedness analysis of Problem 3.1.

Theorem 3.2. (i) *If* (u, p) *is a solution of* (HVI-NS)*, then* u *also solves* (HVI-NS)^{div}.
(ii) *Assume* $f \in V^*$, $H(\psi)$ *and* (3.12)*. Then* (HVI-NS)^{div} *admits a solution* u *satisfying*

$$\|u\|_V \le c_f. \tag{3.20}$$

In addition, under the assumption that

$$0 < \frac{N c_f}{2\mu - c_\psi \lambda_0^{-1}} < 1, \tag{3.21}$$

u *is the unique solution of* (HVI-NS)^{div}*. Furthermore, there exists a unique* $p \in \mathring{Q}$ *such that* (u, p) *solves* (HVI-NS)*. The solution* (u, p) *depends Lipschitz continuously on* f*, i.e., there exists a constant* $\widetilde{c} > 0$ *such that for solutions* u^1 *and* u^2 *of* (HVI-NS) *corresponding to* $f = f^1$ *and* f^2*,*

$$\|u^1 - u^2\|_V + \|p^1 - p^2\|_Q \le \widetilde{c}\,\|f^1 - f^2\|_{V^*}.$$

Proof. (i) It is evident. (ii) Problem 3.1 can be considered as a special case of Problem 2.1 in [38] without the stability term. After a trivial modification for the result from [38, Theorem 2.2], the well-posedness of Problem 3.1 is covered. Moreover, the proof of the Lipschitz continuity of the solution with respect to the right hand side function is similar to that of [38, Theorem 3.2]. □

Remark 3.2. In case the functional ψ is convex, Problems 3.3 and 3.1 reduce to a coupled problem of variational inequality similar to the ones studied in [61] for the leak interface.

Remark 3.3. Arguing as in Remark 12 of Migórski and Ochal [57], or Remark 2.2 of Mahdioui, Ben Aadi and Akhlil [42], we conclude that if $(u, p) \in V \times \mathring{Q}$ is a solution to Problem 3.1 and sufficiently smooth, then (u, p) satisfies the Navier–Stokes equations (1.1) and the condition (1.5).

4. The domain decomposition method

To simulate the two-subregion coupled fluid flow system, we apply the domain decomposition method. In each iteration step of the method, we solve two sub-problems of smaller size. The key question is how to design the decoupling approach for the interface condition $-\sigma_{1,\tau} \in \partial\psi([u_\tau])$. Our idea is to separate the original system into a sub-problem in Ω_1 with a boundary condition of subdifferential type on Γ, and a fluid problem in Ω_2 with Neumann boundary condition on Γ. Below we present the domain decomposition algorithm for (HVI-NS) and provide a convergence analysis. The case of (HVI-S) is similar and simpler.

Note that sufficiently large μ guarantees that both (4.1) and (4.2) admit unique solution (see [37] for a detailed discussion of the unique existence of (4.1)). In the following, we skip the argument on the well-posedness of (4.1) and (4.2), and only show the convergence of the domain decomposition algorithm.

Theorem 4.1. *Let* (u, p) *and* $(u^{(n)}, p^{(n)})$ *be the unique solutions of (HVI-NS) and* (4.1)–(4.2)*, respectively. For* μ *sufficiently large and* θ *sufficiently small, there is a constant* $\eta \in (0, 1)$ *such that* $\|u - u^{(n)}\|_V \le C\eta^n$*. In particular, this implies the convergence:* $\|u - u^{(n)}\|_V \to 0$ *as* $n \to \infty$.

Remark 4.1. It can be seen from the proof below that we need μ sufficiently large so that (4.10) and (4.28) hold, and we need θ sufficiently small so that (4.29) holds.

Proof. The proof is divided into three steps. We first show that $\|u^{(n)}\|_V$ is bounded independent of θ and n. Then, we bound $\|u^{(n)} - u\|_V$, and finally show that $\|u^{(n)} - u\|_V \to 0$ as $n \to \infty$.

Step 1. Given $u_i^{(n-1)} \in V_i$ $(i = 1, 2)$, the existence of a unique weak solution $(u_1^{(n)}, p_1^{(n)}) \in V_1 \times \mathring{Q}_1$ of (4.1) follows from an argument similar to that of the Navier–Stokes hemivariational inequality (cf. [62, Theorem 3.7]). The existence of a unique weak solution $(u_2^{(n)}, p_2^{(n)}) \in V_2 \times \mathring{Q}_2$ of (4.2) follows from a standard argument for the Navier–Stokes equations with the Dirichlet boundary condition (cf. [63, Theorem 2.2, Chapter IV], or [55, Theorem 1.3, Chapter 2]).

Algorithm 1 The domain decomposition algorithm

- Initialization. Choose $\theta \in (0,1)$ and specify a stopping criterion. Let $n=1$, $u_{2,\tau}^{(0)}=0$ and $-\sigma_\tau(u_2^{(0)})=0$.
- Iteration. For $n=1,2,\cdots$ until the stopping criterion is satisfied, solve the boundary value problem in Ω_1:

$$-\nabla\cdot\sigma(u_1^{(n)},p_1^{(n)})+(u_1^{(n)}\cdot\nabla)u_1^{(n)}=f_1 \quad \text{in } \Omega_1, \tag{4.1a}$$

$$\nabla\cdot u_1^{(n)}=0 \quad \text{in } \Omega_1, \tag{4.1b}$$

$$u_1^{(n)}=0 \quad \text{on } \Gamma_1, \tag{4.1c}$$

$$-\sigma_\tau(u_1^{(n)})\in\partial\psi(u_{1,\tau}^{(n)}-u_{2,\tau}^{(n-1)}),\quad u_{1,\nu}^{(n)}=0 \quad \text{on } \Gamma, \tag{4.1d}$$

and the boundary value problem in Ω_2:

$$-\nabla\cdot\sigma(u_2^{(n)},p_2^{(n)})+(u_2^{(n)}\cdot\nabla)u_2^{(n)}=f_2 \quad \text{in } \Omega_2, \tag{4.2a}$$

$$\nabla\cdot u_2^{(n)}=0 \quad \text{in } \Omega_2, \tag{4.2b}$$

$$u_2^{(n)}=0 \quad \text{on } \Gamma_2, \tag{4.2c}$$

$$\sigma_\tau(u_2^{(n)})=(1-\theta)\sigma_\tau(u_2^{(n-1)})+\theta(-\sigma_\tau(u_1^{(n)})),\quad u_{2,\nu}^{(n)}=0 \quad \text{on } \Gamma. \tag{4.2d}$$

We will show the boundedness of $u^{(n)}$. To this end, testing (4.1a) by $v_1\in V_1^{\mathrm{div}}$ and testing (4.2a) by $v_2\in V_2^{\mathrm{div}}$, we have

$$a_1(u_1^{(n)},v_1)+C_1(u_1^{(n)};u_1^{(n)},v_1)=\int_{\Omega_1}f_1\cdot v_1\,dx+\int_\Gamma\sigma_\tau(u_1^{(n)})\cdot v_{1,\tau}\,ds, \tag{4.3a}$$

$$a_2(u_2^{(n)},v_2)+C_2(u_2^{(n)};u_2^{(n)},v_2)=\int_{\Omega_2}f_2\cdot v_2\,dx+\int_\Gamma\sigma_\tau(u_2^{(n)})\cdot v_{2,\tau}\,ds. \tag{4.3b}$$

It follows from (4.1d) and (4.2d) that

$$-\int_\Gamma\sigma_\tau(u_1^{(n)})\cdot v_{1,\tau}\,ds\le\int_\Gamma\psi^0(u_{1,\tau}^{(n)}-u_{2,\tau}^{(n-1)};v_{1,\tau})\,ds, \tag{4.4a}$$

$$\begin{aligned}\int_\Gamma\sigma_\tau(u_2^{(n)})\cdot v_{2,\tau}\,ds &=\int_\Gamma(1-\theta)\sigma_\tau(u_2^{(n-1)})\cdot v_{2,\tau}\,ds-\int_\Gamma\theta\sigma_\tau(u_1^{(n)})\cdot v_{2,\tau}\,ds\\ &=\int_\Gamma(1-\theta)^2\sigma_\tau(u_2^{(n-2)})\cdot v_{2,\tau}\,ds-\int_\Gamma\Big(\theta(1-\theta)\sigma_\tau(u_1^{(n-1)})+\theta\sigma_\tau(u_1^{(n)})\Big)\cdot v_{2,\tau}\,ds\\ &\;\;\vdots\\ &=\int_\Gamma(1-\theta)^n\sigma_\tau(u_2^{(0)})\cdot v_{2,\tau}\,ds-\theta\int_\Gamma\sum_{i=0}^{n-1}(1-\theta)^i\sigma_\tau(u_1^{(n-i)})\cdot v_{2,\tau}\,ds.\end{aligned} \tag{4.4b}$$

Taking $v_1=-u_1^{(n)}$ and in view of $C_1(u_1^{(n)};u_1^{(n)},u_1^{(n)})=0$, we obtain

$$\begin{aligned}2\mu_1\|u_1^{(n)}\|_{V_1}^2&\le\|f_1\|_{V_1^*}\|u_1^{(n)}\|_{V_1}+\int_\Gamma\big(c_0+c_1|u_{1,\tau}^{(n)}-u_{2,\tau}^{(n-1)}|\big)|u_{1,\tau}^{(n)}|\,ds\quad(\text{by Lemma Lemma 3.1})\\ &\le\Big(\|f_1\|_{V_1^*}+c_0|\Gamma|^{\frac12}\lambda_0^{-\frac12}+\lambda_0^{-1}c_1(\|u_1^{(n)}\|_{V_1}+\|u_2^{(n-1)}\|_{V_2})\Big)\|u_1^{(n)}\|_{V_1}\quad(\text{by (3.13)}),\end{aligned}$$

which implies

$$\|u_1^{(n)}\|_{V_1}\le\frac{\|f_1\|_{V_1^*}+c_0|\Gamma|^{\frac12}\lambda_0^{-\frac12}+\lambda_0^{-1}c_1\|u_2^{(n-1)}\|_{V_2}}{2\mu_1-\lambda_0^{-1}c_1}. \tag{4.5}$$

On the other hand, substituting $v_2=u_2^{(n)}$ into (4.3b) and (4.4b), and in view of $C_2(u_2^{(n)};u_2^{(n)},u_2^{(n)})=0$ and $\sigma_\tau(u_2^{(0)})=0$, we calculate as

$$\begin{aligned}2\mu_2\|u_2^{(n)}\|_{V_2}^2&\le\|f_2\|_{V_2^*}\|u_2^{(n)}\|_{V_2}-\theta\sum_{i=0}^{n-1}(1-\theta)^i\int_\Gamma\sigma_\tau(u_1^{(n-i)})\cdot u_{2,\tau}^{(n)}\,ds\\ &\le\|f_2\|_{V_2^*}\|u_2^{(n)}\|_{V_2}+\theta\sum_{i=0}^{n-1}(1-\theta)^i\int_\Gamma\psi^0(u_{1,\tau}^{(n-i)}-u_{2,\tau}^{(n-i-1)})\cdot u_{2,\tau}^{(n)}\,ds\quad(\text{by (4.1d)})\\ &\le\|f_2\|_{V_2^*}\|u_2^{(n)}\|_{V_2}+\theta\sum_{i=0}^{n-1}(1-\theta)^i\big(c_0|\Gamma|^{\frac12}+c_1\lambda_0^{-\frac12}(\|u_1^{(n-i)}\|_{V_1}+\|u_2^{(n-i-1)}\|_{V_2})\big)\lambda_0^{-\frac12}\|u_2^{(n)}\|_{V_2},\end{aligned} \tag{4.6}$$

where we have applied Lemma 3.1 and (3.13) in the last inequality. Inserting (4.5) to the right-hand side of (4.6), we get

$$\begin{aligned} 2\mu_2\|u_2^{(n)}\|_{V_2}^2 \le \|f_2\|_{V_2^*}\|u_2^{(n)}\|_{V_2} + \theta\sum_{i=0}^{n-1}(1-\theta)^i\Big(c_0|\Gamma|^{\frac12}\lambda_0^{-\frac12} + \frac{c_1\lambda_0^{-1}(\|f_1\|_{V_1^*} + c_0|\Gamma|^{\frac12}\lambda_0^{-\frac12})}{2\mu_1 - \lambda_0^{-1}c_1}\Big)\|u_2^{(n)}\|_{V_2} \\ + \theta\sum_{i=0}^{n-1}(1-\theta)^i c_1\lambda_0^{-1}\Big(\frac{c_1\lambda_0^{-1}}{2\mu_1-\lambda_0^{-1}c_1} + 1\Big)\|u_2^{(n-i-1)}\|_{V_2}\|u_2^{(n)}\|_{V_2}, \end{aligned} \tag{4.7}$$

which yields (by $\sum_{i=0}^{n-1}(1-\theta)^i = \theta^{-1}(1-(1-\theta)^n)$)

$$2\mu_2\|u_2^{(n)}\|_{V_2} \le \|f_2\|_{V_2^*} + c_2 + c_3\max_{0\le i\le n-1}\|u_2^{(n-i-1)}\|_{V_2}, \tag{4.8}$$

where

$$c_2 := \Big(c_0|\Gamma|^{\frac12}\lambda_0^{-\frac12} + \frac{c_1\lambda_0^{-1}(\|f_1\|_{V_1^*} + c_0|\Gamma|^{\frac12}\lambda_0^{-\frac12})}{2\mu_1 - \lambda_0^{-1}c_1}\Big), \quad c_3 := c_1\lambda_0^{-1}\Big(\frac{c_1\lambda_0^{-1}}{2\mu_1-\lambda_0^{-1}c_1} + 1\Big). \tag{4.9}$$

Under the assumption (this is satisfied if μ_2 is sufficiently large)

$$c_3 < 2\mu_2, \tag{4.10}$$

we assert that

$$\|u_2^{(i)}\|_{V_2} \le \frac{\|f_2\|_{V_2^*} + c_2}{2\mu_2 - c_3} =: c_4 \qquad (1 \le i \le n). \tag{4.11}$$

This is proved by an induction as follows. In view of $u_2^{(0)} = 0$, it is easy to see from (4.8) that (4.11) holds for $i = 1$. Suppose that $\|u_2^{(i)}\|_{V_2} \le \frac{\|f_2\|_{V_2^*}+c_2}{2\mu_2-c_3}$ holds for $0 \le i \le n-1$. We obtain from (4.8) that

$$2\mu_2\|u_2^{(n)}\|_{V_2} \le \|f_2\|_{V_2^*} + c_2 + c_3\frac{\|f_2\|_{V_2^*} + c_2}{2\mu_2 - c_3} = \frac{(2\mu_2 - c_3 + c_3)(\|f_2\|_{V_2^*} + c_2)}{2\mu_2 - c_3},$$

which concludes (4.11).

It follows from (4.5) and (4.11) that

$$\|u_1^{(n)}\|_{V_1} \le \frac{\|f_1\|_{V_1^*} + c_0|\Gamma|^{\frac12}\lambda_0^{-\frac12} + \lambda_0^{-1}c_1\frac{\|f_2\|_{V_2^*}+c_2}{2\mu_2-c_3}}{2\mu_1 - \lambda_0^{-1}c_1} =: c_5. \tag{4.12}$$

Hence, we have shown the boundedness

$$\|u^{(n)}\|_V \le \|u_1^{(n)}\|_{V_1} + \|u_2^{(n)}\|_{V_2} \le c_4 + c_5. \tag{4.13}$$

Note that the constants $\{c_i\}_{i=2}^5$ depend on c_0, c_1, λ_0, μ_1, μ_2 and $\|f\|_{V^*}$, and are independent of n and θ. Moreover, it is easy to observe that c_f and $\{c_i\}_{i=2}^5$ decrease as μ_1 and μ_2 increase, and c_f, c_4 and c_5 tend to 0 as μ_1 and μ_2 go to ∞.

Step 2. For brevity, denote

$$(e_i^{(n)}, \phi_i^{(n)}) := (u_i - u_i^{(n)}, p_i - p_i^{(n)}) \quad (i = 1, 2), \quad e^{(n)} := (e_1^{(n)}, e_2^{(n)}), \quad p^{(n)} := (p_1^{(n)}, p_2^{(n)}).$$

Our goal is to estimate $(e_1^{(n)}, e_2^{(n)})$. It follows from (1.1), (1.6), (4.1) (4.2) that

$$-\nabla\cdot\sigma(e_1^{(n)}, \phi_1^{(n)}) + (u_1\cdot\nabla)u_1 - (u_1^{(n)}\cdot\nabla)u_1^{(n)} = 0, \quad \nabla\cdot e_1^{(n)} = 0 \quad \text{in } \Omega_1, \tag{4.14a}$$

$$e_{1,\nu}^{(n)} = 0, \quad \sigma_\tau(e_1^{(n)}) = \sigma_\tau(u_1) - \sigma_\tau(u_1^{(n)}),\ -\sigma_\tau(u_1) \in \partial\psi([u_\tau]),\ -\sigma_\tau(u_1^{(n)}) \in \partial\psi(u_{1,\tau}^{(n)} - u_{2,\tau}^{(n-1)}) \quad \text{on } \Gamma, \tag{4.14b}$$

$$-\nabla\cdot\sigma(e_2^{(n)}, \phi_2^{(n)}) + (u_2\cdot\nabla)u_2 - (u_2^{(n)}\cdot\nabla)u_2^{(n)} = 0, \quad \nabla\cdot e_2^{(n)} = 0 \quad \text{in } \Omega_2, \tag{4.14c}$$

$$e_{2,\nu}^{(n)} = 0, \quad \sigma_\tau(e_2^{(n)}) = (1-\theta)\sigma_\tau(e_2^{(n-1)}) + \theta(-\sigma_\tau(e_1^{(n)})) \quad \text{on } \Gamma, \tag{4.14d}$$

$$e_1^{(n)} = 0 \quad \text{on } \Gamma_1, \quad e_2^{(n)} = 0 \quad \text{on } \Gamma_2. \tag{4.14e}$$

Testing the first equation in (4.14a) by $v_1 \in V_1^{\text{div}}$, testing the first equation in (4.14c) by $v_2 \in V_2^{\text{div}}$, applying the boundary conditions, and noting that

$$(u_i\cdot\nabla)u_i - (u_i^{(n)}\cdot\nabla)u_i^{(n)} = (e_i^{(n)}\cdot\nabla)u_i + (u_i^{(n)}\cdot\nabla)e_i^{(n)},$$

we obtain

$$a_1(e_1^{(n)}, v_1) + C_1(e_1^{(n)}; u_1, v_1) + C_1(u_1^{(n)}; e_1^{(n)}, v_1) = \int_\Gamma \Big(\sigma_\tau(u_1) - \sigma_\tau(u_1^{(n)})\Big)\cdot v_{1,\tau}\, ds, \tag{4.15a}$$

$$a_2(e_2^{(n)}, v_2) + C_2(e_2^{(n)}; u_2, v_2) + C_2(u_2^{(n)}; e_2^{(n)}, v_2) = \int_\Gamma \sigma_\tau(e_2^{(n)})\cdot v_{2,\tau}\, ds. \tag{4.15b}$$

Taking $v_1 = e_1^{(n)}$ in (4.15a) and $v_2 = e_2^{(n-1)}$ in (4.15b) with n replaced by $n-1$, and summing up the results, we get (noting that $C(u_i^{(n)}; e_i^{(n)}, e_i^{(n)}) = 0$)

$$2\mu_1 \|e_1^{(n)}\|_{V_1}^2 + 2\mu_2 \|e_2^{(n-1)}\|_{V_2}^2 + C_1(e_1^{(n)}; u_1, e_1^{(n)}) + C_2(e_2^{(n-1)}; u_2, e_2^{(n-1)}) = R_1^{(n)} + R_2^{(n)}, \tag{4.16}$$

where

$$R_1^{(n)} = \int_\Gamma \left(\sigma_\tau(u_1) - \sigma_\tau(u_1^{(n)})\right) \cdot (e_{1,\tau}^{(n)} - e_{2,\tau}^{(n-1)})\, ds, \quad R_2^{(n)} = \int_\Gamma \left(\sigma_\tau(e_1^{(n)}) + \sigma_\tau(e_2^{(n-1)})\right) \cdot e_{2,\tau}^{(n-1)}\, ds.$$

The convection terms on the left-hand side of (4.16) are bounded as follows:

$$|C_1(e_1^{(n)}; u_1, e_1^{(n)})| \le N \|e_1^{(n)}\|_{V_1}^2 \|u_1\|_{V_1}, \tag{4.17a}$$

$$|C_2(e_2^{(n-1)}; u_2, e_2^{(n-1)})| \le N \|e_2^{(n-1)}\|_{V_2}^2 \|u_2\|_{V_2}. \tag{4.17b}$$

Since $e_{1,\tau}^{(n)} - e_{2,\tau}^{(n-1)} = [u_\tau] - (u_{1,\tau}^{(n)} - u_{2,\tau}^{(n-1)})$, we can apply Lemma 3.1(iv) to bound $R_1^{(n)}$:

$$R_1^{(n)} \le c_\psi \|e_{1,\tau}^{(n)} - e_{2,\tau}^{(n-1)}\|_{L^2(\Gamma)}^2. \tag{4.18}$$

On the other hand, by (4.14d),

$$\sigma_\tau(e_1^{(n)}) + \sigma_\tau(e_2^{(n-1)}) = \theta^{-1}(\sigma_\tau(e_2^{(n-1)}) - \sigma_\tau(e_2^{(n)})). \tag{4.19}$$

Using (4.19) and (4.15b), we get

$$\begin{aligned}
R_2^{(n)} &= \theta^{-1} \int_\Gamma \sigma_\tau(e_2^{(n-1)}) \cdot e_{2,\tau}^{(n-1)}\, ds - \theta^{-1} \int_\Gamma \sigma_\tau(e_2^{(n)}) \cdot e_{2,\tau}^{(n-1)}\, ds \\
&= \theta^{-1} a_2(e_2^{(n-1)}, e_2^{(n-1)}) + \theta^{-1} C_2(e_2^{(n-1)}; u_2, e_2^{(n-1)}) + \theta^{-1} C_2(u_2^{(n-1)}; e_2^{(n-1)}, e_2^{(n-1)}) \\
&\quad - \theta^{-1} a_2(e_2^{(n)}, e_2^{(n-1)}) - \theta^{-1} C_2(e_2^{(n)}; u_2, e_2^{(n-1)}) - \theta^{-1} C_2(u_2^{(n)}; e_2^{(n)}, e_2^{(n-1)}) \\
&= R_{21}^{(n)} + R_{22}^{(n)} + R_{23}^{(n)},
\end{aligned}$$

where

$$\begin{aligned}
R_{21}^{(n)} &:= \theta^{-1} a_2(e_2^{(n-1)} - e_2^{(n)}, e_2^{(n-1)}), \\
R_{22}^{(n)} &:= \theta^{-1} C_2(e_2^{(n-1)} - e_2^{(n)}; u_2, e_2^{(n-1)}), \\
R_{23}^{(n)} &:= \theta^{-1} C_2(u_2^{(n)}; e_2^{(n-1)} - e_2^{(n)}, e_2^{(n-1)})
\end{aligned}$$

and we have used $C(u_2^{(n-1)}; e_2^{(n-1)}, e_2^{(n-1)}) = C(u_2^{(n)}; e_2^{(n-1)}, e_2^{(n-1)}) = 0$ in the last step. In view of

$$\begin{aligned}
R_{21}^{(n)} &= \mu_2 \theta^{-1}(\|e_2^{(n-1)}\|_{V_2}^2 - \|e_2^{(n)}\|_{V_2}^2 + \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2}^2), \\
|R_{22}^{(n)}| &\le \theta^{-1} N \|u_2\|_{V_2} \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2} \|e_2^{(n-1)}\|_{V_2}, \\
|R_{23}^{(n)}| &\le \theta^{-1} N \|u_2^{(n)}\|_{V_2} \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2} \|e_2^{(n-1)}\|_{V_2},
\end{aligned}$$

we obtain a bound on $R_2^{(n)}$:

$$\begin{aligned}
R_2^{(n)} &\le \mu_2 \theta^{-1}(\|e_2^{(n-1)}\|_{V_2}^2 - \|e_2^{(n)}\|_{V_2}^2 + \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2}^2) \\
&\quad + \theta^{-1} N(\|u_2\|_{V_2} + \|u_2^{(n)}\|_{V_2}) \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2} \|e_2^{(n-1)}\|_{V_2}.
\end{aligned} \tag{4.20}$$

Using (4.17), (4.18) and (4.20) in (4.16), we find that

$$\begin{aligned}
\frac{\mu_2}{\theta}(\|e_2^{(n-1)}\|_{V_2}^2 - \|e_2^{(n)}\|_{V_2}^2) \ge{}& (2\mu_1 - N\|u_1\|_{V_1}) \|e_1^{(n)}\|_{V_1}^2 + (2\mu_2 - N\|u_2\|_{V_2}) \|e_2^{(n-1)}\|_{V_1}^2 \\
&- \frac{\mu_2}{\theta} \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2}^2 - c_\psi \|e_{1\tau}^{(n)} - e_{2\tau}^{(n-1)}\|_{L^2(\Gamma)}^2 \\
&- \frac{N}{\theta}(\|u_2\|_{V_2} + \|u_2^{(n-1)}\|_{V_2}) \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2} \|e_2^{(n-1)}\|_{V_2}.
\end{aligned}$$

Applying the trace inequality (3.13), we have

$$\begin{aligned}
&\frac{\mu_2}{\theta}(\|e_2^{(n-1)}\|_{V_2}^2 - \|e_2^{(n)}\|_{V_2}^2) \\
&\ge (2\mu_1 - N\|u_1\|_{V_1}) \|e_1^{(n)}\|_{V_1}^2 + (2\mu_2 - N\|u_2\|_{V_2}) \|e_2^{(n-1)}\|_{V_1}^2 \\
&\quad - (\frac{\mu_2}{\theta} + c_\psi \lambda_0^{-1}) \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2}^2 - \frac{N}{\theta}(\|u_2\|_{V_2} + \|u_2^{(n-1)}\|_{V_2}) \|e_2^{(n-1)}\|_{V_2} \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2}.
\end{aligned} \tag{4.21}$$

We intend to find a sufficient condition on μ and θ such that the right-hand side of (4.21) keeps positive so that we can deduce that $\|e_2^{(n)}\|_{V_2}^2 \downarrow 0$ as $n \to \infty$. To this end, we derive from (4.15b) that

$$\begin{aligned}
&\int_\Gamma \big(\sigma_\tau(e_2^{(n-1)}) - \sigma_\tau(e_2^{(n)})\big) \cdot (e_{2,\tau}^{(n-1)} - e_{2,\tau}^{(n)})\, ds \\
&= \int_\Gamma \sigma_\tau(e_2^{(n-1)}) \cdot (e_{2,\tau}^{(n-1)} - e_{2,\tau}^{(n)})\, ds - \int_\Gamma \sigma_\tau(e_2^{(n)}) \cdot (e_{2,\tau}^{(n-1)} - e_{2,\tau}^{(n)})\, ds \\
&= a_2(e_2^{(n-1)}, e_2^{(n-1)} - e_2^{(n)}) + C_2(e_2^{(n-1)}; u_2, e_2^{(n-1)} - e_2^{(n)}) + C_2(u_2^{(n-1)}; e_2^{(n-1)}, e_2^{(n-1)} - e_2^{(n)}) \\
&\quad - a_2(e_2^{(n)}, e_2^{(n-1)} - e_2^{(n)}) - C_2(e_2^{(n)}; u_2, e_2^{(n-1)} - e_2^{(n)}) - C_2(u_2^{(n)}; e_2^{(n)}, e_2^{(n-1)} - e_2^{(n)}) \\
&= 2\mu_2 \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2}^2 + C_2(e_2^{(n-1)} - e_2^{(n)}; u_2^{(n-1)}, e_2^{(n-1)} - e_2^{(n)}) \\
&\geq 2\mu_2 \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2}^2 - N\|u_2^{(n-1)}\|_{V_2} \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2}^2,
\end{aligned} \tag{4.22}$$

where we used $C_2(u_2^{(n)}; e_2^{(n)} - e_2^{(n-1)}, e_2^{(n-1)} - e_2^{(n)}) = 0$ and $u_2^{(n-1)} - u_2^{(n)} = e_2^{(n)} - e_2^{(n-1)}$. Meanwhile, by using (4.19),

$$\int_\Gamma \big(\sigma_\tau(e_2^{(n-1)}) - \sigma_\tau(e_2^{(n)})\big) \cdot (e_{2,\tau}^{(n-1)} - e_{2,\tau}^{(n)})\, ds = \int_\Gamma \theta\big(\sigma_\tau(e_1^{(n)}) + \sigma_\tau(e_2^{(n-1)})\big) \cdot (e_{2,\tau}^{(n-1)} - e_{2,\tau}^{(n)})\, ds = R_3 + R_4, \tag{4.23}$$

where

$$\begin{aligned}
R_3 &= \theta \int_\Gamma \sigma_\tau(e_1^{(n)}) \cdot (e_{2,\tau}^{(n-1)} - e_{2,\tau}^{(n)})\, ds, \\
R_4 &= \theta \int_\Gamma \sigma_\tau(e_2^{(n-1)}) \cdot (e_{2,\tau}^{(n-1)} - e_{2,\tau}^{(n)})\, ds.
\end{aligned}$$

By (4.15b) with n replaced by $(n-1)$, we derive a bound on R_4:

$$\begin{aligned}
R_4 &= \theta\Big(a_2(e_2^{(n-1)}, e_2^{(n-1)} - e_2^{(n)}) + C_2(e_2^{(n-1)}; u_2, e_2^{(n-1)} - e_2^{(n)}) + C_2(u_2^{(n-1)}; e_2^{(n-1)}, e_2^{(n-1)} - e_2^{(n)}) \Big) \\
&\leq \theta\Big(2\,\mu_2 \|e_2^{(n-1)}\|_{V_2} + N(\|u_2\|_{V_2} + \|u_2^{(n-1)}\|_{V_2})\|e_2^{(n-1)}\|_{V_2} \Big) \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2}.
\end{aligned}$$

To bound R_3, denote $\xi := (e_{2,\tau}^{(n-1)} - e_{2,\tau}^{(n)})|_\Gamma \in H_{00}^{\frac{1}{2}}(\Gamma)$. Note that $\xi \cdot n = 0$ on Γ and the zero extension of ξ to $\partial\Omega_1$ or $\partial\Omega_2$ (also denoted by ξ) is continuous (cf. [28, Lemma A.1]). By [64, Theorem IV.1.1], there exists an extension of ξ from $H_{00}^{\frac{1}{2}}(\Gamma)$ to $V_i^{\rm div}$ $(i = 1, 2)$, denoted by $E_i\xi$, such that

$$E_i\xi = \xi \text{ on } \Gamma, \quad \|E_i\xi\|_{V_i} \leq c_E \|\xi\|_{H^{\frac{1}{2}}(\Gamma)} = c_E \|e_{2\tau}^{(n-1)} - e_{2\tau}^{(n)}\|_{H^{\frac{1}{2}}(\Gamma)} \leq c_E c_\gamma \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2}, \tag{4.24}$$

where $c_E, c_\gamma > 0$ are constants. Thus,

$$\begin{aligned}
R_3 &= \theta \int_\Gamma \sigma_\tau(e_1^{(n)}) \cdot \xi\, ds = \theta \int_\Gamma \sigma_\tau(e_1^{(n)}) \cdot E_1\xi\, ds \\
&= \theta\Big(a_1(e_1^{(n)}, E_1\xi) + C(e_1^{(n)}; u_1, E_1\xi) + C(u_1^{(n)}; e_1^{(n)}, E_1\xi) \Big) \quad (\text{by (4.15a)}) \\
&\leq \theta\Big(2\mu_1 \|e_1^{(n)}\|_{V_1} \|E_1\xi\|_{V_1} + N(\|u_1\|_{V_1} + \|u_1^{(n)}\|_{V_1})\|e_1^{(n)}\|_{V_1} \|E_1\xi\|_{V_1} \Big) \\
&\leq \theta\Big(2\mu_1 \|e_1^{(n)}\|_{V_1} + N(\|u_1\|_{V_1} + \|u_1^{(n)}\|_{V_1})\|e_1^{(n)}\|_{V_1} \Big) c_E c_\gamma \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2} \quad (\text{by (4.24)}).
\end{aligned}$$

From a combination of (4.22) and (4.23), together with the above estimates of R_3 and R_4, we obtain

$$\begin{aligned}
&\big(2\mu_2 - N\|u_2^{(n-1)}\|_{V_2}\big) \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2}^2 \\
&\leq \theta\ \Big(2\mu_2 \|e_2^{(n-1)}\|_{V_2} + N(\|u_2\|_{V_2} + \|u_2^{(n-1)}\|_{V_2})\|e_2^{(n-1)}\|_{V_2} \\
&\qquad + 2\mu_1 c_E c_\gamma \|e_1^{(n)}\|_{V_1} + c_E c_\gamma N(\|u_1\|_{V_1} + \|u_1^{(n)}\|_{V_1})\|e_1^{(n)}\|_{V_1} \Big)\ \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2},
\end{aligned}$$

leading to

$$\begin{aligned}
\big(2\mu_2 - N\|u_2^{(n-1)}\|_{V_2}\big) \|e_2^{(n-1)} - e_2^{(n)}\|_{V_2} \leq \theta\ \Big(&\big(2\mu_2 + N(\|u_2\|_{V_2} + \|u_2^{(n-1)}\|_{V_2})\big) \|e_2^{(n-1)}\|_{V_2} \\
&+ c_E c_\gamma \big(2\mu_1 + N(\|u_1\|_{V_1} + \|u_1^{(n)}\|_{V_1})\big) \|e_1^{(n)}\|_{V_1} \Big).
\end{aligned} \tag{4.25}$$

Using (4.25) in (4.21), we have

$$
\begin{aligned}
&\frac{\mu_2}{\theta}(\|e_2^{(n-1)}\|_{V_2}^2 - \|e_2^{(n)}\|_{V_2}^2)\\
\geq &\Big(2\mu_1 - N\|u_1\|_{V_1} - (\frac{\mu_2}{\theta} + c_\psi \lambda_0^{-1})2\theta^2 c_E^2 c_\gamma^2 \big(2\mu_1 + N(\|u_1\|_{V_1} + \|u_1^{(n)}\|_{V_1})\big)^2 (2\mu_2 - N\|u_2^{(n-1)}\|_{V_2})^{-2}\Big)\|e_1^{(n)}\|_{V_1}^2\\
&+ \Big(2\mu_2 - N\|u_2\|_{V_1} - (\frac{\mu_2}{\theta} + c_\psi \lambda_0^{-1})2\theta^2 \big(2\mu_2 + N(\|u_2\|_{V_2} + \|u_2^{(n-1)}\|_{V_2})\big)^2 (2\mu_2 - N\|u_2^{(n-1)}\|_{V_2})^{-2}\Big)\|e_2^{(n-1)}\|_{V_2}^2\\
&- N\big(\|u_2\|_{V_2} + \|u_2^{(n-1)}\|_{V_2}\big)\big(2\mu_2 + N(\|u_2\|_{V_2} + \|u_2^{(n-1)}\|_{V_2})(2\mu_2 - N\|u_2^{(n-1)}\|_{V_2})^{-1}\big)\|e_2^{(n-1)}\|_{V_2}^2\\
&- N\big(\|u_2\|_{V_2} + \|u_2^{(n-1)}\|_{V_2}\big)c_E c_\gamma \big(2\mu_1 + N(\|u_1\|_{V_1} + \|u_1^{(n)}\|_{V_1})(2\mu_2 - N\|u_2^{(n-1)}\|_{V_2})^{-1}\big)\|e_2^{(n-1)}\|_{V_2}\|e_1^{(n)}\|_{V_1}.
\end{aligned}
\tag{4.26}
$$

According to Theorem 3.2 and the result from Step 1, we have the bounds

$$
\|u\|_V \leq c_f, \qquad \|u_2^{(n)}\|_{V_2} \leq c_4, \qquad \|u_1^{(n)}\|_{V_1} \leq c_5.
$$

Hence, we can simplify (4.26) and obtain that

$$
\begin{aligned}
&\frac{\mu_2}{\theta}(\|e_2^{(n-1)}\|_{V_2}^2 - \|e_2^{(n)}\|_{V_2}^2)\\
\geq &\Big(2\mu_1 - Nc_f - (\frac{\mu_2}{\theta} + c_\psi \lambda_0^{-1})2\theta^2 c_E^2 c_\gamma^2 \big(2\mu_1 + N(c_f + c_5)\big)^2 (2\mu_2 - Nc_4)^{-2}\Big)\|e_1^{(n)}\|_{V_1}^2\\
&+ \Big(2\mu_2 - Nc_f - (\frac{\mu_2}{\theta} + c_\psi \lambda_0^{-1})2\theta^2 \big(2\mu_2 + N(c_f + c_5)\big)^2 (2\mu_2 - Nc_4)^{-2}\Big)\|e_2^{(n-1)}\|_{V_2}^2\\
&- N(c_f + c_4)(2\mu_2 + N(c_f + c_4))(2\mu_2 - Nc_4)^{-1}\|e_2^{(n-1)}\|_{V_2}^2\\
&- N(c_f + c_4)c_E c_\gamma (2\mu_1 + N(c_f + c_5))(2\mu_2 - Nc_4)^{-1}\|e_2^{(n-1)}\|_{V_2}\|e_1^{(n)}\|_{V_1}\\
\geq &\Bigg(2\mu_1 - Nc_f - \frac{N(c_f + c_4)c_E c_\gamma}{2(2\mu_2 - Nc_4)}\big(2\mu_1 + N(c_f + c_5)\big)\\
&\quad - \theta(\mu_2 + c_\psi \lambda_0^{-1}\theta)2c_E^2 c_\gamma^2 \big(2\mu_1 + N(c_f + c_5)\big)^2 (2\mu_2 - Nc_4)^{-2} \Bigg) \|e_1^{(n)}\|_{V_1}^2\\
&+ \Bigg(2\mu_2 - Nc_f - \frac{N(c_f + c_4)}{2\mu_2 - Nc_4}\big(2\mu_2 + N(c_f + c_4)\big) - \frac{N(c_f + c_4)c_E c_\gamma}{2(2\mu_2 - Nc_4)}\big(2\mu_1 + N(c_f + c_5)\big)\\
&\quad - \theta(\mu_2 + c_\psi \lambda_0^{-1}\theta)2\big(2\mu_2 + N(c_f + c_4)\big)^2 (2\mu_2 - Nc_4)^{-2} \Bigg) \|e_2^{(n-1)}\|_{V_2}^2,
\end{aligned}
\tag{4.27}
$$

where we have used $\|e_2^{(n-1)}\|_{V_2}\|e_1^{(n)}\|_{V_1} \leq \frac{1}{2}(\|e_2^{(n-1)}\|_{V_2}^2 + \|e_1^{(n)}\|_{V_1}^2)$ to get the last inequality.

Step 3. As commented at the end of *Step 1*, we find that c_f, c_4 and c_5 decrease as μ_1 and μ_2 increase, and c_f, c_4 and c_5 tends to 0 as μ_1 and μ_2 go to ∞. Assume that μ_1 and μ_2 are sufficiently large such that

$$
c_6 := 2\mu_1 - Nc_f - \frac{N(c_f + c_4)c_E c_\gamma}{2(2\mu_2 - Nc_4)}(2\mu_1 + N(c_f + c_5)) > 0, \tag{4.28a}
$$

$$
c_7 := 2\mu_2 - Nc_f - \frac{N(c_f + c_5)}{2\mu_2 - Nc_4}\big(2\mu_2 + N(c_f + c_4) + \frac{1}{2}c_E c_\gamma (2\mu_1 + N(c_f + c_5))\big) > 0. \tag{4.28b}
$$

Furthermore, we can choose small enough θ to guarantee that

$$
c_6 - \theta(\mu_2 + c_\psi \lambda_0^{-1}\theta)2c_E^2 c_\gamma^2 \big(2\mu_1 + N(c_f + c_5)\big)^2 (2\mu_2 - Nc_4)^{-2} > 0, \tag{4.29a}
$$

$$
c_7 - \theta(\mu_2 + c_\psi \lambda_0^{-1}\theta)2\big(2\mu_2 + N(c_f + c_4)\big)^2 (2\mu_2 - Nc_4)^{-2} > 0. \tag{4.29b}
$$

In this case, $\|e_2^{(n)}\|_{V_2}^2$ decreases unless $\|e_1^{(n)}\|_{V_1} = \|e_2^{(n-1)}\|_{V_2} = 0$. Thus $\|e_2^{(n)}\|_{V_2} \downarrow 0$ as $n \to \infty$, which also implies (by (4.27)) $\|e_1^{(n)}\|_{V_1} \to 0$. More preciously, by setting

$$
\eta := 1 - \theta\mu_2^{-1}\Big(c_7 - \theta(\mu_2 + c_\psi \lambda_0^{-1}\theta)2\big(2\mu_2 + N(c_f + c_4)\big)^2 (2\mu_2 - Nc_4)^{-2}\Big) \in (0, 1),
$$

it follows from (4.27) that

$$
\|e_2^{(n)}\|_{V_2}^2 \leq \eta \|e_2^{(n-1)}\|_{V_2}^2 \leq \cdots \leq \eta^n \|e_2^{(0)}\|_{V_2}^2.
$$

Together with (4.27), we assert $\|e_1^{(n)}\|_{V_1}^2 \leq C\eta^n \to 0$ as $n \to \infty$. $\square$

Remark 4.2. For the Problem (HVI-S), analogously to the above argument, we can also prove the convergence of the domain decomposition method under the assumptions of large μ_1, μ_2, and small θ.

5. Mixed finite element method

For simplicity, we assume that both Ω_1 and Ω_2 are polygonal domains, and Γ consists of plot components. For $i = 1, 2$, we introduce a family of quasi-uniform triangulations $\{\mathcal{T}_i^h\}$ to $\overline{\Omega_i}$, and assume that for any h, $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ are compatible on the interface Γ in the sense that the restrictions of $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ on Γ coincide. The triangulations of Ω_1 and Ω_2 have the same mesh size $h = \max_{K\in\mathcal{T}_1^h\cup\mathcal{T}_2^h} \operatorname{diam}(K)$. Let V_i^h and Q_i^h be finite element subspaces of V_i and Q_i associated with the triangulation $\mathcal{T}_i^h$. Denote

$$\begin{aligned}
&V^h := V_1^h \times V_2^h, \quad \mathring{V}^h := \{v^h \in V^h : v^h|_\Gamma = 0\},\\
&Q^h := Q_1^h \times Q_2^h, \quad \mathring{Q}_i^h := Q_i^h \cap L_0^2(\Omega_i), \quad \mathring{Q}^h := \mathring{Q}_1^h \times \mathring{Q}_2^h,\\
&V_i^{h,\mathrm{div}} := \{v_i^h \in V_i^h : b_i(v_i^h, q_i^h) = 0\ \forall\, q_i^h \in Q_i^h\}, \quad V^{h,\mathrm{div}} := V_1^{h,\mathrm{div}} \times V_2^{h,\mathrm{div}}.
\end{aligned}$$

Since $V_i^h \subset V_i$, the coercivity of $a(\cdot,\cdot)$ remains to be valid on V^h:

$$a(v^h, v^h) \ge 2\mu\|v^h\|_V^2 \quad \forall\, v^h \in V^h.$$

We assume that the following inf–sup condition holds for the pair $V_i^h \times Q_h^h$:

$$\sup_{v_i^h\in V_i^h} \frac{b_i(v_i^h, p_i^h)}{\|v_i^h\|_{V_i}} \ge C\|p_i^h\|_{Q_i} \quad \forall\, p_i^h \in \mathring{Q}_i^h. \tag{5.1}$$

Note that the P1b/P1 and P2/P1 pairs (see (5.15) and (5.16) below) satisfy the discrete inf–sup condition (5.1).

5.1. The discrete problems

We consider the finite element approximations of the Navier–Stokes hemivariational inequality.

Problem 5.1 (*HVI-NS^h*)**.** Find $u^h \in V^h$ and $p^h \in \mathring{Q}^h$ such that

$$a(u^h, v^h) + b(v^h, p^h) + C(u^h; u^h, v^h) + \int_\Gamma \psi^0([u_\tau^h]; [v_\tau^h])\, ds \ge \langle f, v^h\rangle \qquad \forall\, v^h \in V^h, \tag{5.2}$$

$$b(u^h, q^h) = 0 \qquad \forall\, q^h \in \mathring{Q}^h. \tag{5.3}$$

A reduced version of Problem 5.1 reads:

Problem 5.2 (*HVI-NS^{h,div}*)**.** Find $u^h \in V^{h,\mathrm{div}}$ such that

$$a(u^h, v^h) + C(u^h; u^h, v^h) + \int_\Gamma \psi^0([u_\tau^h]; [v_\tau^h])\, ds \ge \langle f, v^h\rangle \quad \forall\, v^h \in V^{h,\mathrm{div}}. \tag{5.4}$$

Under the assumptions stated in Theorem 3.2 and (5.1), we can show that Problems 5.1 and 5.2 have unique solutions and the two problems are equivalent. Moreover,

$$\|u^h\|_V \le c_f, \tag{5.5}$$

where c_f is defined by (3.15).

5.2. Error bound

Let us bound the error between the continuous and discrete solutions. Noting that $v|_\Gamma = 0$ for all $v \in \mathring{V}$, we have

$$a(u, v) + b(v, p) + C(u; u, v) = \langle f, v\rangle \quad \forall\, v \in \mathring{V}. \tag{5.6}$$

The discrete analogue of (5.6) is derived from (5.2),

$$a(u^h, v^h) + b(v^h, p^h) + C(u^h; u^h, v^h) = \langle f, v^h\rangle \quad \forall\, v^h \in \mathring{V}^h. \tag{5.7}$$

Let $q^h \in \mathring{Q}^h$ be arbitrary. By (5.1),

$$C\|p^h - q^h\|_Q \le \sup_{v^h\in\mathring{V}^h} \frac{b(v^h, p^h - q^h)}{\|v^h\|_V}. \tag{5.8}$$

For any $v^h \in \mathring{V}^h$, by (5.6) and (5.7),

$$\begin{aligned}
b(v^h, p^h) &= \langle f, v^h\rangle - a(u^h, v^h) - C(u^h; u^h, v^h),\\
b(v^h, p) &= \langle f, v^h\rangle - a(u, v^h) - C(u; u, v^h).
\end{aligned}$$

We see that

$$\begin{aligned}
b(v^h, p^h - p) &= b(v^h, p^h) - b(v^h, p)\\
&= a(u - u^h, v^h) + C(u; u, v^h) - C(u^h; u^h, v^h)\\
&= a(u - u^h, v^h) + C(u; u - u^h, v^h) + C(u - u^h; u^h, v^h).
\end{aligned}$$

Hence, it follows from (5.8) and

$$b(v^h, p^h - q^h) = b(v^h, p^h - p) + b(v^h, p - q^h)$$

that

$$\|p^h - q^h\|_Q \le c\left(1 + \|u\|_V + \|u^h\|_V\right)\|u - u^h\|_V + c\,\|p - q^h\|_Q.$$

Since $\|u\|_V$ and $\|u^h\|_V$ are bounded by the constant c_f,

$$\|p^h - q^h\|_Q \le c\left(\|u - u^h\|_V + \|p - q^h\|_Q\right).$$

Using the triangle inequality

$$\|p - p^h\|_Q \le \|p - q^h\|_Q + \|p^h - q^h\|_Q,$$

we then derive the inequality

$$\|p - p^h\|_Q \le c\left(\|u - u^h\|_V + \|p - q^h\|_Q\right). \tag{5.9}$$

On the other hand, for any $v^h \in V^h$,

$$\begin{aligned} 2\,\mu\,\|u - u^h\|_V^2 &\le a(u - u^h, u - u^h) \\ &= a(u, u - u^h) - a(u^h, u - u^h) \\ &= a(u, u - u^h) - a(u^h, u - v^h) + a(u^h, u^h - v^h). \end{aligned} \tag{5.10}$$

Substituting $v = u^h - u$ into (3.6) yields

$$a(u, u - u^h) \le C(u; u, u^h - u) + b(u^h - u, p) + \int_\Gamma \psi^0([u_\tau]; [u_\tau^h - u_\tau])\,ds - \langle f, u^h - u\rangle.$$

By (5.2) with v^h replaced by $v^h - u^h$,

$$a(u^h, u^h - v^h) \le C(u^h; u^h, v^h - u^h) + b(v^h - u^h, p^h) + \int_\Gamma \psi^0([u_\tau^h]; [v_\tau^h - u_\tau^h])\,ds - \langle f, v^h - u^h\rangle.$$

Also, Substituting $v = u - v^h$ into (3.6), and in view of

$$-a(u^h, u - v^h) = a(u - u^h, u - v^h) + a(u, v^h - u),$$

we get

$$a(u, v^h - u) \le C(u; u, u - v^h) + b(u - v^h, p) + \int_\Gamma \psi^0([u_\tau]; [u_\tau - v_\tau^h])\,ds - \langle f, u - v^h\rangle.$$

Inserting these inequalities into (5.10) results

$$2\,\mu\,\|u - u^h\|_V^2 \le a(u - u^h, u - v^h) + I_c + I_b + I_\psi, \tag{5.11}$$

where

$$\begin{aligned} I_c &= C(u; u, u^h - u) + C(u^h; u^h, v^h - u^h) + C(u; u, u - v^h), \\ I_b &= b(u^h - u, p) + b(v^h - u^h, p^h) + b(u - v^h, p), \\ I_\psi &= \int_\Gamma \left(\psi^0([u_\tau]; [u_\tau^h - u_\tau]) + \psi^0([u_\tau^h]; [v_\tau^h - u_\tau^h]) + \psi^0([u_\tau]; [u_\tau - v_\tau^h])\right) ds. \end{aligned}$$

We rewrite I_c as

$$\begin{aligned} I_c &= C(u; u, u^h - v^h) + C(u^h; u^h, v^h - u^h) \\ &= C(u; u - u^h, u^h - v^h) + C(u, u^h; u^h - v^h) + C(u^h; u^h, v^h - u^h) \\ &= C(u; u - u^h, u^h - v^h) + C(u - u^h; u^h, u^h - v^h) \\ &= C(u; u - u^h, u - v^h) + C(u - u^h; u^h, u^h - v^h), \end{aligned}$$

where we used $c(u, u - u^h, u^h - u) = 0$ in the last step. Then we bound I_c by

$$\begin{aligned} I_c &\le N\,\|u\|_V \|u - u^h\|_V \|u - v^h\|_V + N\,\|u^h\|_V \|u - u^h\|_V \|u^h - v^h\|_V \\ &\le N\,c_f \|u - u^h\|_V \left(\|u - v^h\|_V + \|u^h - v^h\|_V\right). \end{aligned}$$

Using (3.7) and (5.3) repeatedly, we have

$$\begin{aligned} I_b &= b(u^h - u, p - q^h) + b(v^h, p^h) - b(v^h, p) \\ &= b(u^h - u, p - q^h) + b(v^h, p^h - p) \end{aligned}$$

$$= b(u^h - u, p - q^h) + b(v^h - u, p^h - p).$$

Thus,

$$I_b \le c\left(\|u - u^h\|_V \|p - q^h\|_Q + \|u - v^h\|_V \|p - p^h\|_Q\right).$$

By the sub-additivity (iii) in Proposition 2.1,

$$\psi^0([u_\tau^h]; [v_\tau^h - u_\tau^h]) \le \psi^0([u_\tau^h]; [u_\tau - u_\tau^h]) + \psi^0([u_\tau^h]; [v_\tau^h - u_\tau]).$$

As a result, I_ψ can be bounded by

$$\begin{aligned} I_\psi &\le \int_\Gamma \left(\psi^0([u_\tau]; [u_\tau^h - u_\tau]) + \psi^0([u_\tau^h]; [u_\tau - u_\tau^h]) + \psi^0([u_\tau^h]; [v_\tau^h - u_\tau]) + \psi^0([u_\tau]; [u_\tau - v_\tau^h])\right) ds \\ &\le \int_\Gamma \left(c_\psi |[u_\tau - u_\tau^h]|^2 + \left((c_0 + c_1|[u_\tau]|) + (c_0 + c_1|[u_\tau^h]|)\right)|[u_\tau - v_\tau^h]|\right) ds \\ &\le c_\psi \lambda_0^{-1}\|u - u^h\|_V^2 + c\left(1 + \|[u_\tau]\|_{L^2(\Gamma;\mathbb{R}^d)} + \|[u_\tau^h]\|_{L^2(\Gamma;\mathbb{R}^d)}\right)\|[u_\tau - v_\tau^h]\|_{L^2(\Gamma;\mathbb{R}^d)}. \end{aligned}$$

By (3.13), $\|[u_\tau]\|_{L^2(\Gamma;\mathbb{R}^d)} \le \lambda_0^{-1/2}\|u\|_V$ and $\|[u_\tau^h]\|_{L^2(\Gamma;\mathbb{R}^d)} \le \lambda_0^{-1/2}\|u^h\|_V$. By (3.20) and (5.5), $\|u\|_V$ and $\|u^h\|_V$ are bounded by c_f. Hence, from (5.11), we obtain

$$\begin{aligned} 2\mu\|u - u^h\|_V^2 &\le c\|u - u^h\|_V\|u - v^h\|_V + N c_f\|u - u^h\|_V\left(\|u - v^h\|_V + \|u^h - v^h\|_V\right) \\ &\quad + c\left(\|u - u^h\|_V\|p - q^h\|_Q + \|u - v^h\|_V\|p - p^h\|_Q\right) \\ &\quad + c_\psi\lambda_0^{-1}\|u - u^h\|_V^2 + c\|u_\tau - v_\tau^h\|_{L^2(\Gamma;\mathbb{R}^d)}. \end{aligned} \tag{5.12}$$

By the triangle inequality

$$\|u^h - v^h\|_V \le \|u - u^h\|_V + \|u - v^h\|_V.$$

We apply the modified Cauchy inequality

$$x\,y \le \epsilon\,x^2 + y^2/(4\epsilon), \quad \forall x, y \in \mathbb{R},\ \forall \epsilon > 0$$

to the terms $c\|u - u^h\|_V\|u - v^h\|_V$, $c\|u - u^h\|_V\|p - q^h\|_Q$, $c\|u - v^h\|_V\|p - p^h\|_Q$, and derive from (5.12) that for $\epsilon > 0$ arbitrarily small,

$$\begin{aligned} &\left(2\mu - c_\psi\lambda_0^{-1} - N c_f - \epsilon\right)\|u - u^h\|_V^2 \\ &\quad \le c\left(\|u - v^h\|_V^2 + \|p - q^h\|_Q^2 + \|u_\tau - v_\tau^h\|_{L^2(\Gamma;\mathbb{R}^d)}\right) + \epsilon\|p - p^h\|_Q^2. \end{aligned} \tag{5.13}$$

By choosing $\epsilon > 0$ sufficiently small, we can combine (5.13) and (5.9) to get

$$\|u - u^h\|_V^2 + \|p - p^h\|_Q^2 \le c \inf_{(v^h,q^h)\in V^h\times Q^h}\left(\|u - v^h\|_V^2 + \|p - q^h\|_Q^2 + \|u_\tau - v_\tau^h\|_{L^2(\Gamma;\mathbb{R}^d)}\right). \tag{5.14}$$

This Céa's inequality is the starting point for error estimation of the numerical solution (u^h, p^h). In particular, consider a family of regular finite element partitions of the domain $\overline{\Omega_1}\cup\overline{\Omega_2}$, formed as unions of finite element partitions on $\overline{\Omega_1}$ and $\overline{\Omega_2}$ whose restrictions on Γ are identical and such that the partitions are compatible to the boundary splitting $\partial\Omega_i = \Gamma_i \cup \Gamma$, $i = 1, 2$. Let us use P1b/P1 finite elements [65]

$$V_i^h = \{v_i^h \in V_i \cap C^0(\overline{\Omega}_i)^d : v_i^h|_T \in [P_1(T)]^d \oplus B(T)\ \forall T \in \mathcal{T}_i^h\}, \tag{5.15a}$$

$$Q_i^h = \{q_i^h \in Q_i \cap C^0(\overline{\Omega}_i) : q_i^h|_T \in P_1(T)\ \forall T \in \mathcal{T}_i^h\}, \tag{5.15b}$$

or P2/P1 finite elements [63]

$$V_i^h = \{v_i^h \in V_i \cap C^0(\overline{\Omega})^d : v_i^h|_T \in [P_2(T)]^d\ \forall T \in \mathcal{T}_i^h\}, \tag{5.16a}$$

$$Q_i^h = \{q_i^h \in Q \cap C^0(\overline{\Omega}_i) : q_i^h|_T \in P_1(T)\ \forall T \in \mathcal{T}_i^h\}, \tag{5.16b}$$

where $P_k(T)$ represents the space of polynomials of a total degree less than or equal to k in T, and $B(T)$ is the space of bubble functions on T. For these choices, the discrete inf–sup condition (5.1) is satisfied. We can then derive an optimal order error estimate for the P1b/P1 element solution from (5.14) and standard finite element interpolation error bounds, under certain solution regularity assumptions. We write Γ as the union of a finite number of flat components:

$$\Gamma = \cup_{j=1}^{j_0}\gamma_j,$$

where each γ_j is a line segment in 2D or a polygon in 3D.

Theorem 5.1. *Let (u, p) and (u^h, p^h) be the solutions of Problems 3.1* (HVI-NS) *and 5.1 (*HVI-NS^h*) with the P1b/P1 elements (5.15a)–(5.15b). Assume $f \in V^*$, $H(\psi)$, (3.12), (3.21), and the regularities $u \in H^2(\Omega)^d$, $u_\tau|_{\gamma_j} \in H^2(\gamma_j)^d$, $1 \le j \le j_0$, and $p \in H^1(\Omega)$, there exists a constant c depending on u and p such that*

$$\|u - u^h\|_V + \|p - p^h\|_Q \le c\,h. \tag{5.17}$$

6. Numerical experiments

In the numerical experiments, we take the function ψ in the form

$$\psi(z) = \int_0^{|z|} \omega(t)\, dt,$$

where $\omega : [0,\infty) \to \mathbb{R}$ is continuous, $\omega(0) > 0$. Then the slip boundary condition $-\sigma_{1,\tau} \in \partial\psi([u_\tau])$ is equivalent to

$$|\sigma_{1,\tau}| \le \omega(0) \text{ if } [u_\tau] = 0, \quad -\sigma_{1,\tau} = \omega(|[u_\tau]|)\, \frac{[u_\tau]}{|[u_\tau]|} \text{ if } [u_\tau] \ne 0.$$

Introduce a Lagrange multiplier

$$\lambda = \frac{-\sigma_{1,\tau}}{\omega(|[u_\tau]|)}$$

and define a set

$$\Lambda = \left\{\eta \in L^2(\Gamma;\mathbb{R}^d) \,:\, |\eta| \le 1 \text{ a.e. on } \Gamma\right\}.$$

Then Problem 3.1 can be restated as follows.

Problem 6.1. Find $(u,p) \in V \times \mathring{Q}$ and $\lambda \in \Lambda$ such that

$$a(u,v) + b(v,p) + C(u;u,v) + \int_\Gamma \omega(|[u_\tau]|)\, \lambda \cdot [v_\tau]\, ds = \langle f, v\rangle \qquad \forall\, v \in V, \tag{6.1}$$

$$b(u,q) = 0 \qquad \forall\, q \in \mathring{Q}, \tag{6.2}$$

$$\lambda \cdot [u_\tau] - |\lambda|\, |[u_\tau]| = 0 \qquad \text{a.e. on } \Gamma. \tag{6.3}$$

We adopt a projection-type iterative procedure [27] to solve Problem 6.1. The algorithm is presented below. Choose a constant parameter $\rho > 0$ and an initial guess $u^{h,(0)}$. Then for $n = 1,2,\dots$, find $(u^{h,(n)}, p^{h,(n)}) \in V^h \times Q^h$ such that for all $(v^h, q^h) \in V^h \times Q^h$,

$$a(u^{h,(n)}, v^h) + b(v^h, p^{h,(n)}) + C(u^{h,(n-1)}; u^{h,(n)}, v^h) = \langle f, v^h\rangle - \int_{\Gamma_1} \omega(|[u^{h,(n-1)}]_\tau|)\, \lambda^{h,(n)} \cdot [v^h_\tau]\, ds, \tag{6.4a}$$

$$b(u^{h,(n)}, q^h) = 0, \tag{6.4b}$$

and update the Lagrange multiplier:

$$\lambda^{h,(n+1)} = P(\lambda^{h,(n)} + \rho\, [u^{h,(n)}_\tau]), \tag{6.5}$$

where P is the orthogonal projection operator from $\mathbb{R}^d$ to the unit closed ball in $\mathbb{R}^d$. To solve (6.4), we again utilize the projection-iteration scheme stated as follows.

Choose an initial guess $\lambda_0^{h,(n)}$. Then for $l = 0,1,\dots,$

1. find $(u_l^{h,(n)}, p_l^{h,(n)}) \in V^h \times Q^h$ such that, for all $(v^h, q^h) \in V^h \times Q^h$,

$$a(u_l^{h,(n)}, v^h) + b(v^h, p_l^{h,(n)}) + C(u^{h,(n-1)}; u_l^{h,(n)}, v^h) = \langle f, v^h\rangle - \int_{\Gamma_1} \omega(|[u^{h,(n-1)}]_\tau|)\, \lambda_l^{h,(n)} \cdot [v^h_\tau]\, ds, \tag{6.6a}$$

$$b(u_l^{h,(n)}, q^h) = 0; \tag{6.6b}$$

2. update the Lagrange multiplier:

$$\lambda_{l+1}^{h,(n)} = P\left(\lambda_l^{h,(n)} + \rho\omega(|[u^{h,(n)}]_{\tau,l}|)\, [u^{h,(n)}_{\tau,l}]\right);$$

3. iterate (1) and (2) until $\|u_l^{h,(n)} - u_{l-1}^{h,(n)}\|_{L^2(\Omega)} < \varepsilon_2$ and at that time, let $(u^{h,(n)}, p^{h,(n)}, \lambda^{h,(n)})$ be the most recent iterates.

Repeat the above procedure until $\|u^{h,(n)} - u^{h,(n-1)}\|_{L^2(\Omega)} < \varepsilon_1$. In our tests, we take $\varepsilon_1, \varepsilon_2 = 10^{-6}$.

In the examples below, we take

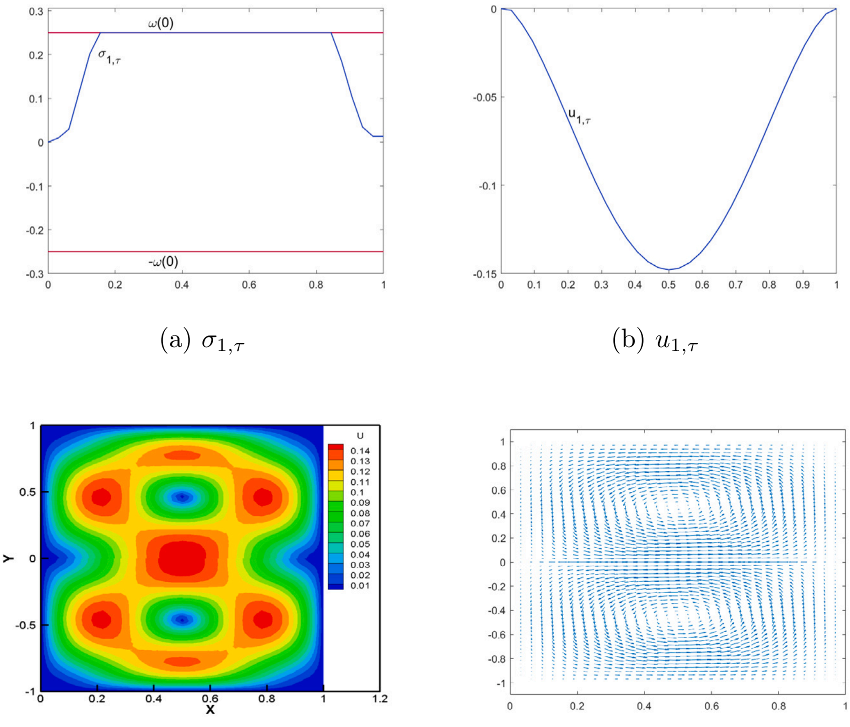
$$\omega(t) = (a-b)\, e^{-\alpha t} + b \quad (a > b). \tag{6.7}$$

to illustrate performance of the numerical method. Let $\Omega = (0,1)\times(-1,1)$ with the subregions $\Omega_1 = (0,1)\times(0,1)$ and $\Omega_2 = (0,1)\times(-1,0)$, the interface $\Gamma = (0,1)\times\{0\}$, and the Dirichlet boundary $\partial\Omega\backslash\Gamma$. Let $\mu_1 = \mu_2 = 1$ and $f_i = -\nabla\cdot\sigma(u_{i0}, p_{i0})$ in Ω_i $(i = 1,2)$, where

$$u_{10}(x,y) = \begin{pmatrix} 20x^2(x-1)^2 y(y-1)(2y-1) \\ -20x(x-1)(2x-1)y^2(y-1)^2 \end{pmatrix}, \qquad p_{10}(x,y) = 20(2x-1)(2y-1),$$

and

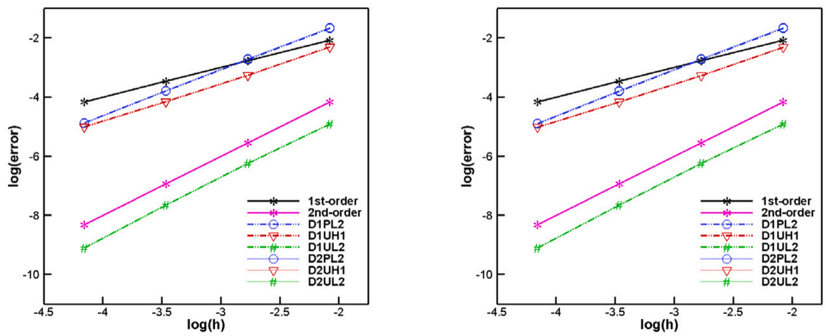
$$u_{20}(x,y) = \begin{pmatrix} 20x^2(x-1)^2 y(y+1)(2y+1) \\ -20x(x-1)(2x-1)y^2(y+1)^2 \end{pmatrix}, \qquad p_{20}(x,y) = 20(2x-1)(2y+1).$$



(a) $\sigma_{1,\tau}$ (b) $u_{1,\tau}$

(c) $|u|$ (d) velocity

Fig. 2. Tangential traction $\sigma_{1,\tau}$ and tangential velocity $u_{1,\tau}$ on the interface Γ; $|u|$ and the velocity u in Ω of (HVI-NS).



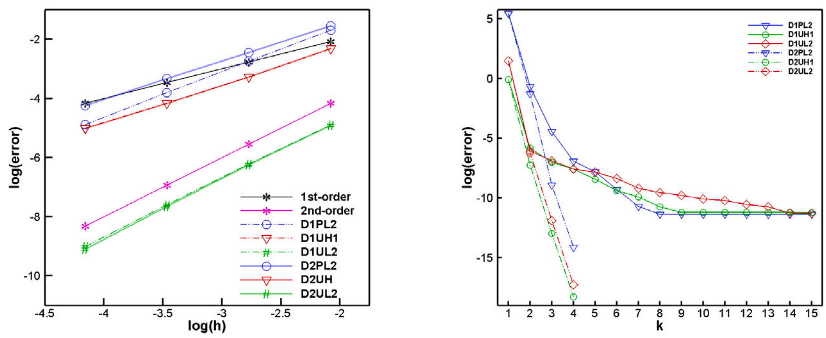
(a) projection-HVI (b) projection-VI

Fig. 3. Convergence behaviors of numerical solutions of (HVI-NS) and (6.8) under the projection iterative procedure.

We adopt a sequence of uniform triangular meshes with the interval $[0,1]$ being split into h^{-1} equal sub-intervals, and use the P1b/P1 finite elements. Since the exact solution is unknown, we take the numerical solution $(u_{1,\text{ref}}, p_{1,\text{ref}}, u_{2,\text{ref}}, p_{2,\text{ref}})$ on a fine mesh $(h = 2^{-8})$ as the reference solution, and compare the solutions (u_{ih}, p_{ih}) on the coarse meshes $(h = 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6})$ with the reference solution. The experimental errors are plotted in Figs. 3 and 4, where the following notation is used in our figures:

$$\text{D1UL2} := \|u_h - u\|_{L^2(\Omega_1)}, \quad \text{D2UL2} := \|u_h - u\|_{L^2(\Omega_2)}, \quad \text{D1UH1} := |u_h - u|_{H^1(\Omega_1)},$$
$$\text{D2UH1} := |u_h - u|_{H^1(\Omega_2)}, \quad \text{D1PL2} := \|p_h - p\|_{L^2_0(\Omega_1)}, \quad \text{D2PL2} := \|p_h - p\|_{L^2_0(\Omega_2)}.$$

Now we apply the discrete projection-type iterative algorithm (6.6) to solve Problem 3.3 ((3.6)–(3.7)) with $a = 0.255$, $b = 0.25$, $\alpha = 10$ in (6.7), and with the initial guess $\lambda_h^{(0)} = 0$ and parameter $\rho = 1$. We plot the figures of the tangential traction $\sigma_{1,\tau}$ and the tangential velocity $u_{1,\tau}$ on the interface Γ, as well as the velocity fields (u_1, u_2) in Fig. 2. The experimental errors are shown in Table



(a) convergence orders (b) iteration errors

Fig. 4. Convergence of numerical solutions of (HVI-NS) under the domain decomposition algorithm.

Table 1
Errors of numerical solutions of (HVI-NS) with interface slip boundary condition (1.6).

h	Ω_1			Ω_2		
	D1UL2	D1UH1	D1PL2	D2UL2	D2UH1	D2PL2
2^{-3}	7.3704e−03	9.9777e−02	1.8587e−01	7.3857e−03	9.8593e−02	1.9202e−01
2^{-4}	1.9469e−03	3.8134e−02	6.5195e−02	1.9429e−03	3.7628e−02	6.6871e−02
2^{-5}	4.7749e−04	1.5663e−02	2.2523e−02	4.7664e−04	1.5454e−02	2.2758e−02
2^{-6}	1.1203e−04	6.7453e−03	7.7155e−03	1.1205e−04	6.6560e−03	7.5848e−03
Order	2.09	1.22	1.55	2.09	1.22	1.59

Table 2
Errors of numerical solutions of (HVI-NS) under the domain decomposition algorithm.

h	Ω_1			Ω_2		
	D1UL2	D1UH1	D1PL2	D2UL2	D2UH1	D2PL2
2^{-3}	7.4826e−03	9.9294e−02	1.8350e−01	7.3857e−03	9.8593e−02	1.8948e−01
2^{-4}	1.9926e−03	3.7981e−02	6.4491e−02	1.9429e−03	3.7628e−02	6.6708e−02
2^{-5}	4.9984e−04	1.5577e−02	2.2206e−02	4.7664e−04	1.5454e−02	2.2747e−02
2^{-6}	1.2255e−04	6.6916e−03	7.5269e−03	1.1205e−04	6.6560e−03	7.5845e−03
Order	2.03	1.22	1.56	2.09	1.22	1.58

1 and Fig. 3(a). We see the $\mathcal{O}(h)$ convergence of $\|u_h - u\|_{H^1(\Omega)}$; while $\mathcal{O}(h^{1.5})$ of the $\|p_h - p\|_Q$ which is commonly observed for the P1b/P1 finite elements in numerical tests. Moreover, we observe the $\mathcal{O}(h^2)$-convergence for $\|u_h - u\|_{L^2(\Omega)}$.

In addition, if we fix the friction function $\omega(t)$ in (6.7) as a positive constant, e.g., $\omega \equiv 0.255$, $\psi(x,\cdot)$ in (1.6b) is convex and the interface condition degenerates into (1.5), then the variational formulation of (1.1) turns into an inequality:

$$\begin{cases} a(u, v-u) + b(v-u, p) + C(u; u, v-u) + \int_\Gamma \omega|[v_\tau]|\,ds - \int_\Gamma \omega|[u_\tau]|\,ds \geq (f, v-u) \quad \forall\, v \in V, \\ \qquad b(u,q) = 0 \quad \forall\, q \in \mathring{Q}. \end{cases} \tag{6.8}$$

This inequality can be solved via the projection iteration [25,27,61]. The convergence behaviors are shown in Fig. 3(b). Figures of the tangential traction, the tangential velocity and the velocity field are omitted, which are similar to that for the case of the hemivariational inequality.

In the following, we examine the applicability of the domain decomposition algorithm. We set the parameter $\theta = 0.35$ and the initial value $(u_{2h}^{(0)}, p_{2h}^{(0)}) = (0,0)$, and carry out the simulation with the same nonconstant $\omega(t)$ as mentioned above. The experimental errors (Table 2) and convergence behavior (Fig. 4(a)) are almost the same as that shown in Table 1 and Fig. 3(a) obtained without using domain decomposition method. Figures of velocity, $\sigma_{1,\tau}$ and $u_{1,\tau}$ are similar to those obtained as aforementioned. We only plot the iteration errors $\|u_h^{(k)} - u_h^{(k-1)}\|$ and $\|p_h^{(k)} - p_h^{(k-1)}\|$ in Fig. 4(b), which decreases exponentially fast and coincides with the theoretical prediction.

7. Conclusions

In this work, the finite element method is applied to solve model problems of two viscous fluids with the nonlinear slip interface condition of friction type. The nonsmooth and nonmonotone property of the slip interface condition leads to a NS/NS hemivariational inequality. Well-posedness of the hemivariational inequality is proved, so is the convergence of the domain decomposition algorithm. Optimal order error estimates are derived for the mixed finite element method with the P1b/P1 element under appropriate solution regularity assumptions and numerical tests are given to illustrate the theoretical result. In a future study, the nonstationary hemivariational inequality and efficient decoupling algorithms will be considered.

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