

On Well-posedness of a Nonstationary Stokes Hemivariational Inequality

Weimin Han¹ and Shengda Zeng²

Dedicated to Professor Zuhair Nashed on the occasion of his 90th birthday

Abstract. This paper is devoted to the well-posedness analysis of a nonstationary Stokes hemivariational inequality for an incompressible fluid flow described by the Stokes equations subject to a nonsmooth boundary condition of friction type described by the Clarke subdifferential. In a recent paper [19], well-posedness of the nonstationary Stokes hemivariational inequality is studied for both the velocity and pressure fields. The solution existence is shown through a limiting procedure based on temporally semi-discrete approximations for both the velocity and pressure fields. In this paper, a refined well-posedness analysis is provided on the nonstationary Stokes hemivariational inequality under more natural assumptions on the problem data. The solution existence is first shown for the velocity field through a limiting procedure based on temporally semi-discrete approximations of a reduced problem and then the pressure field is recovered with the help of an inf-sup property. In this way, assumptions on the source term and the initial velocity needed in [19] are weakened, and a compatibility condition on initial values of the data is dropped. Moreover, several hemivariational inequalities are introduced for the mathematical model and their equivalence is explored.

Keywords. Nonstationary Stokes equations, hemivariational inequality, well-posedness

1 Introduction

Since the pioneering work of Panagiotopoulos ([29]), there has been extensive research on modeling, analysis, numerical solution and applications of hemivariational inequalities and the more general variational-hemivariational inequalities. Several research monographs are available for detailed expositions of mathematical theories and applications of variational-hemivariational inequalities, e.g., just to mention a few recent ones, [3, 14, 26, 27, 31]. In most references in the area, abstract surjectivity results on pseudomonotone operators are needed to investigate the well-posedness of variational-hemivariational inequalities. For an approach without the need of the notion and results of pseudomonotone operators, the reader is referred to the book [14] for an accessible coverage of the mathematical theory

¹Department of Mathematics, University of Iowa, Iowa City, IA 52242-1410, USA. E-mail: weimin-han@uiowa.edu. The work of this author was partially supported by Simons Foundation Collaboration Grants, No. 850737.

²National Center for Applied Mathematics in Chongqing, and School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China. E-mail: zengshengda@163.com. The work of this author was supported by the National Natural Science Foundation of China under Grant No. 12371312, and the Natural Science Foundation of Chongqing under Grant No. CSTB2024NSCQ-JQX0033.

for variational-hemivariational inequalities, based on the techniques developed in [12, 13]. Since there are no analytic solution formulas for variational-hemivariational inequalities from applications, numerical methods are needed to solve the problems. An early comprehensive reference on the finite element method for solving hemivariational inequalities is [20]. There has been substantial progress on numerical analysis of variational-hemivariational inequalities in the recent years, and the reader is referred to the two survey papers [18, 16] and the references therein. See also the survey paper [15] for an elementary level overview of mathematical theory and numerical analysis of stationary variational-hemivariational inequalities for beginners of the area.

Nonstationary Stokes variational inequalities are used to describe incompressible fluid flow problems involving a non-leak nonsmooth monotone slip condition of friction type. They have been studied in a number of papers, e.g., [8, 10, 23]. When a relevant superpotential function in these papers is no longer assumed to be convex, a hemivariational inequality arises. This paper is devoted to the study of a nonstationary Stokes hemivariational inequality, i.e., a hemivariational inequality of the time-dependent Stokes equations for an incompressible fluid flow subject to a non-leak nonsmooth non-monotone slip condition of friction type. In the recent paper [19], the well-posedness of the nonstationary Stokes hemivariational inequality is proved under a compatibility condition on initial values of the data. The solution existence is shown through a limiting procedure based on temporally semi-discrete approximations for both the velocity and pressure fields. In this paper, the solution existence is first shown for the velocity field through a limiting procedure based on temporally semi-discrete approximations of a reduced problem and then the pressure field is recovered with the help of an inf-sup property. In this way, the compatibility condition on initial values of the data needed in [19] is dropped. We note in passing that in [19], numerical analysis is extended to the nonstationary Stokes hemivariational inequality. The Rothe method has been used in studying a variety of time-dependent problems, see for example [30, 22, 2, 34]. Numerical methods have been studied in a number of papers for solving stationary mixed hemivariational inequalities in fluid mechanics, e.g., [9, 24] for a stationary Stokes hemivariational inequality, [17] for a stationary Navier-Stokes hemivariational inequality.

The organization of this paper is as follows. In Section 2 we present some definitions and auxiliary material. Several formulations of the problem are introduced and their equivalence is explored in Section 3; assumptions on the data are also stated in the section. In Section 4, reduced formulations of the problem for the velocity variable are presented in which the pressure variable is eliminated. In Section 5, the existence and uniqueness of the velocity variable are investigated for the reduced formulations. In Section 6, the pressure component of the solution for the problem is recovered. In Section 7, Lipschitz continuous dependence of the velocity solution on the source function and the initial value is shown. In Section 8, we provide a concluding remark on the main contribution of this paper.

2 Preliminaries

Let V be a Banach space with the norm $\|\cdot\|_V$. Denote by V^* its topological dual, and by $\langle \cdot, \cdot \rangle_{V^* \times V}$ the duality pairing between V^* and V . Weak convergence will be indicated by the symbol $\rightharpoonup$. The symbol 2^{V^*} represents the set of all subsets of V^* .

We first recall the notions of the generalized directional derivative and the generalized subdifferential of a locally Lipschitz function. These notions are needed in expressing hemivariational inequalities. Let

$\Psi: V \rightarrow \mathbb{R}$ be a locally Lipschitz function. Then the generalized directional derivative (in the sense of Clarke) of Ψ at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u; v) = \limsup_{w \rightarrow u, \lambda \rightarrow 0} \frac{\Psi(w + \lambda v) - \Psi(w)}{\lambda},$$

and the generalized subdifferential (in the sense of Clarke) of Ψ at u is defined by

$$\partial\Psi(u) = \{\xi \in V^* \mid \Psi^0(u; v) \geq \langle \xi, v \rangle_{V^* \times V} \ \forall v \in V\}.$$

For properties and detailed discussions of the generalized directional derivative and the generalized subdifferential of a locally Lipschitz function, the reader is referred to [5, 6] or [26, Section 3.2].

Given a finite time interval $I = (0, T)$, for $1 \leq p \leq \infty$, we will use the Bochner space $L^p(I; V)$. For $1 \leq q < \infty$, we denote by $BV^q(I; V)$ the space of functions of bounded total variation on I of exponent q . This space consists of functions $v: \bar{I} \rightarrow V$ such that

$$\|v\|_{BV^q(I; V)}^q = \sup_{\pi \in \mathcal{P}} \sum_{i=1}^{n(\pi)} \|v(a_i) - v(a_{i-1})\|_V^q < \infty,$$

where $\mathcal{P}$ is the set of all finite partitions π of $\bar{I} = [0, T]$: $0 = a_0 < a_1 < \dots < a_{n(\pi)} = T$, $n(\pi)$ being a positive integer.

Given Banach spaces V_1 and V_2 such that $V_1 \subset V_2$, define

$$M^{p,q}(I; V_1, V_2) = L^p(I; V_1) \cap BV^q(I; V_2)$$

which is a Banach space for $1 \leq p, q < \infty$ with the norm given by $\|\cdot\|_{L^p(I; V_1)} + \|\cdot\|_{BV^q(I; V_2)}$.

The following important result is proved in [22].

Theorem 2.1 *Let $1 \leq p, q < \infty$. Let $V_1 \subset V_2 \subset V_3$ be real Banach spaces such that V_1 is reflexive, the embedding $V_1 \subset V_2$ is compact and the embedding $V_2 \subset V_3$ is continuous. Then a bounded subset of $M^{p,q}(I; V_1, V_3)$ is relatively compact in $L^p(I; V_2)$.*

We will apply the modified Cauchy-Schwarz inequality on several occasions in Section 5, with a proper choice of $\epsilon > 0$,

$$ab \leq \epsilon a^2 + c_\epsilon b^2 \quad \forall a, b \in \mathbb{R}, \epsilon > 0, c_\epsilon = 1/(4\epsilon). \quad (2.1)$$

The following version of Aubin-Cellina convergence theorem ([1, p. 60]) will be used.

Theorem 2.2 *Let $F: X \rightarrow 2^Y$ be a set-valued function between two Banach spaces X and Y . Assume F is upper semicontinuous from X into Y_w and the values of F are nonempty, closed and convex subsets of Y . Let $x_n: (0, T) \rightarrow X$, $y_n: (0, T) \rightarrow Y$, $n \in \mathbb{N}$, be measurable functions such that x_n converges almost everywhere on $(0, T)$ to a function $x: (0, T) \rightarrow X$ and y_n converges weakly in $L^1(0, T; Y)$ to $y: (0, T) \rightarrow Y$. If $y_n(t) \in F(x_n(t))$ for all $n \in \mathbb{N}$ and almost all $t \in (0, T)$, then $y(t) \in F(x(t))$ for a.e. $t \in (0, T)$.*

3 Mathematical model

Let $d = 2$ or 3 be the dimension of the spatial domain for the problem under consideration. Denote the Euclidean norm in $\mathbb{R}^d$ by $|\cdot|$. Denote by $\mathbb{S}^d$ the space of symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of symmetric matrices of order d .

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Its boundary Γ is split into two non-trivial non-overlapping parts: $\Gamma = \Gamma_D \cup \Gamma_S$ with Γ_D relatively closed and Γ_S relatively open. Let $[0, T]$ be the time interval for a given $T > 0$. Since the boundary is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu} = (\nu_1, \dots, \nu_d)^T$ exists a.e. on Γ . For a vector $\mathbf{v}$, we denote by $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$ its normal component and $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$ its tangential component on Γ .

We consider the motion of an incompressible fluid flow in Ω , modeled by the Stokes equations. The unknown variables are the velocity field $\mathbf{u}$ and the pressure field p of the fluid. Denote by $\mu > 0$ the viscosity of the fluid, by $\mathbf{f}$ the density of the volume force applied to the fluid, and by $\mathbf{u}_0: \Omega \rightarrow \mathbb{R}^d$ be the initial velocity. The stress tensor is $\boldsymbol{\sigma} = 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) - p \mathbf{I}$, where $\boldsymbol{\varepsilon}(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$ and $\mathbf{I}$ is the identity tensor. The normal and tangential components of the stress on the boundary are $\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. We impose the homogeneous Dirichlet boundary condition on Γ_D and a non-smooth, possibly non-monotone, friction condition with a potential function ψ on Γ_S . The potential function ψ is assumed to be locally Lipschitz continuous. The classical pointwise formulation of the problem is the following:

$$\mathbf{u}' - 2\mu \operatorname{div} \boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } (0, T) \times \Omega, \quad (3.1)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } (0, T) \times \Omega, \quad (3.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } (0, T) \times \Gamma_D, \quad (3.3)$$

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi(\mathbf{u}_\tau) \quad \text{on } (0, T) \times \Gamma_S, \quad (3.4)$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega. \quad (3.5)$$

In (3.4), $\partial\psi$ denotes the generalized subdifferential of ψ in the sense of Clarke.

The problem (3.1)–(3.5) will be studied in weak forms. For this purpose, we introduce the function spaces

$$\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^d \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D, v_\nu = 0 \text{ on } \Gamma_S\}, \quad (3.6)$$

$$Q = L_0^2(\Omega) = \{q \in L^2(\Omega) \mid (q, 1)_{L^2(\Omega)} = 0\}, \quad (3.7)$$

$$\mathbf{H} = L^2(\Omega)^d, \quad (3.8)$$

where $(\cdot, \cdot)_{L^2(\Omega)}$ is the canonical inner product in $L^2(\Omega)$, and

$$\mathcal{V} = L^2(0, T; \mathbf{V}), \quad (3.9)$$

$$\mathcal{W} = \{\mathbf{v} \in \mathcal{V} \mid \mathbf{v}' \in \mathcal{V}^*\}, \quad (3.10)$$

$$\mathcal{Q} = L^2(0, T; Q), \quad (3.11)$$

$$\mathcal{H} = L^2(0, T; \mathbf{H}). \quad (3.12)$$

Since $|\Gamma_D| > 0$, Korn's inequality holds: there exists a constant $c > 0$, depending only on Ω and Γ_D , such that

$$\|\mathbf{v}\|_{H^1(\Omega)^d} \leq c \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{L^2(\Omega)^{d \times d}} \quad \forall \mathbf{v} \in \mathbf{V}.$$

See e.g. [28, p. 79] for a proof of this inequality. Consequently, $\mathbf{V}$ is a Hilbert space with the inner product

$$(\mathbf{u}, \mathbf{v})_{\mathbf{V}} := \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V},$$

and the induced norm $\|\mathbf{v}\|_{\mathbf{V}} = (\mathbf{v}, \mathbf{v})_{\mathbf{V}}^{1/2}$ is equivalent to the standard $H^1(\Omega)^d$ -norm over $\mathbf{V}$. The spaces Q , $\mathbf{H}$, $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Q}$ and $\mathcal{H}$ are all Hilbert spaces with their standard inner products. We will also need the following subspace of $\mathbf{V}$:

$$\mathbf{V}_0 := H_0^1(\Omega)^d. \quad (3.13)$$

We have the trace inequality

$$\|\mathbf{v}_{\tau}\|_{L^2(\Gamma_S)^d} \leq \lambda_{\tau}^{-1/2} \|\mathbf{v}\|_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3.14)$$

where $\lambda_{\tau} > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\mathbf{u} \in \mathbf{V}, \quad (\mathbf{u}, \mathbf{v})_{\mathbf{V}} = \lambda (\mathbf{u}_{\tau}, \mathbf{v}_{\tau})_{L^2(\Gamma_S)^d} \quad \forall \mathbf{v} \in \mathbf{V}.$$

Define the bilinear forms

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}, \quad (3.15)$$

$$b(\mathbf{v}, q) = \int_{\Omega} q \operatorname{div} \mathbf{v} \, dx \quad \forall \mathbf{v} \in \mathbf{V}, q \in Q. \quad (3.16)$$

Obviously, the bilinear form $a(\cdot, \cdot) : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ is symmetric, continuous, and $\mathbf{V}$ -elliptic:

$$\begin{aligned} |a(\mathbf{u}, \mathbf{v})| &\leq 2\mu \|\mathbf{u}\|_{\mathbf{V}} \|\mathbf{v}\|_{\mathbf{V}} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}, \\ a(\mathbf{v}, \mathbf{v}) &= 2\mu \|\mathbf{v}\|_{\mathbf{V}}^2 \quad \forall \mathbf{v} \in \mathbf{V}. \end{aligned}$$

The bilinear form $b(\cdot, \cdot) : \mathbf{V} \times Q \rightarrow \mathbb{R}$ is continuous: for a constant $c_b > 0$,

$$|b(\mathbf{v}, q)| \leq c_b \|\mathbf{v}\|_{\mathbf{V}} \|q\|_Q \quad \forall \mathbf{v} \in \mathbf{V}, q \in Q.$$

Moreover, we have the inf-sup property ([11, 21, 32]): for a positive constant $\alpha_b > 0$,

$$\sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}_0} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{V}}} \geq \alpha_b \|q\|_Q \quad \forall q \in Q. \quad (3.17)$$

Define operators $A : \mathbf{V} \rightarrow \mathbf{V}^*$ and $B : \mathbf{V} \rightarrow Q^*$ by

$$\begin{aligned} \langle A\mathbf{u}, \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} &= a(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}, \\ \langle B\mathbf{v}, q \rangle_{Q^* \times Q} &= b(\mathbf{v}, q) \quad \forall \mathbf{v} \in \mathbf{V}, q \in Q. \end{aligned}$$

Then $A \in \mathcal{L}(\mathbf{V}; \mathbf{V}^*)$ is self-adjoint and

$$\langle A\mathbf{v}, \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} = 2\mu \|\mathbf{v}\|_{\mathbf{V}}^2 \quad \forall \mathbf{v} \in \mathbf{V}.$$

Concerning the data, we assume

$$\mathbf{f} \in \mathcal{V}^*, \quad \mathbf{u}_0 \in \mathbf{H}, \quad (3.18)$$

and

$H(\psi)$: $\psi: \Gamma_S \times (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}$ is such that

- (i) $\psi(\cdot, \cdot, \boldsymbol{\xi})$ is measurable on $\Gamma_S \times (0, T)$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and there exists $\boldsymbol{\xi}_0 \in L^2(\Gamma_S)^d$ such that $\psi(\cdot, t, \boldsymbol{\xi}_0(\cdot)) \in L^1(\Gamma_S)$ for a.e. $t \in (0, T)$;
- (ii) $\psi(\mathbf{x}, t, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $(\mathbf{x}, t) \in \Gamma_S \times (0, T)$;
- (iii) $|\boldsymbol{\eta}| \leq c_0(1 + |\boldsymbol{\xi}|)$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, $\boldsymbol{\eta} \in \partial\psi(\mathbf{x}, t, \boldsymbol{\xi})$, a.e. $(\mathbf{x}, t) \in \Gamma_S \times (0, T)$ with $c_0 > 0$;
- (iv) $(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq -\alpha_\psi |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2$ for all $\boldsymbol{\eta}_i \in \partial\psi(\mathbf{x}, t, \boldsymbol{\xi}_i)$, $\boldsymbol{\xi}_i \in \mathbb{R}^d$, $i = 1, 2$, a.e. $(\mathbf{x}, t) \in \Gamma_S \times (0, T)$ with a constant $\alpha_\psi > 0$.

For simplicity in writing, we will use the notation $I_{\Gamma_S}(z)$ for the integral of a function z over Γ_S .

The following result holds ([19, Lemma 3.1]).

Lemma 3.1 *Assume $H(\psi)$. Then,*

$$I_{\Gamma_S}(\psi^0(\mathbf{v}_{1,\tau}; \mathbf{v}_{2,\tau} - \mathbf{v}_{1,\tau}) + \psi^0(\mathbf{v}_{2,\tau}; \mathbf{v}_{1,\tau} - \mathbf{v}_{2,\tau})) \leq \alpha_\psi \lambda_\tau^{-1} \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{V}}^2 \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}.$$

Through a standard procedure, we get the following weak formulation of the problem (3.1)–(3.5).

Problem 3.2 *Find $\mathbf{u} \in \mathcal{W}$ and $p \in \mathcal{Q}$ such that for a.e. $t \in (0, T)$,*

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p(t)) + I_{\Gamma_S}(\psi^0(\mathbf{u}_\tau(t); \mathbf{v}_\tau)) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3.19)$$

$$b(\mathbf{u}(t), q) = 0 \quad \forall q \in \mathcal{Q}, \quad (3.20)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (3.21)$$

Note that since $\mathcal{W} \subset C([0, T]; \mathbf{H})$, the initial condition (3.21) is well-defined.

By allowing $\mathbf{v}$ and q to depend on t and integrating (3.19) and (3.20) with respect to t , we obtain another weak formulation of the problem (3.1)–(3.5).

Problem 3.3 *Find $\mathbf{u} \in \mathcal{W}$ and $p \in \mathcal{Q}$ such that*

$$\begin{aligned} & \int_0^T [\langle \mathbf{u}'(t), \mathbf{v}(t) \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}(t)) - b(\mathbf{v}(t), p(t)) + I_{\Gamma_S}(\psi^0(\mathbf{u}_\tau(t); \mathbf{v}_\tau(t)))] dt \\ & \geq \int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{\mathbf{V}^* \times \mathbf{V}} dt \quad \forall \mathbf{v} \in \mathcal{V}, \end{aligned} \quad (3.22)$$

$$\int_0^T b(\mathbf{u}(t), q(t)) dt = 0 \quad \forall q \in \mathcal{Q}, \quad (3.23)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (3.24)$$

It turns out that the above two problems are equivalent in the sense that a pair of functions $(\mathbf{u}, p) \in \mathcal{W} \times \mathcal{Q}$ is a solution of one problem if and only if it is a solution of the other ([19, Proposition 3.4]).

Proposition 3.4 *Problem 3.2 and Problem 3.3 are equivalent.*

To proceed further, we need function spaces on Γ_S :

$$\mathbf{S} = L^2(\Gamma_S)^d, \quad (3.25)$$

$$\mathcal{S} = L^2(0, T; \mathbf{S}). \quad (3.26)$$

Define a functional

$$\Psi(\mathbf{z}) = I_{\Gamma_S}(\psi(\mathbf{z})), \quad \mathbf{z} \in \mathbf{S}. \quad (3.27)$$

Denote by $\gamma: \mathbf{v} \mapsto \mathbf{v}_\tau$ a trace operator over $\mathbf{V}$. Note that $\gamma \in \mathcal{L}(\mathbf{V}; \mathbf{S})$ is compact. Since (cf. [26, Theorem 3.47 (iv)])

$$\Psi^0(\gamma \mathbf{v}_1; \gamma \mathbf{v}_2) \leq I_{\Gamma_S}(\psi^0(\gamma \mathbf{v}_1; \gamma \mathbf{v}_2)) \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V},$$

we deduce from Lemma 3.1 that

$$\Psi^0(\gamma \mathbf{v}_1; \gamma \mathbf{v}_2 - \gamma \mathbf{v}_1) + \Psi^0(\gamma \mathbf{v}_2; \gamma \mathbf{v}_1 - \gamma \mathbf{v}_2) \leq \alpha_\psi \lambda_\tau^{-1} \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{V}}^2 \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}. \quad (3.28)$$

Now introduce an auxiliary problem:

Problem 3.5 *Find $\mathbf{u} \in \mathcal{W}$ and $p \in \mathcal{Q}$ such that for a.e. $t \in (0, T)$,*

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p(t)) + \Psi^0(\gamma \mathbf{u}(t); \gamma \mathbf{v}) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3.29)$$

$$b(\mathbf{u}(t), q) = 0 \quad \forall q \in \mathcal{Q}, \quad (3.30)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (3.31)$$

Similar to Problem 3.3, we have a global version of Problem 3.5.

Problem 3.6 *Find $\mathbf{u} \in \mathcal{W}$ and $p \in \mathcal{Q}$ such that*

$$\begin{aligned} & \int_0^T [\langle \mathbf{u}'(t), \mathbf{v}(t) \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}(t)) - b(\mathbf{v}(t), p(t)) + \Psi^0(\gamma \mathbf{u}(t); \gamma \mathbf{v}(t))] dt \\ & \geq \int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{\mathbf{V}^* \times \mathbf{V}} dt \quad \forall \mathbf{v} \in \mathcal{V}, \end{aligned} \quad (3.32)$$

$$\int_0^T b(\mathbf{u}(t), q(t)) dt = 0 \quad \forall q \in \mathcal{Q}, \quad (3.33)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (3.34)$$

By adapting the proof of Proposition 3.4 in [19], we can show the next result.

Proposition 3.7 *Problem 3.5 and Problem 3.6 are equivalent.*

A relation between the two pairs of problems, Problems 3.2 and 3.3 vs. Problems 3.5 and 3.6, is the following.

Proposition 3.8 *Any solution of Problem 3.5 is a solution of Problem 3.2. If Problem 3.5 has a solution and a solution of Problem 3.2 is unique, then both problems admit a unique solution and the solution is the same to the two problems.*

Proof. The first statement follows from the property

$$\Psi^0(\gamma \mathbf{u}(t); \gamma \mathbf{v}) \leq I_{\Gamma_S}(\psi^0(\gamma \mathbf{u}(t); \gamma \mathbf{v}(t))).$$

The second statement is obvious. ■

Starting with Problem 3.5, we consider the next inclusion problem.

Problem 3.9 *Find $\mathbf{u} \in \mathcal{W}$ and $p \in \mathcal{Q}$ such that for a.e. $t \in (0, T)$,*

$$\mathbf{u}'(t) + A\mathbf{u}(t) - B^*p(t) + \gamma^* \partial \Psi(\gamma \mathbf{u}(t)) \ni \mathbf{f}(t), \quad (3.35)$$

$$B\mathbf{u}(t) = 0, \quad (3.36)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (3.37)$$

This problem can be equivalently rewritten as the following.

Problem 3.10 *Find $\mathbf{u} \in \mathcal{W}$, $p \in \mathcal{Q}$ and $\boldsymbol{\xi} \in \mathcal{S}^*$ such that for a.e. $t \in (0, T)$,*

$$\mathbf{u}'(t) + A\mathbf{u}(t) - B^*p(t) + \gamma^* \boldsymbol{\xi}(t) = \mathbf{f}(t), \quad (3.38)$$

$$\boldsymbol{\xi}(t) \in \partial \Psi(\gamma \mathbf{u}(t)), \quad (3.39)$$

$$B\mathbf{u}(t) = 0, \quad (3.40)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (3.41)$$

We have the next result.

Proposition 3.11 *If $(\mathbf{u}, p) \in \mathcal{W} \times \mathcal{Q}$ is a solution of Problem 3.9, or if $(\mathbf{u}, p, \boldsymbol{\xi}) \in \mathcal{W} \times \mathcal{Q} \times \mathcal{S}^*$ is a solution of Problem 3.10, then $(\mathbf{u}, p) \in \mathcal{W} \times \mathcal{Q}$ is a solution of Problem 3.5.*

Proof. Note that for all $\boldsymbol{\xi}(t) \in \partial \Psi(\gamma \mathbf{u}(t))$, we have

$$\langle \gamma^* \boldsymbol{\xi}(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} \leq \Psi^0(\gamma \mathbf{u}(t); \gamma \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}.$$

Hence, (3.35) implies (3.29). ■

4 Reduced formulations for the velocity field

In this section, we consider reduced forms of the problems from last section by eliminating the pressure variable p . For this purpose, we introduce the following subspaces of $\mathbf{V}$, $\mathcal{V}$ and $\mathcal{W}$:

$$\mathbf{V}_{\text{div}} = \{\mathbf{v} \in \mathbf{V} \mid \text{div} \mathbf{v} = 0 \text{ in } \Omega\}, \quad (4.1)$$

$$\mathcal{V}_{\text{div}} = L^2(0, T; \mathbf{V}_{\text{div}}), \quad (4.2)$$

$$\mathcal{W}_{\text{div}} = \{\mathbf{v} \in \mathcal{V}_{\text{div}} \mid \mathbf{v}' \in \mathcal{V}_{\text{div}}^*\}. \quad (4.3)$$

Endowed by the inner products and norms in $\mathbf{V}$, $\mathcal{V}$ and $\mathcal{W}$, these subspaces are themselves Hilbert spaces. Since $\mathbf{V}_{\text{div}} \subset \mathbf{V}$, we have $\mathbf{V}^* \subset \mathbf{V}_{\text{div}}^*$ and the duality pairing $\langle \cdot, \cdot \rangle_{\mathbf{V}^* \times \mathbf{V}}$ is automatically extended to $\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}$ with the property

$$\langle \mathbf{g}, \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} = \langle \mathbf{g}, \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} \quad \forall \mathbf{g} \in \mathbf{V}^*, \mathbf{v} \in \mathbf{V}_{\text{div}}. \quad (4.4)$$

A similar comment holds regarding the duality pairing on $\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}$. For $\mathbf{g} \in \mathbf{V}^*$ and $\mathbf{v} \in \mathbf{V}_{\text{div}}$, since $\|\mathbf{g}\|_{\mathbf{V}_{\text{div}}^*} \leq \|\mathbf{g}\|_{\mathbf{V}^*}$ and $\|\mathbf{v}\|_{\mathbf{V}_{\text{div}}} = \|\mathbf{v}\|_{\mathbf{V}}$, we have the inequality

$$|\langle \mathbf{g}, \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}}| \leq \|\mathbf{g}\|_{\mathbf{V}^*} \|\mathbf{v}\|_{\mathbf{V}}.$$

Denote by γ_{div} the restriction of $\gamma \in \mathcal{L}(\mathbf{V}; \mathbf{S})$ on $\mathbf{V}_{\text{div}}$: $\gamma_{\text{div}} \in \mathcal{L}(\mathbf{V}_{\text{div}}; \mathbf{S})$. The reduced forms of Problem 3.5 and Problem 3.6 are the following.

Problem 4.1 Find $\mathbf{u} \in \mathcal{W}_{\text{div}}$ such that for a.e. $t \in (0, T)$,

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} + a(\mathbf{u}(t), \mathbf{v}) + \Psi^0(\gamma_{\text{div}} \mathbf{u}(t); \gamma_{\text{div}} \mathbf{v}) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}}, \quad (4.5)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (4.6)$$

Problem 4.2 Find $\mathbf{u} \in \mathcal{W}_{\text{div}}$ such that

$$\begin{aligned} & \int_0^T [\langle \mathbf{u}'(t), \mathbf{v}(t) \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} + a(\mathbf{u}(t), \mathbf{v}(t)) + \Psi^0(\gamma_{\text{div}} \mathbf{u}(t); \gamma_{\text{div}} \mathbf{v}(t))] dt \\ & \geq \int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} dt \quad \forall \mathbf{v} \in \mathcal{V}_{\text{div}} \end{aligned} \quad (4.7)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (4.8)$$

Similar to Proposition 3.7, we can show that Problem 4.1 and Problem 4.2 are equivalent.

The counterpart of Problem 3.9 is

Problem 4.3 Find $\mathbf{u} \in \mathcal{W}_{\text{div}}$ such that for a.e. $t \in (0, T)$,

$$\mathbf{u}'(t) + A\mathbf{u}(t) + \gamma_{\text{div}}^* \partial \Psi(\gamma_{\text{div}} \mathbf{u}(t)) \ni \mathbf{f}(t), \quad (4.9)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (4.10)$$

This problem is interpreted as

Problem 4.4 Find $\mathbf{u} \in \mathcal{W}_{\text{div}}$ and $\boldsymbol{\xi} \in \mathcal{S}^*$ such that for a.e. $t \in (0, T)$,

$$\mathbf{u}'(t) + A\mathbf{u}(t) + \gamma_{\text{div}}^* \boldsymbol{\xi}(t) = \mathbf{f}(t) \quad \text{in } \mathbf{V}_{\text{div}}^*, \quad (4.11)$$

$$\boldsymbol{\xi}(t) \in \partial\Psi(\gamma_{\text{div}} \mathbf{u}(t)), \quad (4.12)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (4.13)$$

Similar to Proposition 3.11, we claim that if $(\mathbf{u}, \boldsymbol{\xi}) \in \mathcal{W}_{\text{div}} \times \mathcal{S}^*$ is a solution of Problem 4.4, then $\mathbf{u}$ is a solution of Problem 4.1.

5 Existence and uniqueness for the velocity field

In this section, we investigate the existence and uniqueness of a solution to Problem 4.4. The main effort is on the existence. For this purpose, we consider a temporally semi-discrete approximation of Problem 4.4 based on the backward Euler difference for the time derivative. In the literature, such an approximation is known as the Rothe method. For a fixed positive integer $N \in \mathbb{N}$, define the time step-size $k = T/N$. Denote the nodes $t_n = nk$, $0 \leq n \leq N$. Introduce the piecewise constant interpolant of $\mathbf{f}$ by

$$\mathbf{f}_n^k = \frac{1}{k} \int_{t_{n-1}}^{t_n} \mathbf{f}(t) dt, \quad 1 \leq n \leq N.$$

It is easy to verify that

$$\|\mathbf{f}_n^k\|_{\mathbf{V}^*}^2 \leq \frac{1}{k} \int_{t_{n-1}}^{t_n} \|\mathbf{f}(t)\|_{\mathbf{V}^*}^2 dt, \quad 1 \leq n \leq N,$$

and consequently,

$$k \sum_{j=1}^n \|\mathbf{f}_j^k\|_{\mathbf{V}^*}^2 \leq \int_0^{t_n} \|\mathbf{f}(t)\|_{\mathbf{V}^*}^2 dt \leq \|\mathbf{f}\|_{\mathbf{V}^*}^2, \quad 1 \leq n \leq N. \quad (5.1)$$

The initial value $\mathbf{u}_0 \in \mathbf{H}$ can be approximated by a sequence $\{\mathbf{u}_{k,0}\} \subset \mathbf{V}$ such that $\mathbf{u}_{k,0} \rightarrow \mathbf{u}_0$ in $\mathbf{H}$ and $\|\mathbf{u}_{k,0}\|_{\mathbf{V}} \leq ck^{-1/2}$ for some constant $c > 0$, cf. [30, Theorem 8.9].

Then, the semi-discrete approximation of Problem 4.4 is the following.

Problem 5.1 Find $\mathbf{u}^k := \{\mathbf{u}_n^k\}_{n=0}^N \subset \mathbf{V}_{\text{div}}$ and $\boldsymbol{\xi}^k := \{\boldsymbol{\xi}_n^k\}_{n=1}^N \subset \mathcal{S}^*$ such that for $n = 1, \dots, N$,

$$\frac{1}{k}(\mathbf{u}_n^k - \mathbf{u}_{n-1}^k) + A\mathbf{u}_n^k + \gamma_{\text{div}}^* \boldsymbol{\xi}_n^k = \mathbf{f}_n^k, \quad (5.2)$$

$$\boldsymbol{\xi}_n^k \in \partial\Psi(\gamma_{\text{div}} \mathbf{u}_n^k), \quad (5.3)$$

and

$$\mathbf{u}_0^k = \mathbf{u}_{k,0}. \quad (5.4)$$

We assume

$$m := 2\mu - \alpha_\psi \lambda_\tau^{-1} > 0. \quad (5.5)$$

This is called a smallness condition in the literature since the condition sets an upper bound on the value α_ψ which reflects the strength of the non-convexity of ψ .

The problem (5.2)–(5.3) is equivalent to

$$\frac{1}{k} \langle \mathbf{u}_n^k - \mathbf{u}_{n-1}^k, \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} + a(\mathbf{u}_n^k, \mathbf{v}) + \Psi^0(\gamma_{\text{div}} \mathbf{u}_n^k; \gamma_{\text{div}} \mathbf{v}) \geq \langle \mathbf{f}_n^k, \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}}. \quad (5.6)$$

The equivalence is due to the facts that

$$\Psi^0(\gamma_{\text{div}} \mathbf{u}_n^k; \gamma_{\text{div}} \mathbf{v}) \geq \langle \boldsymbol{\xi}_n^k, \gamma_{\text{div}} \mathbf{v} \rangle_{\mathbf{S}^* \times \mathbf{S}} \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}}$$

for any $\boldsymbol{\xi}_n^k \in \partial \Psi(\gamma_{\text{div}} \mathbf{u}_n^k)$, and given $\mathbf{z} \in \mathbf{S}$,

$$\Psi^0(\gamma_{\text{div}} \mathbf{u}_n^k; \mathbf{z}) = \max\{\langle \boldsymbol{\zeta}, \mathbf{z} \rangle_{\mathbf{S}^* \times \mathbf{S}} \mid \boldsymbol{\zeta} \in \partial \Psi(\gamma_{\text{div}} \mathbf{u}_n^k)\} = \langle \boldsymbol{\xi}_n^k, \mathbf{z} \rangle_{\mathbf{S}^* \times \mathbf{S}}$$

for one element $\boldsymbol{\xi}_n^k \in \partial \Psi(\gamma_{\text{div}} \mathbf{u}_n^k)$. Then,

$$\Psi^0(\gamma_{\text{div}} \mathbf{u}_n^k; \gamma_{\text{div}} \mathbf{v}) = \langle \boldsymbol{\xi}_n^k, \gamma_{\text{div}} \mathbf{v} \rangle_{\mathbf{S}^* \times \mathbf{S}} \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}}.$$

Rewrite (5.6) as

$$\langle \mathbf{u}_n^k, \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} + k a(\mathbf{u}_n^k, \mathbf{v}) + k \Psi^0(\gamma_{\text{div}} \mathbf{u}_n^k; \gamma_{\text{div}} \mathbf{v}) \geq \langle \tilde{\mathbf{f}}_n^k, \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}}. \quad (5.7)$$

where

$$\tilde{\mathbf{f}}_n^k = k \mathbf{f}_n^k + \mathbf{u}_{n-1}^k.$$

By [14, Theorem 5.30], if

$$\alpha_\psi \lambda_\tau^{-1} k < 1 + 2\mu k, \quad (5.8)$$

then (5.7) has a unique solution $\mathbf{u}_n^k \in \mathbf{V}_{\text{div}}$, which is the minimizer of the functional

$$\frac{1}{2} \|\mathbf{v}\|_{\mathbf{H}}^2 + k\mu \|\mathbf{v}\|_{\mathbf{V}}^2 + k \Psi(\gamma_{\text{div}} \mathbf{v}) - \langle \tilde{\mathbf{f}}_n^k, \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}}$$

over $\mathbf{V}_{\text{div}}$.

The condition (5.8) is guaranteed by the assumption (5.5); alternatively, it is guaranteed if $k < \lambda_\tau / \alpha_\psi$ without the assumption (5.5).

Summarizing, we have the following result on Problem 5.1.

Theorem 5.2 *Assume (3.18), $H(\psi)$ and (5.5). Then, Problem 5.1 has a solution $(\mathbf{u}^k, \boldsymbol{\xi}^k) \subset \mathbf{V}_{\text{div}} \times \mathbf{S}^*$ and $\mathbf{u}^k$ is unique.*

Remark 5.3 *On uniqueness of $\boldsymbol{\xi}^k$: The equation (5.2) implies that $\gamma_{\text{div}}^* \boldsymbol{\xi}_n^k$ is unique, i.e., the value $\langle \boldsymbol{\xi}_n^k, \gamma_{\text{div}} \mathbf{v} \rangle_{\mathbf{S}^* \times \mathbf{S}} = \langle \boldsymbol{\xi}_n^k, \mathbf{v}_\tau \rangle_{\mathbf{S}^* \times \mathbf{S}}$ is uniquely defined with respect to $\mathbf{v}_\tau$ for $\mathbf{v} \in \mathbf{V}_{\text{div}}$. In other words, the normal component of $\boldsymbol{\xi}_n^k$ does not matter in (5.2) and in terms of the tangential component of $\boldsymbol{\xi}_n^k$, it is unique. Alternatively, we can choose $\boldsymbol{\xi}_n^k$ such that its normal component vanishes, or equivalently, choose $\boldsymbol{\xi}_n^k$ to be the minimal-norm element of the second solution component among all solutions $(\mathbf{u}_n^k, \boldsymbol{\xi}_n^k)$ of the problem (5.2)–(5.3).*

Then, we turn to a derivation of upper bounds on the semi-discrete solutions.

Lemma 5.4 Assume (3.18), $H(\psi)$ and (5.5). Then, there is a constant $c > 0$ such that

$$\max_{0 \leq n \leq N} \|\mathbf{u}_n^k\|_{\mathbf{H}} \leq c, \quad (5.9)$$

$$\sum_{n=1}^N \|\mathbf{u}_n^k - \mathbf{u}_{n-1}^k\|_{\mathbf{H}}^2 \leq c, \quad (5.10)$$

$$k \sum_{n=1}^N \|\mathbf{u}_n^k\|_{\mathbf{V}}^2 \leq c, \quad (5.11)$$

$$\|\boldsymbol{\xi}_n^k\|_{\mathbf{S}^*} \leq c (1 + \|\mathbf{u}_n^k\|_{\mathbf{V}}). \quad (5.12)$$

Proof. Apply (5.2) to $\mathbf{v} = \mathbf{u}_n^k$:

$$\frac{1}{k} (\mathbf{u}_n^k - \mathbf{u}_{n-1}^k, \mathbf{u}_n^k)_{\mathbf{H}} + a(\mathbf{u}_n^k, \mathbf{u}_n^k) + \langle \boldsymbol{\xi}_n^k, \gamma_{\text{div}} \mathbf{u}_n^k \rangle_{\mathbf{S}^* \times \mathbf{S}} = \langle \mathbf{f}_n^k, \mathbf{u}_n^k \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}}. \quad (5.13)$$

For the first term on the left hand side,

$$(\mathbf{u}_n^k - \mathbf{u}_{n-1}^k, \mathbf{u}_n^k)_{\mathbf{H}} = \|\mathbf{u}_n^k\|_{\mathbf{H}}^2 - \|\mathbf{u}_{n-1}^k\|_{\mathbf{H}}^2 + \|\mathbf{u}_n^k - \mathbf{u}_{n-1}^k\|_{\mathbf{H}}^2.$$

For the third term on the left hand side, using (5.3), (3.28) and $H(\psi)$,

$$\begin{aligned} \langle \boldsymbol{\xi}_n^k, \gamma_{\text{div}} \mathbf{u}_n^k \rangle_{\mathbf{S}^* \times \mathbf{S}} &= -\langle \boldsymbol{\xi}_n^k, -\gamma_{\text{div}} \mathbf{u}_n^k \rangle_{\mathbf{S}^* \times \mathbf{S}} \\ &\geq -\Psi^0(\gamma_{\text{div}} \mathbf{u}_n^k; -\gamma_{\text{div}} \mathbf{u}_n^k) \\ &= -[\Psi^0(\gamma_{\text{div}} \mathbf{u}_n^k; -\gamma_{\text{div}} \mathbf{u}_n^k) + \Psi^0(\mathbf{0}; \gamma_{\text{div}} \mathbf{u}_n^k)] + \Psi^0(\mathbf{0}; \gamma_{\text{div}} \mathbf{u}_n^k) \\ &\geq -\alpha_\psi \lambda_\tau^{-1} \|\mathbf{u}_n^k\|_{\mathbf{V}}^2 - c_1 - c_2 \|\mathbf{u}_n^k\|_{\mathbf{V}}. \end{aligned}$$

Then apply the modified Cauchy-Schwarz inequality (2.1) to bound the term $c_2 \|\mathbf{u}_n^k\|_{\mathbf{V}}$ to find that for any $\epsilon > 0$, there is an ϵ -dependent constant c_3 such that

$$\langle \boldsymbol{\xi}_n^k, \gamma_{\text{div}} \mathbf{u}_n^k \rangle_{\mathbf{S}^* \times \mathbf{S}} \geq -(\alpha_\psi \lambda_\tau^{-1} + \epsilon) \|\mathbf{u}_n^k\|_{\mathbf{V}}^2 - c_3.$$

For the right hand side of (5.13),

$$\langle \mathbf{f}_n^k, \mathbf{u}_n^k \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} \leq \|\mathbf{f}_n^k\|_{\mathbf{V}^*} \|\mathbf{u}_n^k\|_{\mathbf{V}},$$

and then apply the modified Cauchy-Schwarz inequality to get an ϵ -dependent constant c such that

$$\langle \mathbf{f}_n^k, \mathbf{u}_n^k \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} \leq \epsilon \|\mathbf{u}_n^k\|_{\mathbf{V}}^2 + c \|\mathbf{f}_n^k\|_{\mathbf{V}^*}^2.$$

Choose $\epsilon = m/4$ for m defined in (5.5). Then, we derive from (5.13) that

$$\frac{1}{2k} (\|\mathbf{u}_n^k\|_{\mathbf{H}}^2 - \|\mathbf{u}_{n-1}^k\|_{\mathbf{H}}^2 + \|\mathbf{u}_n^k - \mathbf{u}_{n-1}^k\|_{\mathbf{H}}^2) + \frac{m}{2} \|\mathbf{u}_n^k\|_{\mathbf{V}}^2 \leq \frac{c_4}{2} (1 + \|\mathbf{f}_n^k\|_{\mathbf{V}^*}^2),$$

or

$$\|\mathbf{u}_n^k\|_{\mathbf{H}}^2 - \|\mathbf{u}_{n-1}^k\|_{\mathbf{H}}^2 + \|\mathbf{u}_n^k - \mathbf{u}_{n-1}^k\|_{\mathbf{H}}^2 + m k \|\mathbf{u}_n^k\|_{\mathbf{V}}^2 \leq c_4 k (1 + \|\mathbf{f}_n^k\|_{\mathbf{V}^*}^2).$$

Change n to j in the above inequality and add the inequality for j from 1 to n :

$$\begin{aligned} \|\mathbf{u}_n^k\|_{\mathbf{H}}^2 + \sum_{j=1}^n \|\mathbf{u}_j^k - \mathbf{u}_{j-1}^k\|_{\mathbf{H}}^2 + m k \sum_{j=1}^n \|\mathbf{u}_j^k\|_{\mathbf{V}}^2 \\ \leq \|\mathbf{u}_{k,0}\|_{\mathbf{H}}^2 + c_4 n k + c_4 k \sum_{j=1}^n \|\mathbf{f}_j^k\|_{\mathbf{V}^*}^2. \end{aligned}$$

Recall that $\mathbf{u}_{k,0} \rightarrow \mathbf{u}_0$ in $\mathbf{H}$, implying the uniform boundedness of $\|\mathbf{u}_{k,0}\|_{\mathbf{H}}$ with respect to k . Note that $n k \leq T$ for any $1 \leq n \leq N$. By (5.1), the third term on the right side of the above inequality is also uniformly bounded with respect to k . Hence, (5.9), (5.10) and (5.11) hold.

From (5.3) and assumptions on Ψ ,

$$\|\boldsymbol{\xi}_n^k\|_{\mathbf{S}^*} \leq c (1 + \|\gamma_{\text{div}} \mathbf{u}_n^k\|_{\mathbf{S}}) \leq c (1 + \|\mathbf{u}_n^k\|_{\mathbf{V}}).$$

Hence, (5.12) holds. ■

We turn to a convergence analysis of the Rothe method. Define a continuous piecewise linear function $\mathbf{u}_k: [0, T] \rightarrow \mathbf{V}_{\text{div}}$ and three piecewise constant functions $\bar{\mathbf{u}}_k: [0, T] \rightarrow \mathbf{V}_{\text{div}}$, $\bar{\boldsymbol{\xi}}_k: [0, T] \rightarrow \mathbf{S}^*$, and $\bar{\mathbf{f}}_k: [0, T] \rightarrow \mathbf{V}^*$ by

$$\mathbf{u}_k(t) = \mathbf{u}_n^k + \left(\frac{t}{k} - n\right) (\mathbf{u}_n^k - \mathbf{u}_{n-1}^k), \quad \bar{\mathbf{u}}_k(t) = \mathbf{u}_n^k, \quad \bar{\boldsymbol{\xi}}_k(t) = \boldsymbol{\xi}_n^k, \quad \bar{\mathbf{f}}_k(t) = \mathbf{f}_n^k$$

for $t \in (t_{n-1}, t_n]$ if $2 \leq n \leq N$, and for $t \in [0, t_1]$ if $n = 1$. By (5.1), we have

$$\|\bar{\mathbf{f}}_k\|_{\mathbf{V}^*} \leq \|\mathbf{f}\|_{\mathbf{V}^*}. \quad (5.14)$$

Also, the following convergence holds ([4]):

$$\bar{\mathbf{f}}_k \rightarrow \mathbf{f} \quad \text{in } \mathbf{V}^*. \quad (5.15)$$

Rewrite (5.2) and (5.3) as, a.e. $t \in (0, T)$,

$$\mathbf{u}'_k(t) + A\bar{\mathbf{u}}_k(t) + \gamma_{\text{div}}^* \bar{\boldsymbol{\xi}}_k(t) = \bar{\mathbf{f}}_k(t), \quad (5.16)$$

$$\bar{\boldsymbol{\xi}}_k(t) \in \partial\Psi(\gamma_{\text{div}} \bar{\mathbf{u}}_k(t)). \quad (5.17)$$

Define Nemytskii operators $\mathcal{A}: \mathcal{V}_{\text{div}} \rightarrow \mathcal{V}_{\text{div}}^*$ and $\tilde{\gamma}: \mathcal{V}_{\text{div}} \rightarrow \mathcal{S}$ by

$$(\mathcal{A}\mathbf{v})(t) = A\mathbf{v}(t),$$

$$(\tilde{\gamma}\mathbf{v})(t) = \gamma_{\text{div}} \mathbf{v}(t).$$

Then, $\mathcal{A} \in \mathcal{L}(\mathcal{V}_{\text{div}}; \mathcal{V}_{\text{div}}^*)$, $\tilde{\gamma} \in \mathcal{L}(\mathcal{V}_{\text{div}}; \mathcal{S})$. We derive from (5.16)–(5.17) that for a.e. $t \in (0, T)$,

$$(\mathbf{u}'_k(t), \mathbf{v})_{\mathbf{H}} + \langle A\bar{\mathbf{u}}_k(t), \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}} + \langle \bar{\boldsymbol{\xi}}_k(t), \gamma_{\text{div}} \mathbf{v} \rangle_{\mathbf{S}^* \times \mathbf{S}} = \langle \bar{\mathbf{f}}_k(t), \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}} \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}},$$

$$\bar{\boldsymbol{\xi}}_k(t) \in \partial\Psi(\gamma_{\text{div}} \bar{\mathbf{u}}_k(t)),$$

or

$$(\mathbf{u}'_k, \mathbf{v})_{\mathcal{H}} + \langle \mathcal{A}\bar{\mathbf{u}}_k, \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}} + \langle \bar{\boldsymbol{\xi}}_k, \tilde{\gamma}\mathbf{v} \rangle_{\mathcal{S}^* \times \mathcal{S}} = \langle \bar{\mathbf{f}}_k, \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}} \quad \forall \mathbf{v} \in \mathcal{V}_{\text{div}}, \quad (5.18)$$

$$\bar{\boldsymbol{\xi}}_k(t) \in \partial\Psi(\gamma_{\text{div}} \bar{\mathbf{u}}_k(t)), \quad \text{a.e. } t \in (0, T). \quad (5.19)$$

Let us establish some bounds on $\{\bar{\mathbf{u}}_k\}$, $\{\mathbf{u}_k\}$, and $\{\bar{\boldsymbol{\xi}}_k\}$ next.

Lemma 5.5 *Under the assumptions (3.18), $H(\psi)$ and (5.5), there exists a constant $c > 0$ such that*

$$\|\bar{\mathbf{u}}_k\|_{L^\infty(0,T;\mathbf{H})} \leq c, \quad (5.20)$$

$$\|\bar{\mathbf{u}}_k\|_{\mathcal{V}} \leq c, \quad (5.21)$$

$$\|\bar{\mathbf{u}}_k\|_{M^{2,2}(0,T;\mathbf{V},\mathbf{V}^*)} \leq c, \quad (5.22)$$

$$\|\mathbf{u}_k\|_{C([0,T];\mathbf{H})} \leq c, \quad (5.23)$$

$$\|\mathbf{u}_k\|_{\mathcal{V}} \leq c, \quad (5.24)$$

$$\|\mathbf{u}'_k\|_{\mathcal{V}^*} \leq c, \quad (5.25)$$

$$\|\bar{\boldsymbol{\xi}}_k\|_{\mathcal{S}^*} \leq c. \quad (5.26)$$

Proof. The bound (5.20) is a direct consequence of (5.9). The bound (5.21) follows from (5.11):

$$\|\bar{\mathbf{u}}_k\|_{\mathcal{V}}^2 = \sum_{n=1}^N \|\mathbf{u}_n^k\|_{\mathbf{V}}^2 k \leq c.$$

For any $t \in (t_{n-1}, t_n]$ if $2 \leq n \leq N$, or for any $t \in [0, t_1]$ if $n = 1$, by (5.9),

$$\|\mathbf{u}_k(t)\|_{\mathbf{H}} \leq \max \{ \|\mathbf{u}_n^k\|_{\mathbf{H}}, \|\mathbf{u}_{n-1}^k\|_{\mathbf{H}} \} \leq c.$$

Notice that $\mathbf{u}_k(t)$ is continuous in t . So (5.23) holds.

The bound (5.24) is established from

$$\|\mathbf{u}_k\|_{\mathcal{V}}^2 = \int_0^T \|\mathbf{u}_k(t)\|_{\mathbf{V}}^2 dt = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left\| \mathbf{u}_n^k + \left(\frac{t}{k} - n \right) (\mathbf{u}_n^k - \mathbf{u}_{n-1}^k) \right\|_{\mathbf{V}}^2 dt \leq c \sum_{n=0}^N k \|\mathbf{u}_n^k\|_{\mathbf{V}}^2$$

which is uniformly bounded with respect to k by (5.11).

By (5.12),

$$\|\bar{\boldsymbol{\xi}}_k\|_{\mathcal{S}^*}^2 = \sum_{n=1}^N k \|\boldsymbol{\xi}_n^k\|_{\mathcal{S}^*}^2 \leq c k \sum_{n=1}^N (1 + \|\mathbf{u}_n^k\|_{\mathbf{V}}^2) \leq c + c k \sum_{n=1}^N \|\mathbf{u}_n^k\|_{\mathbf{V}}^2.$$

Then, applying (5.11), we derive (5.26).

From (5.16),

$$\mathbf{u}'_k(t) = \bar{\mathbf{f}}_k(t) - A\bar{\mathbf{u}}_k(t) - \gamma_{\text{div}}^* \bar{\boldsymbol{\xi}}_k(t) \quad \text{a.e. } t \in (0, T).$$

Then,

$$\|\mathbf{u}'_k(t)\|_{\mathcal{V}^*} \leq \|\bar{\mathbf{f}}_k(t)\|_{\mathcal{V}^*} + \|A\| \|\bar{\mathbf{u}}_k(t)\|_{\mathbf{V}} + \|\gamma_{\text{div}}^*\| \|\bar{\boldsymbol{\xi}}_k(t)\|_{\mathcal{S}^*}^2 \quad \text{a.e. } t \in (0, T).$$

Consequently, for some constant $\bar{c} > 0$,

$$\|\mathbf{u}'_k\|_{\mathcal{V}^*} \leq \bar{c} [\|\bar{\mathbf{f}}_k\|_{\mathcal{V}^*} + \|\bar{\mathbf{u}}_k\|_{\mathcal{V}} + \|\bar{\boldsymbol{\xi}}_k\|_{\mathcal{S}^*}],$$

which is bounded by a constant c due to (5.14), (5.21) and (5.26). Thus, (5.25) holds.

Finally, we derive the bound (5.22). By definition,

$$\|\bar{\mathbf{u}}_k\|_{BV^2(0,T;\mathbf{V}^*)}^2 = \sup_{0=s_0 < s_1 < \dots < s_{i_0}=T} \sum_{i=1}^{i_0} \|\bar{\mathbf{u}}_k(s_i) - \bar{\mathbf{u}}_k(s_{i-1})\|_{\mathbf{V}^*}^2,$$

where the supremum is taken with respect to all partitions of $[0, T]$. For a particular partition $0 = s_0 < s_1 < \dots < s_{i_0} = T$, we have positive integers $m_1 \leq m_2 \leq \dots \leq m_{i_0} = N$ such that $s_i \in ((m_i - 1)k, m_i k]$, $1 \leq i \leq i_0$. Then,

$$\bar{\mathbf{u}}_k(s_i) - \bar{\mathbf{u}}_k(s_{i-1}) = \mathbf{u}_{m_i}^k - \mathbf{u}_{m_{i-1}}^k.$$

By the triangle inequality,

$$\|\bar{\mathbf{u}}_k(s_i) - \bar{\mathbf{u}}_k(s_{i-1})\|_{\mathbf{V}^*} = \|\mathbf{u}_{m_i}^k - \mathbf{u}_{m_{i-1}}^k\|_{\mathbf{V}^*} \leq \sum_{j=m_{i-1}+1}^{m_i} \|\mathbf{u}_j^k - \mathbf{u}_{j-1}^k\|_{\mathbf{V}^*}$$

and so,

$$\|\bar{\mathbf{u}}_k(s_i) - \bar{\mathbf{u}}_k(s_{i-1})\|_{\mathbf{V}^*}^2 \leq (m_i - m_{i-1}) \sum_{j=m_{i-1}+1}^{m_i} \|\mathbf{u}_j^k - \mathbf{u}_{j-1}^k\|_{\mathbf{V}^*}^2 \leq (m_i - m_{i-1}) \sum_{j=1}^N \|\mathbf{u}_j^k - \mathbf{u}_{j-1}^k\|_{\mathbf{V}^*}^2.$$

As a result,

$$\begin{aligned} \sum_{i=1}^{i_0} \|\bar{\mathbf{u}}_k(s_i) - \bar{\mathbf{u}}_k(s_{i-1})\|_{\mathbf{V}^*}^2 &\leq \sum_{i=1}^{i_0} (m_i - m_{i-1}) \sum_{j=1}^N \|\mathbf{u}_j^k - \mathbf{u}_{j-1}^k\|_{\mathbf{V}^*}^2 \\ &= N \sum_{j=1}^N \|\mathbf{u}_j^k - \mathbf{u}_{j-1}^k\|_{\mathbf{V}^*}^2 \\ &= T k \sum_{j=1}^N \|\mathbf{u}_j^k - \mathbf{u}_{j-1}^k\|_{\mathbf{V}^*}^2 \\ &= T \|\mathbf{u}'_k\|_{\mathbf{V}^*}^2 \end{aligned}$$

which is uniformly bounded with respect to k by (5.25). Thus,

$$\|\bar{\mathbf{u}}_k\|_{M^{2,2}(0,T;\mathbf{V},\mathbf{V}^*)} = \|\bar{\mathbf{u}}_k\|_{\mathcal{V}} + \|\bar{\mathbf{u}}_k\|_{BV^2(0,T;\mathbf{V}^*)} \leq \mathcal{C},$$

i.e., (5.22) holds. ■

Lemma 5.6 Assume (3.18), $H(\psi)$ and (5.5). For a subsequence $\{k\}$, we have, as $k \rightarrow 0$,

$$\bar{\mathbf{u}}_k \rightharpoonup \mathbf{u} \quad \text{in } \mathcal{V}_{\text{div}}, \quad (5.27)$$

$$\bar{\mathbf{u}}_k \rightarrow \mathbf{u} \quad \text{in } \mathcal{H}, \quad (5.28)$$

$$\bar{\mathbf{u}}_{k,\tau} \rightarrow \mathbf{u}_\tau \quad \text{in } \mathcal{S} \quad (5.29)$$

$$\mathbf{u}_k \rightharpoonup \mathbf{u} \quad \text{in } \mathcal{V}_{\text{div}}, \quad (5.30)$$

$$\mathbf{u}'_k \rightharpoonup \mathbf{u}' \quad \text{in } \mathcal{V}^*, \quad (5.31)$$

$$\bar{\boldsymbol{\xi}}_k \rightharpoonup \boldsymbol{\xi} \quad \text{in } \mathcal{S}^*, \quad (5.32)$$

and $(\mathbf{u}, \boldsymbol{\xi})$ solves Problem 4.4.

Proof. From the bounds in Lemma 5.5, we can assume that, passing to a subsequence if necessary, there exist $\bar{\mathbf{u}} \in \mathcal{V}_{\text{div}} \cap L^\infty(0, T; \mathbf{H})$, $\mathbf{u} \in \mathcal{V}_{\text{div}} \cap L^\infty(0, T; \mathbf{H})$, $\tilde{\mathbf{u}} \in \mathcal{V}^*$, and $\boldsymbol{\xi} \in \mathcal{S}^*$ such that as $k \rightarrow 0$,

$$\bar{\mathbf{u}}_k \rightharpoonup \bar{\mathbf{u}} \text{ in } \mathcal{V} \text{ and } \bar{\mathbf{u}}_k \rightharpoonup^* \bar{\mathbf{u}} \text{ in } L^\infty(0, T; \mathbf{H}), \quad (5.33)$$

$$\mathbf{u}_k \rightharpoonup \mathbf{u} \text{ in } \mathcal{V} \text{ and } \mathbf{u}_k \rightharpoonup^* \mathbf{u} \text{ in } L^\infty(0, T; \mathbf{H}), \quad (5.34)$$

$$\mathbf{u}'_k \rightharpoonup \tilde{\mathbf{u}} \text{ in } \mathcal{V}^*, \quad (5.35)$$

$$\bar{\boldsymbol{\xi}}_k \rightharpoonup \boldsymbol{\xi} \text{ in } \mathcal{S}^*. \quad (5.36)$$

First we show that $\bar{\mathbf{u}} = \mathbf{u}$. From

$$\|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_{\mathcal{V}^*}^2 = \sum_{i=1}^N \int_{(i-1)k}^{ik} (ik - t)^2 \left\| \frac{\mathbf{u}_{k,i} - \mathbf{u}_{k,i-1}}{k} \right\|_{\mathcal{V}^*}^2 dt = \frac{k^2}{3} \|\mathbf{u}'_k\|_{\mathcal{V}^*}^2,$$

we know that $\bar{\mathbf{u}}_k - \mathbf{u}_k \rightarrow 0$ in $\mathcal{V}^*$ as $k \rightarrow 0$. On the other hand, by (5.33) and (5.34), $\bar{\mathbf{u}}_k - \mathbf{u}_k \rightharpoonup \bar{\mathbf{u}} - \mathbf{u}$ in $\mathcal{V}$. Since the embedding $\mathcal{V} \subset \mathcal{V}^*$ is continuous, we also have $\bar{\mathbf{u}}_k - \mathbf{u}_k \rightharpoonup \bar{\mathbf{u}} - \mathbf{u}$ in $\mathcal{V}^*$. Hence, $\bar{\mathbf{u}} - \mathbf{u} = 0$, i.e. $\bar{\mathbf{u}} = \mathbf{u}$. Since $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $\mathcal{V}$ and $\mathbf{u}'_k \rightharpoonup \tilde{\mathbf{u}}$ in $\mathcal{V}^*$, we conclude (cf. [33, Proposition 23.19]) that $\tilde{\mathbf{u}} = \mathbf{u}'$. Thus, for all $\mathbf{v} \in \mathcal{V}_{\text{div}}$, we obtain

$$\langle \mathbf{u}'_k, \mathbf{v} \rangle_{\mathcal{H}} = \langle \mathbf{u}'_k, \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}} \rightarrow \langle \mathbf{u}', \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}}. \quad (5.37)$$

Since $\mathcal{A}: \mathcal{V}_{\text{div}} \rightarrow \mathcal{V}_{\text{div}}^*$ is linear and continuous, it is weakly continuous. Thus, $\bar{\mathbf{u}}_k \rightharpoonup \mathbf{u}$ in $\mathcal{V}_{\text{div}}$ implies that

$$\langle \mathcal{A}\bar{\mathbf{u}}_k, \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}} \rightarrow \langle \mathcal{A}\mathbf{u}, \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}}. \quad (5.38)$$

From (5.36) we get

$$\langle \bar{\boldsymbol{\xi}}_k, \tilde{\gamma}\mathbf{v} \rangle_{\mathcal{S}^* \times \mathcal{S}} \rightarrow \langle \boldsymbol{\xi}, \tilde{\gamma}\mathbf{v} \rangle_{\mathcal{S}^* \times \mathcal{S}}. \quad (5.39)$$

It follows from (5.15) that $\bar{\mathbf{f}}_k \rightarrow \mathbf{f}$ in $\mathcal{V}^*$. Thus,

$$\langle \bar{\mathbf{f}}_k, \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}} \rightarrow \langle \mathbf{f}, \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}}. \quad (5.40)$$

Using (5.37)–(5.40), we can pass to the limit $k \rightarrow 0$ in (5.18) and obtain

$$\langle \mathbf{u}', \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}} + \langle \mathcal{A}\mathbf{u}, \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}} + \langle \boldsymbol{\xi}, \tilde{\gamma}\mathbf{v} \rangle_{\mathcal{S}^* \times \mathcal{S}} = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathcal{V}_{\text{div}}^* \times \mathcal{V}_{\text{div}}} \quad \forall \mathbf{v} \in \mathcal{V}_{\text{div}}. \quad (5.41)$$

From this, we deduce (4.11) in $\mathbf{V}_{\text{div}}^*$ for a.e. $t \in (0, T)$.

Recall the bound (5.22). Applying Theorem 2.1 and Sobolev embedding theorems, we know that, up to a further subsequence if necessary, $\bar{\mathbf{u}}_k \rightarrow \mathbf{u}$ in $L^2(0, T; H^\delta(\Omega)^d)$ for some number $\delta \in (1/2, 1)$. Since $\gamma: H^\delta(\Omega)^d \rightarrow \mathcal{S}$ is compact, for a further subsequence if needed, we have $\gamma_{\text{div}}\bar{\mathbf{u}}_k(t) \rightarrow \gamma_{\text{div}}\mathbf{u}(t)$ in $\mathcal{S}$ for a.e. $t \in (0, T)$. Since $\partial\psi: \mathcal{S} \rightarrow 2^{\mathcal{S}^*}$ has nonempty, closed and convex values and is upper semicontinuous from $\mathcal{S}$ endowed with strong topology into $\mathcal{S}^*$ endowed with weak topology (cf. [7, Proposition 5.6.10]), we deduce from (5.36), (5.19) and Theorem 2.2 that

$$\boldsymbol{\xi}(t) \in \partial\psi(\gamma_{\text{div}}\mathbf{u}(t)) \quad \text{a.e. } t \in (0, T). \quad (5.42)$$

Finally, we pass to the limit with the initial conditions on the function $\mathbf{u}_k$. Since $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $\mathcal{V}$, $\mathbf{u}'_k \rightharpoonup \mathbf{u}'$ in $\mathcal{V}^*$, we have $\mathbf{u}_k(t) \rightharpoonup \mathbf{u}(t)$ in $\mathbf{H}$ for all $t \in [0, T]$ (cf. [25, Lemma 4(b)]). Therefore, $\mathbf{u}_{k,0} = \mathbf{u}_k(0) \rightharpoonup \mathbf{u}(0)$ in $\mathbf{H}$. Since $\mathbf{u}_{k,0} \rightarrow \mathbf{u}_0$ in $\mathbf{H}$, we have $\mathbf{u}(0) = \mathbf{u}_0$.

In conclusion, $(\mathbf{u}, \boldsymbol{\xi})$ is a solution of Problem 4.4. ■

Summarizing, we have the next two results.

Theorem 5.7 Assume (3.18), $H(\psi)$ and (5.5). Then, Problem 4.4 has a solution $(\mathbf{u}, \boldsymbol{\xi}) \in \mathcal{W}_{\text{div}} \times \mathcal{S}^*$ and the solution component $\mathbf{u}$ is unique.

Proof. Existence of a solution to Problem 4.4 follows from Lemma 5.6.

Let us show the uniqueness of the solution component $\mathbf{u}$. Suppose both $(\mathbf{u}_1, \boldsymbol{\xi}_1), (\mathbf{u}_2, \boldsymbol{\xi}_2) \in \mathcal{W}_{\text{div}} \times \mathcal{S}^*$ are solutions of Problem 4.4. Then for a.e. $t \in (0, T)$,

$$\langle \mathbf{u}'_1(t), \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} + a(\mathbf{u}_1(t), \mathbf{v}) + \langle \boldsymbol{\xi}_1(t), \gamma_{\text{div}} \mathbf{v} \rangle_{\mathcal{S}^* \times \mathcal{S}} = \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}}, \quad (5.43)$$

$$\boldsymbol{\xi}_1(t) \in \partial \Psi(\gamma_{\text{div}} \mathbf{u}_1(t)), \quad (5.44)$$

$$\langle \mathbf{u}'_2(t), \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} + a(\mathbf{u}_2(t), \mathbf{v}) + \langle \boldsymbol{\xi}_2(t), \gamma_{\text{div}} \mathbf{v} \rangle_{\mathcal{S}^* \times \mathcal{S}} = \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}}, \quad (5.45)$$

$$\boldsymbol{\xi}_2(t) \in \partial \Psi(\gamma_{\text{div}} \mathbf{u}_2(t)), \quad (5.46)$$

and

$$\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0. \quad (5.47)$$

Take $\mathbf{v} = \mathbf{u}_2(t) - \mathbf{u}_1(t)$ in (5.43), $\mathbf{v} = \mathbf{u}_1(t) - \mathbf{u}_2(t)$ in (5.45), and add the two inequalities to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}}^2 + a(\mathbf{u}_1(t) - \mathbf{u}_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t)) \\ + \langle \boldsymbol{\xi}_1(t) - \boldsymbol{\xi}_2(t), \gamma_{\text{div}} \mathbf{u}_1(t) - \gamma_{\text{div}} \mathbf{u}_2(t) \rangle_{\mathcal{S}^* \times \mathcal{S}} = 0 \end{aligned} \quad (5.48)$$

for a.e. $t \in (0, T)$. Then, with $m = 2\mu - \alpha_\psi \lambda_\tau^{-1}$ defined in (5.5),

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}}^2 + m \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{V}}^2 \leq 0, \quad \text{a.e. } t \in (0, T). \quad (5.49)$$

Integrate the inequality from 0 to $t \in (0, T)$ to obtain

$$\frac{1}{2} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}}^2 + m \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_{\mathbf{V}}^2 ds \leq 0, \quad t \in (0, T).$$

Since $m > 0$ by assumption, we deduce from the above inequality that

$$\mathbf{u}_1(t) - \mathbf{u}_2(t) = \mathbf{0} \text{ for } t \in (0, T).$$

Therefore, we have the uniqueness of $\mathbf{u}$. ■

Theorem 5.8 Assume (3.18), $H(\psi)$ and (5.5). Then, Problem 4.1 has a unique solution $\mathbf{u} \in \mathcal{W}_{\text{div}}$.

Proof. Theorem 5.7 guarantees the existence of a solution $\mathbf{u} \in \mathcal{W}_{\text{div}}$ for Problem 4.1.

For the uniqueness, assume $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{W}_{\text{div}}$ are two solutions of Problem 4.1. Then, similar to (5.48), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}}^2 + a(\mathbf{u}_1(t) - \mathbf{u}_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t)) \\ \leq \Psi^0(\gamma_{\text{div}} \mathbf{u}_1(t); \gamma_{\text{div}} \mathbf{u}_2(t) - \gamma_{\text{div}} \mathbf{u}_1(t)) + \Psi^0(\gamma_{\text{div}} \mathbf{u}_2(t); \gamma_{\text{div}} \mathbf{u}_1(t) - \gamma_{\text{div}} \mathbf{u}_2(t)) \end{aligned}$$

for a.e. $t \in (0, T)$. We again have the inequality (5.49) from which, the uniqueness follows. ■

6 Existence and uniqueness of the pressure field

We have already shown in Theorem 5.8 that Problem 4.1 has a unique solution $\mathbf{u} \in \mathcal{W}_{\text{div}}$. Since

$$\Psi^0(\gamma_{\text{div}}\mathbf{u}(t); \gamma_{\text{div}}\mathbf{v}) \leq I_{\Gamma_S}(\psi^0(\gamma_{\text{div}}\mathbf{u}(t); \gamma_{\text{div}}\mathbf{v})),$$

we apply Theorem 5.8 to know that there is an element $\mathbf{u} \in \mathcal{W}_{\text{div}}$ such that for a.e. $t \in (0, T)$,

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} + a(\mathbf{u}(t), \mathbf{v}) + I_{\Gamma_S}(\psi^0(\gamma_{\text{div}}\mathbf{u}(t); \gamma_{\text{div}}\mathbf{v})) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}_{\text{div}}^* \times \mathbf{V}_{\text{div}}} \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}}.$$

Moreover, similar to the proof of Theorem 5.8, we can show that the element $\mathbf{u} \in \mathcal{W}_{\text{div}}$ is unique.

From Lemma 5.6, we know that $\mathbf{u}' \in \mathcal{V}^*$. Moreover, $\mathcal{A}\mathbf{u}$ is an element in $\mathcal{V}^*$ since $\mathbf{u} \in \mathcal{V}$. Also, for a.e. $t \in [0, T]$, $\gamma_{\text{div}}^*\boldsymbol{\xi}(t)$ can be viewed an element in $\mathbf{V}^*$ through the formula

$$\langle \gamma_{\text{div}}^*\boldsymbol{\xi}(t), \mathbf{v} \rangle = \langle \boldsymbol{\xi}(t), \gamma\mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}.$$

In addition, by assumption (3.18), $\mathbf{f}(t) \in \mathbf{V}^*$ for a.e. $t \in (0, T)$. Therefore, there exists a unique solution $\mathbf{u} \in \mathcal{W}_{\text{div}}$ to the following problem.

Problem 6.1 Find $\mathbf{u} \in \mathcal{W}_{\text{div}}$ such that for a.e. $t \in (0, T)$,

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}) + I_{\Gamma_S}(\psi^0(\gamma\mathbf{u}(t); \gamma\mathbf{v})) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}_{\text{div}}, \quad (6.1)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (6.2)$$

We deduce from (6.1) that

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}) = \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}_{0,\text{div}} := \mathbf{V}_0 \cap \mathbf{V}_{\text{div}}. \quad (6.3)$$

Thanks to the inf-sup property (3.17), there exists a function $p(t) \in Q$ such that

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p(t)) = \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}_0. \quad (6.4)$$

Thus,

$$b(\mathbf{v}, p(t)) = \langle \mathbf{u}'(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}) - \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}_0,$$

leading to

$$b(\mathbf{v}, p(t)) \leq (\|\mathbf{u}'(t)\|_{\mathbf{V}^*} + 2\mu\|\mathbf{u}(t)\|_{\mathbf{V}} + \|\mathbf{f}(t)\|_{\mathbf{V}^*})\|\mathbf{v}\|_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}_0.$$

Hence, with an application of the inf-sup property (3.17),

$$\alpha_b\|p(t)\|_Q \leq \|\mathbf{u}'(t)\|_{\mathbf{V}^*} + 2\mu\|\mathbf{u}(t)\|_{\mathbf{V}} + \|\mathbf{f}(t)\|_{\mathbf{V}^*}.$$

Consequently,

$$p \in Q.$$

For an arbitrary $\mathbf{v} \in \mathbf{V}$, by the inf-sup property, there exists a function $\mathbf{v}_1 \in \mathbf{V}_0$ such that

$$b(\mathbf{v}_1, q) = b(\mathbf{v}, q) \quad \forall q \in Q. \quad (6.5)$$

Define

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1.$$

Then $\mathbf{v}_2 \in \mathbf{V}$ and

$$b(\mathbf{v}_2, q) = 0 \quad \forall q \in Q.$$

Hence, $\mathbf{v}_2 \in \mathbf{V}_{\text{div}}$. By (6.1),

$$\langle \mathbf{u}'(t), \mathbf{v}_2 \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}_2) + I_{\Gamma_S}(\psi^0(\gamma \mathbf{u}(t); \gamma \mathbf{v}_2)) \geq \langle \mathbf{f}(t), \mathbf{v}_2 \rangle_{\mathbf{V}^* \times \mathbf{V}},$$

i.e.,

$$\langle \mathbf{u}'(t), \mathbf{v} - \mathbf{v}_1 \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v} - \mathbf{v}_1) + I_{\Gamma_S}(\psi^0(\gamma \mathbf{u}(t); \gamma(\mathbf{v} - \mathbf{v}_1))) \geq \langle \mathbf{f}(t), \mathbf{v} - \mathbf{v}_1 \rangle_{\mathbf{V}^* \times \mathbf{V}}.$$

Note that $\mathbf{v}_1 \in \mathbf{V}_0$ and so $\gamma \mathbf{v}_1 = \mathbf{0}$ on Γ_S . As a result,

$$\langle \mathbf{u}'(t), \mathbf{v} - \mathbf{v}_1 \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v} - \mathbf{v}_1) + I_{\Gamma_S}(\psi^0(\gamma \mathbf{u}(t); \gamma \mathbf{v})) \geq \langle \mathbf{f}(t), \mathbf{v} - \mathbf{v}_1 \rangle_{\mathbf{V}^* \times \mathbf{V}}.$$

By (6.4),

$$\langle \mathbf{u}'(t), \mathbf{v}_1 \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}_1) - b(\mathbf{v}_1, p(t)) = \langle \mathbf{f}(t), \mathbf{v}_1 \rangle_{\mathbf{V}^* \times \mathbf{V}}.$$

By (6.5),

$$b(\mathbf{v}_1, p(t)) = b(\mathbf{v}, p(t)).$$

Thus,

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p(t)) + I_{\Gamma_S}(\psi^0(\gamma \mathbf{u}(t); \gamma \mathbf{v})) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}}.$$

Therefore, $(\mathbf{u}, p)$ is a solution of Problem 3.2.

We already know that the solution component $\mathbf{u}$ is unique. To show the uniqueness of p , assume $(\mathbf{u}, p_1)$ and $(\mathbf{u}, p_2)$ are two solutions of Problem 3.2. By (6.4), for all $\mathbf{v} \in \mathbf{V}_0$,

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p_1(t)) = \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}}, \quad (6.6)$$

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}} + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p_2(t)) = \langle \mathbf{f}(t), \mathbf{v} \rangle_{\mathbf{V}^* \times \mathbf{V}}. \quad (6.7)$$

Subtract (6.7) from (6.6) to get

$$b(\mathbf{v}, p_1(t) - p_2(t)) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_0.$$

Applying the inf-sup property (3.17), we have

$$\alpha_b \|p_1(t) - p_2(t)\|_Q \leq \sup_{\mathbf{v} \in \mathbf{V}_0} \frac{b(\mathbf{v}, p_1(t) - p_2(t))}{\|\mathbf{v}\|_{\mathbf{V}}} = 0.$$

Hence, $p_1(t) - p_2(t) = 0$ and the solution component p is unique.

In conclusion, we have proved the following existence and uniqueness result.

Theorem 6.2 Assume (3.18), $H(\psi)$ and (5.5). Then, Problem 3.2 has a unique solution $(\mathbf{u}, p) \in \mathcal{W} \times Q$.

7 Lipschitz continuous dependence on data

In this section, we explore the Lipschitz continuous dependence of the solution $\mathbf{u}$ on the source function $\mathbf{f}$ and the initial value $\mathbf{u}_0$.

Theorem 7.1 Assume (3.18), $H(\psi)$ and (5.5), and let $(\mathbf{u}, p) \in \mathcal{W} \times \mathcal{Q}$ be the solution of Problem 3.2. Then, the mapping $(\mathbf{f}, \mathbf{u}_0) \mapsto \mathbf{u}$ is Lipschitz continuous from $\mathcal{V}^* \times \mathbf{H}$ to $\mathcal{V} \cap C(0, T; \mathbf{H})$.

Proof. For $i = 1, 2$, let $(\mathbf{f}_i, \mathbf{u}_{i,0}) \in \mathcal{V}^* \times \mathbf{H}$ and let $(\mathbf{u}_i, p_i) \in \mathcal{W} \times \mathcal{Q}$ be the corresponding solution of Problem 3.2. From the definition of the problem, for $i = 1, 2$ and for a.e. $t \in (0, T)$,

$$\langle \mathbf{u}'_i(t), \mathbf{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} + a(\mathbf{u}_i(t), \mathbf{v}) - b(\mathbf{v}, p_i(t)) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{i,\tau}(t); \mathbf{v}_\tau)) \geq \langle \mathbf{f}_i(t), \mathbf{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} \quad \forall \mathbf{v} \in \mathcal{V}, \quad (7.1)$$

$$b(\mathbf{u}_i(t), q) = 0 \quad \forall q \in \mathcal{Q}, \quad (7.2)$$

and

$$\mathbf{u}_i(0) = \mathbf{u}_{i,0}. \quad (7.3)$$

Take $\mathbf{v} = \mathbf{u}_2(t) - \mathbf{u}_1(t)$ in (7.1) for $i = 1$, take $\mathbf{v} = \mathbf{u}_1(t) - \mathbf{u}_2(t)$ in (7.1) for $i = 2$, and add the two inequalities to obtain, for a.e. $t \in (0, T)$,

$$\begin{aligned} & \langle \mathbf{u}'_1(t) - \mathbf{u}'_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} + a(\mathbf{u}_1(t) - \mathbf{u}_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t)) \\ & \leq I_{\Gamma_S}(\psi^0(\mathbf{u}_{1,\tau}(t), \mathbf{u}_{2,\tau}(t) - \mathbf{u}_{1,\tau}(t)) + \psi^0(\mathbf{u}_{2,\tau}(t), \mathbf{u}_{1,\tau}(t) - \mathbf{u}_{2,\tau}(t))) \\ & \quad + b(\mathbf{u}_1(t) - \mathbf{u}_2(t), p_1(t) - p_2(t)) + \langle \mathbf{f}_1(t) - \mathbf{f}_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t) \rangle_{\mathcal{V}^* \times \mathcal{V}}. \end{aligned}$$

It follows from (7.2) that $b(\mathbf{u}_1(t) - \mathbf{u}_2(t), p_1(t) - p_2(t)) = 0$. Note that

$$\begin{aligned} & \langle \mathbf{u}'_1(t) - \mathbf{u}'_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} = \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}}^2, \\ & a(\mathbf{u}_1(t) - \mathbf{u}_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t)) = 2\mu \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathcal{V}}^2, \\ & I_{\Gamma_S}(\psi^0(\mathbf{u}_{1,\tau}(t), \mathbf{u}_{2,\tau}(t) - \mathbf{u}_{1,\tau}(t)) + \psi^0(\mathbf{u}_{2,\tau}(t), \mathbf{u}_{1,\tau}(t) - \mathbf{u}_{2,\tau}(t))) \leq \alpha_\psi \lambda_\tau^{-1} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathcal{V}}^2, \\ & \langle \mathbf{f}_1(t) - \mathbf{f}_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t) \rangle_{\mathcal{V}^* \times \mathcal{V}} \leq \|\mathbf{f}_1(t) - \mathbf{f}_2(t)\|_{\mathcal{V}^*} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathcal{V}}. \end{aligned}$$

Hence,

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}}^2 + m \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathcal{V}}^2 \leq \|\mathbf{f}_1(t) - \mathbf{f}_2(t)\|_{\mathcal{V}^*} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathcal{V}},$$

where m is defined in (5.5). Applying the inequality

$$\|\mathbf{f}_1(t) - \mathbf{f}_2(t)\|_{\mathcal{V}^*} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathcal{V}} \leq \frac{m}{2} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathcal{V}}^2 + \frac{1}{2m} \|\mathbf{f}_1(t) - \mathbf{f}_2(t)\|_{\mathcal{V}^*}^2,$$

we further derive that for a.e. $t \in (0, T)$,

$$\frac{d}{dt} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}}^2 + m \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathcal{V}}^2 \leq \frac{1}{m} \|\mathbf{f}_1(t) - \mathbf{f}_2(t)\|_{\mathcal{V}^*}^2.$$

Integrating the inequality from 0 to t , we get that for any $t \in [0, T]$,

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}}^2 + m \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_{\mathcal{V}}^2 ds \leq \|\mathbf{u}_{1,0} - \mathbf{u}_{2,0}\|_{\mathbf{H}}^2 + \frac{1}{m} \int_0^t \|\mathbf{f}_1(s) - \mathbf{f}_2(s)\|_{\mathcal{V}^*}^2 ds. \quad (7.4)$$

The stated Lipschitz continuous dependence follows from the inequality (7.4). ■

8 Concluding remark

To the best of our knowledge, the reference [19] is the first paper that provides an existence and uniqueness result for both the velocity and pressure fields of a nonstationary Stokes hemivariational inequality for an incompressible fluid flow subject to a nonsmooth non-monotone boundary condition of friction type. The well-posedness analysis in [19] is carried out under the following assumptions on the data

$$\mathbf{u}_0 \in \mathbf{V}, \quad \operatorname{div} \mathbf{u}_0 = 0 \text{ in } \Omega, \quad \mathbf{f} \in H^1(0, T; \mathbf{V}^*) \quad (8.1)$$

together with a compatibility condition on initial values of the data. This paper provides a different approach for the well-posedness analysis of the nonstationary Stokes hemivariational inequality so that the assumptions (8.1) are replaced by the more natural ones:

$$\mathbf{u}_0 \in \mathbf{H}, \quad \mathbf{f} \in L^2(0, T; \mathbf{V}^*);$$

moreover, the compatibility condition on initial values of the data is dropped.

We also comment that the analysis in [19] on a fully discrete scheme for solving the nonstationary Stokes hemivariational inequality is still valid without the compatibility condition on initial values of the data and the condition $\operatorname{div} \mathbf{u}_0 = 0$ in Ω .

References

- [1] J. P. Aubin and A. Cellina, *Differential Inclusions. Set-Valued Maps and Viability Theory*. Springer, Berlin, 1984.
- [2] K. Bartosz, X. Cheng, P. Kalita, Y. Yu and C. Zheng, Rothe method for parabolic variational-hemivariational inequalities, *Journal of Mathematical Analysis and Applications* **423** (2015), 841–862.
- [3] S. Carl and V. K. Le, *Multi-Valued Variational Inequalities and Inclusions*, Springer, New York, 2021.
- [4] C. Carstensen and J. Gwinner, A theory of discretisation for nonlinear evolution inequalities applied to parabolic Signorini problems, *Annali di Matematica Pura ed Applicata* **177** (1999), 363–394.
- [5] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, Interscience, New York, 1983.
- [6] F. H. Clarke, Y.S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, New York, 1998.
- [7] Z. Denkowski, S. Migórski, and N. S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic, New York, 2003.
- [8] J. K. Djoko, On the time approximation of the Stokes equations with nonlinear slip boundary conditions, *International Journal of Numerical Analysis and Modeling, Series B* **11** (2014), 34–53.

- [9] C. Fang, K. Czuprynski, W. Han, X. L. Cheng, and X. Dai, Finite element method for a stationary Stokes hemivariational inequality with slip boundary condition, *IMA Journal of Numerical Analysis* **40** (2020), 2696–2716.
- [10] H. Fujita, A coherent analysis of Stokes flows under boundary conditions of friction type, *Journal of Computational and Applied Mathematics* **149** (2002), 57–69.
- [11] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Springer-Verlag, Berlin, 1986.
- [12] W. Han, Minimization principles for elliptic hemivariational inequalities, *Nonlinear Analysis: Real World Applications* **54** (2020), paper no. 103114.
- [13] W. Han, A revisit of elliptic variational-hemivariational inequalities, *Numerical Functional Analysis and Optimization* **42** (2021), 371–395.
- [14] W. Han, *An Introduction to Theory and Applications of Stationary Variational-Hemivariational Inequalities*, Springer, New York, 2024.
- [15] W. Han, Variational-hemivariational inequalities: a brief survey on mathematical theory and numerical analysis, to appear in *Approximation Theory and Special Functions*.
- [16] W. Han, F. Feng, F. Wang, and J. Huang, Numerical analysis of hemivariational inequalities with applications in contact mechanics, *Advances in Applied Mechanics* **60** (2025), 113–178.
- [17] W. Han, F. Jing, and Y. Yao, Stabilized mixed finite element methods for a Navier–Stokes hemivariational inequality, *BIT Numerical Mathematics* **63** (2023), article number 46.
- [18] W. Han and M. Sofonea, Numerical analysis of hemivariational inequalities in contact mechanics, *Acta Numerica* **28** (2019), 175–286.
- [19] W. Han, Y. Yao, and S. Zeng, Well-posedness and numerical analysis of a nonstationary Stokes hemivariational inequality, to appear in *Mathematics and Mechanics of Solids*.
- [20] J. Haslinger, M. Miettinen, and P. D. Panagiotopoulos, *Finite Element Method for Hemivariational Inequalities: Theory, Methods and Applications*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1999.
- [21] V. John, *Finite Element Methods for Incompressible Flow Problems*, Springer, 2016.
- [22] P. Kalita, Convergence of Rothe scheme for hemivariational inequalities of parabolic type, *International Journal of Numerical Analysis and Modeling* **10** (2013), 445–465.
- [23] Y. Li and K. Li, Penalty finite element method for Stokes problem with nonlinear slip boundary conditions, *Applied Mathematics and Computation* **204** (2008), 216–226.
- [24] M. Ling, W. Han, and S. Zeng, A pressure projection stabilized mixed finite element method for a Stokes hemivariational inequality, *Journal of Scientific Computing* **92** (2022), article number 13.
- [25] S. Migórski and A. Ochal, Quasistatic hemivariational inequality via vanishing acceleration approach, *SIAM Journal on Mathematical Analysis* **41** (2009), 1415–1435.

- [26] S. Migórski, A. Ochal, and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics 26, Springer, New York, 2013.
- [27] Z. Naniewicz and P. D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Dekker, New York, 1995.
- [28] J. Nečas and I. Hlavaček, *Mathematical Theory of Elastic and Elastoplastic Bodies: An Introduction*, Elsevier, Amsterdam, 1981.
- [29] P. D. Panagiotopoulos, Nonconvex energy functions, hemivariational inequalities and substationary principles, *Acta Mechanica* **42** (1983), 160–183.
- [30] T. Roubicek, *Nonlinear Partial Differential Equations with Applications*, second edition, Birkhäuser Verlag, Basel, Boston, Berlin, 2013.
- [31] M. Sofonea and S. Migórski, *Variational-Hemivariational Inequalities with Applications*, second edition, CRC Press, Boca Raton, FL, 2025.
- [32] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holland, Amsterdam, 1979.
- [33] E. Zeidler, *Nonlinear Functional Analysis and Applications. IIA*, Springer, New York, 1990.
- [34] S. Zeng, S. Migórski, and V. T. Nguyen, A class of hyperbolic variational-hemivariational inequalities without damping terms, *Advances in Nonlinear Analysis* **11** (2022), 1287–1306.