

Well-posedness and numerical analysis of a nonstationary Stokes hemivariational inequality

Weimin Han¹ , Yuan Yao¹  and Shengda Zeng² 

Abstract

A nonstationary mixed hemivariational inequality is studied for an incompressible fluid flow described by the Stokes equations subject to a nonsmooth boundary condition of friction type described by the Clarke subdifferential. The solution existence is shown through a limiting procedure based on temporally semi-discrete approximations. Uniqueness of the solution and its continuous dependence on data are also established. Fully discrete numerical methods are introduced to solve the nonstationary mixed hemivariational inequality. The backward Euler scheme is applied to discretize the time derivative, and mixed finite element methods are used for the spatial discretization. An error bound is derived for the numerical solution of the unknown velocity. Numerical results are reported on computer simulations of some examples.

Keywords

Stokes equations, hemivariational inequality, well-posedness, backward Euler scheme, mixed finite element method, error estimate

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1. Introduction

This paper is devoted to the study of a nonstationary Stokes hemivariational inequality, i.e., a hemivariational inequality of the time-dependent Stokes equations for an incompressible fluid flow subject to a non-leak nonsmooth non-monotone slip condition of friction type. Research on hemivariational inequalities, or more generally, variational-hemivariational inequalities, started in early 1980s with the pioneering work of Panagiotopoulos [1]. Since then, there has been extensive research on modeling, analysis, numerical solution, and applications of variational-hemivariational inequalities. The reader is referred to several research monographs [2–5] on mathematical theories and applications of variational-hemivariational inequalities where abstract surjectivity results on pseudomonotone operators are applied, and to the new book [6] for a more accessible coverage of the mathematical theory without the need of the notion and results of pseudomonotone operators. Numerical methods are needed to solve variational-hemivariational inequalities from applications. An early comprehensive reference on

¹Department of Mathematics, The University of Iowa, Iowa City, IA, USA

²National Center for Applied Mathematics in Chongqing, and School of Mathematical Sciences, Chongqing Normal University, Chongqing, China

Corresponding author:

Weimin Han, Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1410, USA.

Email: weimin-han@uiowa.edu

the subject is Haslinger et al. [7]. More recent advances on the subject are presented in the two survey papers [8,9] and the references therein.

Variational inequalities for the time-dependent Stokes equations have been studied in several papers [10–12]. When a relevant superpotential function in these papers is no longer assumed to be convex, a hemivariational inequality arises. To show the solution existence of the nonstationary Stokes hemivariational inequality, we use a sequence of temporally semi-discrete approximation problems, known as the Rothe method in some references, and prove that as the time step-size approaches zero, the solution of the semi-discrete problem converges to a limit which is the unique solution of the hemivariational inequality. The Rothe method has been used in studying a variety of time-dependent problems [13–16]. Numerical methods have been studied in a number of papers for solving stationary mixed hemivariational inequalities in fluid mechanics, e.g., [17,18] for a stationary Stokes hemivariational inequality and [19] for a stationary Navier–Stokes hemivariational inequality. In this paper, we extend the numerical analysis to the nonstationary Stokes hemivariational inequality.

The organization of this paper is as follows. In Section 2, we introduce some definitions and auxiliary material. The problem setup and some assumptions on the data are presented in Section 3. In Section 4, we show the solution existence and uniqueness. In Section 5, we analyze a fully discrete method for solving the nonstationary Stokes hemivariational inequality. The time derivative is discretized with the backward Euler difference, and the spatial discretization is achieved by the mixed finite element method with a pair of finite element spaces satisfying the discrete inf-sup condition. In particular, we derive an error bound for the numerical solution of the velocity variable when the MINI (P1b/P1) element pair is used, under certain solution regularity assumptions; the error estimate is of optimal order with respect to the time step-size. In Section 6, we report numerical results on the performance of the numerical method.

2. Preliminaries

For a normed space V , we denote by $\|\cdot\|_V$ its norm, by V^* its topological dual, and by $\langle \cdot, \cdot \rangle_{V^* \times V}$ the duality pairing between V^* and V . Weak convergence will be indicated by the symbol $\rightharpoonup$. We denote the Euclidean norm in $\mathbb{R}^d$ by $|\cdot|$, d being a positive integer. In addition, we denote by $\mathbb{S}^d$ the space of symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of symmetric matrices of order d . The symbol 2^{V^*} represents the set of all subsets of V^* . We will always assume V is a Banach space, unless stated otherwise.

Let $\Psi: V \rightarrow \mathbb{R}$ be a locally Lipschitz function. Then, the generalized directional derivative (in the sense of Clarke) of Ψ at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u; v) = \limsup_{w \rightarrow u, \lambda \rightarrow 0} \frac{\Psi(w + \lambda v) - \Psi(w)}{\lambda},$$

and the generalized gradient or subdifferential (in the sense of Clarke) of Ψ at u is defined by

$$\partial\Psi(u) = \{\zeta \in V^* \mid \Psi^0(u; v) \geq \langle \zeta, v \rangle_{V^* \times V} \forall v \in V\}.$$

For properties and detailed discussions of the generalized directional derivative and the generalized subdifferential of a locally Lipschitz function, the reader is referred to Clarke [20] and Clarke et al. [21] or [3, Section 3.2]. We list some properties in the next result.

Proposition 2.1. *Assume that $\Psi: V \rightarrow \mathbb{R}$ is a locally Lipschitz function. Then, the following statements are valid.*

- (i) $\Psi^0(u; \lambda v) = \lambda \Psi^0(u; v) \forall \lambda \geq 0, u, v \in V$.
 $\Psi^0(u; -v) = (-\Psi)^0(u; v) \forall u, v \in V$.
- (ii) $\Psi^0(u; v_1 + v_2) \leq \Psi^0(u; v_1) + \Psi^0(u; v_2) \forall u, v_1, v_2 \in V$.
- (iii) $\Psi^0(u; v) = \max \{\langle u^*, v \rangle \mid u^* \in \partial\Psi(u)\} \forall u, v \in V$.
- (iv) If $u_n \rightarrow u$ and $v_n \rightarrow v$ in V , then

$$\limsup_{n \rightarrow \infty} \Psi^0(u_n; v_n) \leq \Psi^0(u; v).$$

- (v) For every $u \in V$, $\partial\Psi(u)$ is nonempty, convex, and weakly* compact in V^* .
- (vi) If $u_n \rightarrow u$ in V , $u_n^* \in \partial\Psi(u_n)$, and $u_n^* \rightarrow u^*$ weakly* in V^* , then $u^* \in \partial\Psi(u)$.
- (vii) If $\Psi: V \rightarrow \mathbb{R}$ is convex, then for all $u \in V$, the generalized subdifferential $\partial\Psi(u)$ coincides with the convex subdifferential $\partial\Psi(u)$.

For a Banach space V and a finite time interval $I = (0, T)$, we will use the spaces $L^p(I; V)$, $1 \leq p \leq \infty$. Denote by $BV(I; V)$ the space of functions of bounded total variation on I defined as follows. Let π denote a finite partition of $\bar{I} = [0, T]$: $0 = a_0 < a_1 < \dots < a_{n(\pi)} = T$, where $n(\pi)$ is a positive integer, and let $\mathcal{P}$ be the collection of all such partitions. Then, we define the total variation of a function $v: \bar{I} \rightarrow V$ as

$$\|v\|_{BV(I; V)} = \sup_{\pi \in \mathcal{P}} \sum_{i=1}^{n(\pi)} \|v(a_i) - v(a_{i-1})\|_V.$$

In general, for $1 \leq q < \infty$, we define

$$\|v\|_{BV^q(I; V)}^q = \sup_{\pi \in \mathcal{P}} \sum_{i=1}^{n(\pi)} \|v(a_i) - v(a_{i-1})\|_V^q$$

and let $BV^q(I; V)$ be the space of functions $v: \bar{I} \rightarrow V$ such that the norm $\|v\|_{BV^q(I; V)} < \infty$.

For Banach spaces V_1 and V_2 such that $V_1 \subset V_2$, we introduce a space

$$M^{p,q}(I; V_1, V_2) = L^p(I; V_1) \cap BV^q(I; V_2)$$

which is a Banach space for $1 \leq p, q < \infty$ with the norm given by $\|\cdot\|_{L^p(I; V_1)} + \|\cdot\|_{BV^q(I; V_2)}$.

The following result is crucial in proving the convergence of the Rothe method.

Theorem 2.2. (Kalita [14]) *Let $1 \leq p, q < \infty$. Let $V_1 \subset V_2 \subset V_3$ be real Banach spaces such that V_1 is reflexive, the embedding $V_1 \subset V_2$ is compact and the embedding $V_2 \subset V_3$ is continuous. Then, a bounded subset of $M^{p,q}(I; V_1, V_3)$ is relatively compact in $L^p(I; V_2)$.*

We will need the modified Cauchy–Schwarz inequality, with a proper choice of $\delta > 0$,

$$ab \leq \delta a^2 + c_\delta b^2 \quad \forall a, b \in \mathbb{R}, \delta > 0, c_\delta = 1/(4\delta). \quad (1)$$

3. Mathematical model

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a Lipschitz domain. Its boundary Γ is split into two non-trivial non-overlapping parts: $\Gamma = \Gamma_D \cup \Gamma_S$ with Γ_D relatively closed and Γ_S relatively open. Let $[0, T]$ be the time interval for a given $T > 0$. Since the boundary is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu} = (\nu_1, \dots, \nu_d)^T$ exists a.e. on Γ . For a vector $\mathbf{v}$, we denote by $\nu_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$ its normal component and $\mathbf{v}_\tau = \mathbf{v} - \nu_\nu \boldsymbol{\nu}$ its tangential component on Γ .

We consider the motion of an incompressible fluid flow in Ω . We use $\mathbf{u}$ and p for the velocity and pressure fields of the fluid. Denote by $\mu > 0$ the viscosity of the fluid. We use the Stokes equations to model the flow. Let $\mathbf{f}$ be the density of the volume force applied to the fluid and let $\mathbf{u}_0: \Omega \rightarrow \mathbb{R}^d$ be the initial velocity. Denote the stress tensor $\boldsymbol{\sigma} = 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) - p\mathbf{I}$, where $\boldsymbol{\varepsilon}(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$ and $\mathbf{I}$ is the identity tensor. The normal and tangential components of the stress on the boundary are $\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. We impose the homogeneous Dirichlet boundary condition on Γ_D and a non-smooth, possibly non-monotone, friction condition with a potential function ψ on Γ_S . The potential function ψ is

assumed to be locally Lipschitz continuous. The classical pointwise formulation of the problem is

$$\mathbf{u}' - 2\mu \operatorname{div} \boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } (0, T) \times \Omega, \quad (2)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } (0, T) \times \Omega, \quad (3)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } (0, T) \times \Gamma_D, \quad (4)$$

$$u_\nu = 0, \quad -\sigma_\tau \in \partial\psi(\mathbf{u}_\tau) \quad \text{on } (0, T) \times \Gamma_S, \quad (5)$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega. \quad (6)$$

In (5), $\partial\psi$ denotes the generalized subdifferential of ψ in the sense of Clarke, reviewed in Section 2.

We introduce weak formulations of the problem (2)–(6) in this section. For this purpose, define the following function spaces:

$$\mathbf{V} = \left\{ \mathbf{v} \in H^1(\Omega)^d \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D, \ v_\nu = 0 \text{ on } \Gamma_S \right\}, \quad (7)$$

$$\mathcal{Q} = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \mid (q, 1)_{L^2(\Omega)} = 0 \right\}, \quad (8)$$

$$\mathbf{H} = L^2(\Omega)^d, \quad (9)$$

and

$$\mathcal{V} = L^2(0, T; \mathbf{V}), \quad (10)$$

$$\mathcal{W} = \left\{ \mathbf{v} \in \mathcal{V} \mid \mathbf{v}' \in \mathcal{V}^* \right\}, \quad (11)$$

$$\mathcal{Q} = L^2(0, T; \mathcal{Q}). \quad (12)$$

Since $|\Gamma_D| > 0$, Korn's inequality holds: for a constant $c > 0$, depending on Ω and Γ_D ,

$$\|\mathbf{v}\|_{H^1(\Omega)^d} \leq c \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{L^2(\Omega)^{d \times d}} \quad \forall \mathbf{v} \in \mathbf{V}.$$

Its proof can be found in numerous publications, e.g., [22, p. 79]. Consequently, $\mathbf{V}$ is a Hilbert space with the inner product

$$(\mathbf{u}, \mathbf{v})_{\mathbf{V}} := \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V},$$

and the induced norm $\|\mathbf{v}\|_{\mathbf{V}} = (\mathbf{v}, \mathbf{v})_{\mathbf{V}}^{1/2}$ is equivalent to the standard $H^1(\Omega)^d$ -norm over $\mathbf{V}$. The spaces $\mathcal{Q}$, $\mathbf{H}$, $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{Q}$ are all Hilbert spaces with their standard inner products. We will also need the following subspace of $\mathbf{V}$:

$$\mathbf{V}_0 := H_0^1(\Omega)^d. \quad (13)$$

We have the trace inequality

$$\|\mathbf{v}_\tau\|_{L^2(\Gamma_S)^d} \leq \lambda_\tau^{-1/2} \|\mathbf{v}\|_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}, \quad (14)$$

where $\lambda_\tau > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\mathbf{u} \in \mathbf{V}, \quad (\mathbf{u}, \mathbf{v})_{\mathbf{V}} = \lambda (\mathbf{u}_\tau, \mathbf{v}_\tau)_{L^2(\Gamma_S)^d} \quad \forall \mathbf{v} \in \mathbf{V}.$$

Define the bilinear forms

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}, \quad (15)$$

$$b(\mathbf{v}, q) = \int_{\Omega} q \operatorname{div} \mathbf{v} \, dx \quad \forall \mathbf{v} \in \mathbf{V}, \ q \in \mathcal{Q}. \quad (16)$$

Obviously, the bilinear form $a(\cdot, \cdot): V \times V \rightarrow \mathbb{R}$ is symmetric, continuous, and V -elliptic:

$$\begin{aligned} |a(\mathbf{u}, \mathbf{v})| &\leq 2\mu \|\mathbf{u}\|_V \|\mathbf{v}\|_V \quad \forall \mathbf{u}, \mathbf{v} \in V, \\ a(\mathbf{v}, \mathbf{v}) &= 2\mu \|\mathbf{v}\|_V^2 \quad \forall \mathbf{v} \in V. \end{aligned}$$

The bilinear form $b(\cdot, \cdot): V \times Q \rightarrow \mathbb{R}$ is continuous: for a constant $c_b > 0$,

$$|b(\mathbf{v}, q)| \leq c_b \|\mathbf{v}\|_V \|q\|_Q \quad \forall \mathbf{v} \in V, q \in Q.$$

Moreover, we have the inf-sup property [23–25]: for a positive constant $\alpha_b > 0$,

$$\sup_{\mathbf{0} \neq \mathbf{v} \in V_0} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_V} \geq \alpha_b \|q\|_Q \quad \forall q \in Q. \quad (17)$$

Define operators $A: V \rightarrow V^*$ and $B: V \rightarrow Q^*$ by

$$\begin{aligned} \langle A\mathbf{u}, \mathbf{v} \rangle &= a(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in V, \\ \langle B\mathbf{v}, q \rangle &= b(\mathbf{v}, q) \quad \forall \mathbf{v} \in V, q \in Q. \end{aligned}$$

Then, $A \in \mathcal{L}(V; V^*)$ is self-adjoint and

$$\langle A\mathbf{v}, \mathbf{v} \rangle = 2\mu \|\mathbf{v}\|_V^2 \quad \forall \mathbf{v} \in V.$$

Concerning the data, we assume

$$\mathbf{f} \in V^*, \quad \mathbf{u}_0 \in V, \quad (18)$$

and

$H(\psi): \psi: \Gamma_S \times (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}$ is such that

- (i) $\psi(\cdot, \cdot, \xi)$ is measurable on $\Gamma_S \times (0, T)$ for all $\xi \in \mathbb{R}^d$ and there exists $\xi_0 \in L^2(\Gamma_S)^d$ such that $\psi(\cdot, t, \xi_0(\cdot)) \in L^1(\Gamma_S)$ for a.e. $t \in (0, T)$;
- (ii) $\psi(\mathbf{x}, t, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $(\mathbf{x}, t) \in \Gamma_S \times (0, T)$;
- (iii) $|\eta| \leq c_0(1 + |\xi|)$ for all $\xi \in \mathbb{R}^d$, $\eta \in \partial\psi(\mathbf{x}, t, \xi)$, a.e. $(\mathbf{x}, t) \in \Gamma_S \times (0, T)$ with $c_0 > 0$;
- (iv) $(\eta_1 - \eta_2)(\xi_1 - \xi_2) \geq -\alpha_\psi |\xi_1 - \xi_2|^2$ for all $\eta_i \in \partial\psi(\mathbf{x}, t, \xi_i)$, $\xi_i \in \mathbb{R}^d$, $i = 1, 2$, a.e. $(\mathbf{x}, t) \in \Gamma_S \times (0, T)$ with a constant $\alpha_\psi > 0$.

For simplicity in writing, we will use $I_{\Gamma_S}(z)$ to mean the integral of a function z over Γ_S .

Lemma 3.1. Assume $H(\psi)$. Then,

$$I_{\Gamma_S}(\psi^0(\mathbf{v}_{1,\tau}; \mathbf{v}_{2,\tau} - \mathbf{v}_{1,\tau})) + \psi^0(\mathbf{v}_{2,\tau}; \mathbf{v}_{1,\tau} - \mathbf{v}_{2,\tau}) \leq \alpha_\psi \lambda_\tau^{-1} \|\mathbf{v}_1 - \mathbf{v}_2\|_V^2 \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in V.$$

Proof. The condition $H(\psi)$ (iv) is equivalent to

$$\psi^0(\mathbf{z}_1; \mathbf{z}_2 - \mathbf{z}_1) + \psi^0(\mathbf{z}_2; \mathbf{z}_1 - \mathbf{z}_2) \leq \alpha_\psi |\mathbf{z}_1 - \mathbf{z}_2|^2 \quad \forall \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^d,$$

cf. [5, p. 124] or [6, Proposition 2.42]. Hence,

$$I_{\Gamma_S}(\psi^0(\mathbf{v}_{1,\tau}; \mathbf{v}_{2,\tau} - \mathbf{v}_{1,\tau})) + \psi^0(\mathbf{v}_{2,\tau}; \mathbf{v}_{1,\tau} - \mathbf{v}_{2,\tau}) \leq \alpha_\psi I_{\Gamma_S}(|\mathbf{v}_{1,\tau} - \mathbf{v}_{2,\tau}|^2),$$

which is further bounded by $\alpha_\psi \lambda_\tau^{-1} \|\mathbf{v}_1 - \mathbf{v}_2\|_V^2$ thanks to (14). $\square$

Through a standard procedure, we get the following weak formulation of the problem (2)–(6).

Problem 3.2. Find $\mathbf{u} \in \mathcal{W}$ and $p \in \mathcal{Q}$ such that for a.e. $t \in (0, T)$,

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p(t)) + I_{\Gamma_S}(\psi^0(\mathbf{u}_\tau(t); \mathbf{v}_\tau)) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}, \quad (19)$$

$$b(\mathbf{u}(t), q) = 0 \quad \forall q \in \mathcal{Q}, \quad (20)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (21)$$

Note that in (19), $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $\mathbf{V}^*$ and $\mathbf{V}$.

By allowing $\mathbf{v}$ and q to depend on t and integrating (19) and (20) with respect to t , we obtain another weak formulation of the problem (2)–(6).

Problem 3.3. Find $\mathbf{u} \in \mathcal{W}$ and $p \in \mathcal{Q}$ such that

$$\begin{aligned} & \int_0^T \left[\langle \mathbf{u}'(t), \mathbf{v}(t) \rangle + a(\mathbf{u}(t), \mathbf{v}(t)) - b(\mathbf{v}(t), p(t)) + I_{\Gamma_S}(\psi^0(\mathbf{u}_\tau(t); \mathbf{v}_\tau(t))) \right] dt \\ & \geq \int_0^T \langle \mathbf{f}(t), \mathbf{v}(t) \rangle dt \quad \forall \mathbf{v} \in \mathcal{V}, \end{aligned} \quad (22)$$

$$\int_0^T b(\mathbf{u}(t), q(t)) dt = 0 \quad \forall q \in \mathcal{Q}, \quad (23)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (24)$$

Proposition 3.4. Problem 3.2 and Problem 3.3 are equivalent.

Proof. Let $(\mathbf{u}, p)$ be a solution of Problem 3.2. For arbitrarily fixed $\mathbf{v} \in \mathcal{V}$ and $q \in \mathcal{Q}$, we have, for a.e. $t \in (0, T)$,

$$\begin{aligned} & \langle \mathbf{u}'(t), \mathbf{v}(t) \rangle + a(\mathbf{u}(t), \mathbf{v}(t)) - b(\mathbf{v}(t), p(t)) + I_{\Gamma_S}(\psi^0(\mathbf{u}_\tau(t); \mathbf{v}_\tau(t))) \geq \langle \mathbf{f}(t), \mathbf{v}(t) \rangle, \\ & b(\mathbf{u}(t), q(t)) = 0. \end{aligned}$$

Integrating these two relations for t from 0 to T , we derive (22) and (23).

Conversely, for any solution $(\mathbf{u}, p)$ of Problem 3.3, let us prove that $(\mathbf{u}, p)$ is a solution of Problem 3.2. We argue by contradiction. Suppose there exist a subset $\Delta \subset [0, T]$ with a positive measure and an element $\mathbf{w} \in \mathbf{V}$ such that

$$\langle \mathbf{u}'(t), \mathbf{w} \rangle + a(\mathbf{u}(t), \mathbf{w}) - b(\mathbf{w}, p(t)) + I_{\Gamma_S}(\psi^0(\mathbf{u}_\tau(t); \mathbf{w}_\tau)) < \langle \mathbf{f}(t), \mathbf{w} \rangle \quad \forall t \in \Delta. \quad (25)$$

Define

$$\mathbf{v}(t) = \begin{cases} \mathbf{w} & \text{if } t \in \Delta, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then, $\mathbf{v} \in \mathcal{V}$ and we use this $\mathbf{v}$ in (22) to obtain

$$\int_{\Delta} \left[\langle \mathbf{u}'(t), \mathbf{w} \rangle + a(\mathbf{u}(t), \mathbf{w}) - b(\mathbf{w}, p(t)) + I_{\Gamma_S}(\psi^0(\mathbf{u}_\tau(t); \mathbf{w}_\tau)) \right] dt \geq \int_{\Delta} \langle \mathbf{f}(t), \mathbf{w} \rangle dt.$$

This contradicts (25). Similarly, (23) leads to (20) for a.e. $t \in (0, T)$. $\square$

4. Existence and uniqueness

We show the existence of a solution to Problem 3.2. This is achieved through the consideration of a temporally semi-discrete approximation of Problem 3.2 based on the backward Euler difference for the time derivative; such an approximation is also known as the Rothe method. For a fixed positive integer $N \in \mathbb{N}$, define the time step-size $k = T/N$. Denote the nodes $t_n = nk$, $0 \leq n \leq N$. Introduce the piecewise constant interpolant of f by

$$f_n^k = \frac{1}{k} \int_{t_{n-1}}^{t_n} f(t) dt, \quad n = 1, \dots, N.$$

The semi-discrete approximation of Problem 3.2 is the following.

Problem 4.1. Find $\{u_n^k\}_{n=0}^N \subset V$ and $\{p_n^k\}_{n=1}^N \subset Q$ such that for $n = 1, \dots, N$,

$$\frac{1}{k}(u_n^k - u_{n-1}^k, v)_H + a(u_n^k, v) - b(v, p_n^k) + I_{\Gamma_S}(\psi^0(u_n^k; v_\tau)) \geq \langle f_n^k, v \rangle \quad \forall v \in V, \quad (26)$$

$$b(u_n^k, q) = 0 \quad \forall q \in Q, \quad (27)$$

and

$$u_0^k = u_0. \quad (28)$$

From now on, we assume the smallness condition

$$m := 2\mu - \alpha_\psi \lambda_\tau^{-1} > 0. \quad (29)$$

Proposition 4.2. Assume $H(\psi)$, (18) and (29). Then, Problem 4.1 admits a unique solution $\{u_n^k\}_{n=0}^N \subset V$ and $\{p_n^k\}_{n=1}^N \subset Q$.

Proof. We prove the result by an induction. For $n = 0$, the element u_0^k is defined by (28). For an n between 1 and N , assume $u_{n-1}^k \in V$ is known. We rewrite (26) as

$$(u_n^k, v)_H + k a(u_n^k, v) - k b(v, p_n^k) + k I_{\Gamma_S}(\psi^0(u_n^k; v_\tau)) \geq (u_{n-1}^k, v)_H + k \langle f_n^k, v \rangle \quad \forall v \in V. \quad (30)$$

Applying [26, Theorem 4.9] or [6, Theorem 7.14], we know that under the smallness condition (29), the problem (30) and (27) has a solution $(u_n^k, p_n^k) \in V \times Q$ and u_n^k is unique.

To show the uniqueness of p_n^k , we notice the following consequence of (30):

$$(u_n^k, v)_H + k a(u_n^k, v) - k b(v, p_n^k) = (u_{n-1}^k, v)_H + k \langle f_n^k, v \rangle \quad \forall v \in V_0. \quad (31)$$

Assume both $(u_n^k, p_n^k), (u_n^k, \tilde{p}_n^k) \in V \times Q$ satisfy (30) and (27). Then, (31) is valid with p_n^k by $\tilde{p}_n^k$:

$$(u_n^k, v)_H + k a(u_n^k, v) - k b(v, \tilde{p}_n^k) = (u_{n-1}^k, v)_H + k \langle f_n^k, v \rangle \quad \forall v \in V_0. \quad (32)$$

Subtract (32) from (31),

$$b(v, \tilde{p}_n^k - p_n^k) = 0 \quad \forall v \in V_0.$$

By the inf-sup property (17),

$$\alpha_b \|\tilde{p}_n^k - p_n^k\|_Q \leq \sup_{0 \neq v \in V_0} \frac{b(v, \tilde{p}_n^k - p_n^k)}{\|v\|_V} = 0.$$

Hence, $\|\tilde{p}_n^k - p_n^k\|_Q = 0$, i.e., the solution component p_n^k is unique. $\square$

Next, we explore the uniform boundedness of the solution of Problem 4.1 with respect to k . Define $\Psi: L^2(\Gamma_S)^d \rightarrow \mathbb{R}$ by the formula $\Psi(\mathbf{z}) = I_{\Gamma_S}(\psi(\mathbf{z}))$ for $\mathbf{z} \in L^2(\Gamma_S)^d$. Denote by $\gamma: \mathbf{V} \ni \mathbf{v} \mapsto \mathbf{v}_\tau \in L^2(\Gamma_S)^d$ the tangential trace operator and denote its adjoint by $\gamma^* \in \mathcal{L}(L^2(\Gamma_S)^d, \mathbf{V}^*)$. In the rest of the section, the assumption (18) on the source function $\mathbf{f}$ and the initial value $\mathbf{u}_0$ is strengthened to the following. (H_0) $\mathbf{f} \in H^1(0, T; \mathbf{V}^*)$, $\mathbf{u}_0 \in \mathbf{V}$, $\operatorname{div} \mathbf{u}_0 = 0$ in Ω , and there exists $\boldsymbol{\eta}_0 \in \partial\Psi(\mathbf{u}_{0,\tau})$ such that

$$\boldsymbol{\omega}_0 := A\mathbf{u}_0 + \gamma^*\boldsymbol{\eta}_0 - \mathbf{f}(0) \in \mathbf{H}.$$

Lemma 4.3. Assume $H(\psi)$, (H_0) and (29). Then, there exists a constant $c > 0$ such that for all $k > 0$, the solution of Problem 4.1 satisfies

$$\max_{1 \leq n \leq N} \left\| \frac{\mathbf{u}_n^k - \mathbf{u}_{n-1}^k}{k} \right\|_{\mathbf{H}} \leq c, \quad (33)$$

$$k \sum_{n=1}^N \left\| \frac{\mathbf{u}_n^k - \mathbf{u}_{n-1}^k}{k} \right\|_{\mathbf{V}}^2 \leq c. \quad (34)$$

Proof. First, consider (26) with $n = 1$,

$$\frac{1}{k}(\mathbf{u}_1^k - \mathbf{u}_0, \mathbf{v})_{\mathbf{H}} + a(\mathbf{u}_1^k, \mathbf{v}) - b(\mathbf{v}, p_1^k) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{1,\tau}^k; \mathbf{v}_\tau)) \geq \langle \mathbf{f}_1^k, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}.$$

Choose $\mathbf{v} = \mathbf{u}_0 - \mathbf{u}_1^k$ to get

$$\begin{aligned} & -\frac{1}{k} \|\mathbf{u}_1^k - \mathbf{u}_0\|_{\mathbf{H}}^2 + a(\mathbf{u}_1^k, \mathbf{u}_0 - \mathbf{u}_1^k) - b(\mathbf{u}_0 - \mathbf{u}_1^k, p_1^k) \\ & + I_{\Gamma_S}(\psi^0(\mathbf{u}_{1,\tau}^k; \mathbf{u}_{0,\tau} - \mathbf{u}_{1,\tau}^k)) \geq \langle \mathbf{f}_1^k, \mathbf{u}_0 - \mathbf{u}_1^k \rangle. \end{aligned} \quad (35)$$

Since $\operatorname{div} \mathbf{u}_0 = 0$ in Ω by (H_0) , we have $b(\mathbf{u}_0, p_1^k) = 0$. By (27), $b(\mathbf{u}_1, p_1^k) = 0$. We can rewrite the inequality (35) as

$$\begin{aligned} & \frac{1}{k} \|\mathbf{u}_1^k - \mathbf{u}_0\|_{\mathbf{H}}^2 + a(\mathbf{u}_1^k - \mathbf{u}_0, \mathbf{u}_1^k - \mathbf{u}_0) \\ & \leq -a(\mathbf{u}_0, \mathbf{u}_1^k - \mathbf{u}_0) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{1,\tau}^k; \mathbf{u}_{0,\tau} - \mathbf{u}_{1,\tau}^k)) + \langle \mathbf{f}_1^k, \mathbf{u}_1^k - \mathbf{u}_0 \rangle. \end{aligned}$$

Write

$$\begin{aligned} I_{\Gamma_S}(\psi^0(\mathbf{u}_{1,\tau}^k; \mathbf{u}_{0,\tau} - \mathbf{u}_{1,\tau}^k)) &= I_{\Gamma_S}(\psi^0(\mathbf{u}_{1,\tau}^k; \mathbf{u}_{0,\tau} - \mathbf{u}_{1,\tau}^k) + \psi^0(\mathbf{u}_{0,\tau}; \mathbf{u}_{1,\tau}^k - \mathbf{u}_{0,\tau})) \\ &\quad - I_{\Gamma_S}(\psi^0(\mathbf{u}_{0,\tau}; \mathbf{u}_{1,\tau}^k - \mathbf{u}_{0,\tau})). \end{aligned}$$

Apply Lemma 3.1,

$$I_{\Gamma_S}(\psi^0(\mathbf{u}_{1,\tau}^k; \mathbf{u}_{0,\tau} - \mathbf{u}_{1,\tau}^k)) \leq \alpha_\psi \lambda_\tau^{-1} \|\mathbf{u}_1^k - \mathbf{u}_0\|_{\mathbf{V}}^2 - I_{\Gamma_S}(\psi^0(\mathbf{u}_{0,\tau}; \mathbf{u}_{1,\tau}^k - \mathbf{u}_{0,\tau})).$$

Write

$$\begin{aligned} \langle \mathbf{f}_1^k, \mathbf{u}_1^k - \mathbf{u}_0 \rangle &= \langle \mathbf{f}_1^k - \mathbf{f}(0), \mathbf{u}_1^k - \mathbf{u}_0 \rangle + \langle \mathbf{f}(0), \mathbf{u}_1^k - \mathbf{u}_0 \rangle \\ &\leq \|\mathbf{f}_1^k - \mathbf{f}(0)\|_{\mathbf{V}^*} \|\mathbf{u}_1^k - \mathbf{u}_0\|_{\mathbf{V}} + \langle \mathbf{f}(0), \mathbf{u}_1^k - \mathbf{u}_0 \rangle. \end{aligned}$$

Apply the modified Cauchy–Schwarz inequality (1) with $\delta = m/2$,

$$\|f_1^k - f(0)\|_{V^*} \|u_1^k - u_0\|_V \leq (m/2) \|u_1^k - u_0\|_V^2 + c \|f_1^k - f(0)\|_{V^*}^2.$$

Then, we have

$$\begin{aligned} \frac{1}{k} \|u_1^k - u_0\|_H^2 + \frac{m}{2} \|u_1^k - u_0\|_V^2 &\leq c \|f_1^k - f(0)\|_{V^*}^2 - a(u_0, u_1^k - u_0) \\ &\quad - I_{\Gamma_S}(\psi^0(u_{0,\tau}; u_{1,\tau}^k - u_{0,\tau})) + \langle f(0), u_1^k - u_0 \rangle. \end{aligned} \quad (36)$$

Under the assumption (H_0) , we have

$$\begin{aligned} &-a(u_0, u_1^k - u_0) - I_{\Gamma_S}(\psi^0(u_{0,\tau}; u_{1,\tau}^k - u_{0,\tau})) + \langle f(0), u_1^k - u_0 \rangle \\ &\leq -a(u_0, u_1^k - u_0) - \Psi^0(u_{0,\tau}; u_{1,\tau}^k - u_{0,\tau}) + \langle f(0), u_1^k - u_0 \rangle \\ &\leq -(\omega_0, u_1^k - u_0)_H \leq \|\omega_0\|_H \|u_1^k - u_0\|_H \leq \frac{1}{2k} \|u_1^k - u_0\|_H^2 + \frac{k}{2} \|\omega_0\|_H^2. \end{aligned}$$

Thus, from (36),

$$\|u_1^k - u_0\|_H^2 + m k \|u_1^k - u_0\|_V^2 \leq \frac{k^2}{2} \|\omega_0\|_H^2 + c k \|f_1^k - f(0)\|_{V^*}^2. \quad (37)$$

Next, for $2 \leq n \leq N$, we take $v = u_{n-1}^k - u_n^k$ in (26),

$$\begin{aligned} &-\frac{1}{k} \|u_n^k - u_{n-1}^k\|_H^2 + a(u_n^k, u_{n-1}^k - u_n^k) - b(u_{n-1}^k - u_n^k, p_n^k) \\ &\quad + I_{\Gamma_S}(\psi^0(u_{n,\tau}^k; u_{n-1,\tau}^k - u_{n,\tau}^k)) \geq \langle f_n^k, u_{n-1}^k - u_n^k \rangle. \end{aligned} \quad (38)$$

In (26), replace n by $(n-1)$ and take $v = u_n^k - u_{n-1}^k$,

$$\begin{aligned} &\frac{1}{k} (u_{n-1}^k - u_{n-2}^k, u_n^k - u_{n-1}^k)_H + a(u_{n-1}^k, u_n^k - u_{n-1}^k) - b(u_n^k - u_{n-1}^k, p_{n-1}^k) \\ &\quad + I_{\Gamma_S}(\psi^0(u_{n-1,\tau}^k; u_{n,\tau}^k - u_{n-1,\tau}^k)) \geq \langle f_{n-1}^k, u_n^k - u_{n-1}^k \rangle. \end{aligned} \quad (39)$$

Note that $b(u_{n-1}^k - u_n^k, p_n^k) = b(u_n^k - u_{n-1}^k, p_{n-1}^k) = 0$ from (27) for any n between 1 and N . Add (38) and (39),

$$\begin{aligned} &\frac{1}{k} \|u_n^k - u_{n-1}^k\|_H^2 + a(u_n^k - u_{n-1}^k, u_n^k - u_{n-1}^k) \\ &\leq \frac{1}{k} (u_{n-1}^k - u_{n-2}^k, u_n^k - u_{n-1}^k)_H + \langle f_n^k - f_{n-1}^k, u_n^k - u_{n-1}^k \rangle \\ &\quad + I_{\Gamma_S}(\psi^0(u_{n,\tau}^k; u_{n-1,\tau}^k - u_{n,\tau}^k) + \psi^0(u_{n-1,\tau}^k; u_{n,\tau}^k - u_{n-1,\tau}^k)). \end{aligned}$$

Then,

$$\begin{aligned} \frac{1}{k} \|u_n^k - u_{n-1}^k\|_H^2 + 2\mu \|u_n^k - u_{n-1}^k\|_V^2 &\leq \frac{1}{k} \|u_{n-1}^k - u_{n-2}^k\|_H \|u_n^k - u_{n-1}^k\|_H \\ &\quad + \|f_n^k - f_{n-1}^k\|_{V^*} \|u_n^k - u_{n-1}^k\|_V \\ &\quad + \alpha_\psi \lambda_\tau^{-1} \|u_n^k - u_{n-1}^k\|_V^2. \end{aligned} \quad (40)$$

By the Cauchy–Schwarz inequality,

$$\|u_{n-1}^k - u_{n-2}^k\|_H \|u_n^k - u_{n-1}^k\|_H \leq \frac{1}{2} \|u_{n-1}^k - u_{n-2}^k\|_H^2 + \frac{1}{2} \|u_n^k - u_{n-1}^k\|_H^2.$$

Applying the modified Cauchy–Schwarz inequality (1) with $\delta = m/2$, we have

$$\|f_n^k - f_{n-1}^k\|_{V^*} \|u_n^k - u_{n-1}^k\|_V \leq (m/2) \|u_n^k - u_{n-1}^k\|_V^2 + c \|f_n^k - f_{n-1}^k\|_{V^*}^2.$$

Hence, we derive from equation (40) that

$$\|u_n^k - u_{n-1}^k\|_H^2 + m k \|u_n^k - u_{n-1}^k\|_V^2 \leq \|u_{n-1}^k - u_{n-2}^k\|_H^2 + c k \|f_n^k - f_{n-1}^k\|_{V^*}^2. \quad (41)$$

Now change n to j in (41) and add the inequality for $j = 2, \dots, n$ with (37) to obtain

$$\|u_n^k - u_{n-1}^k\|_H^2 + m k \sum_{j=1}^n \|u_j^k - u_{j-1}^k\|_V^2 \leq \frac{k^2}{2} \|\omega_0\|_H^2 + c k \sum_{j=1}^n \|f_j^k - f_{j-1}^k\|_{V^*}^2. \quad (42)$$

By Bartosz et al. [15],

$$\sum_{j=1}^n \|f_j^k - f_{j-1}^k\|_{V^*}^2 \leq \sum_{j=1}^N \|f_j^k - f_{j-1}^k\|_{V^*}^2 \leq k \|f'\|_{V^*}^2.$$

Hence, from equation (42),

$$\left\| \frac{u_n^k - u_{n-1}^k}{k} \right\|_H^2 + m k \sum_{j=1}^n \left\| \frac{u_j^k - u_{j-1}^k}{k} \right\|_V^2 \leq \frac{1}{2} \|\omega_0\|_H^2 + c \|f'\|_{V^*}^2,$$

which implies (33) and (34). □

Corollary 4.4. Assume $H(\psi)$, (H_0) and equation (29). There exists a constant $c > 0$ such that

$$\max_{1 \leq n \leq N} \|u_n^k\|_H \leq c, \quad (43)$$

$$\sum_{n=1}^N \|u_n^k - u_{n-1}^k\|_H \leq c, \quad (44)$$

$$\max_{1 \leq n \leq N} \|u_n^k\|_V \leq c, \quad (45)$$

$$\max_{1 \leq n \leq N} \|p_n^k\|_Q \leq c. \quad (46)$$

Proof. The bound (44) follows straightforward from (33). The bound (43) follows from the bound (45). To prove (45), we start with

$$u_n^k = u_0 + k \sum_{j=1}^n \frac{u_j^k - u_{j-1}^k}{k},$$

$$\|u_n^k\|_V \leq \|u_0\|_V + k \sum_{j=1}^n \left\| \frac{u_j^k - u_{j-1}^k}{k} \right\|_V.$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} k \sum_{j=1}^n \left\| \frac{\mathbf{u}_j^k - \mathbf{u}_{j-1}^k}{k} \right\|_{\mathbf{V}} &\leq \left(k \sum_{j=1}^n 1 \right)^{1/2} \left(k \sum_{j=1}^n \left\| \frac{\mathbf{u}_j^k - \mathbf{u}_{j-1}^k}{k} \right\|_{\mathbf{V}}^2 \right)^{1/2} \\ &\leq T^{1/2} \left(k \sum_{j=1}^N \left\| \frac{\mathbf{u}_j^k - \mathbf{u}_{j-1}^k}{k} \right\|_{\mathbf{V}}^2 \right)^{1/2}, \end{aligned}$$

which is bounded by a constant, cf. (34). Hence, (45) holds.

From (26), we have

$$b(\mathbf{v}, p_n^k) = \frac{1}{k} (\mathbf{u}_n^k - \mathbf{u}_{n-1}^k, \mathbf{v})_{\mathbf{H}} + a(\mathbf{u}_n^k, \mathbf{v}) - \langle \mathbf{f}_n^k, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}_0.$$

Applying the inf-sup condition (17),

$$\|p_n^k\|_{\mathcal{Q}} \leq c \sup_{\mathbf{v} \in \mathbf{V}_0} \frac{b(\mathbf{v}, p_n^k)}{\|\mathbf{v}\|_{\mathbf{V}}} \leq c \left(\left\| \frac{\mathbf{u}_n^k - \mathbf{u}_{n-1}^k}{k} \right\|_{\mathbf{H}} + \|\mathbf{u}_n^k\|_{\mathbf{V}} + \|\mathbf{f}_n^k\|_{\mathbf{V}^*} \right).$$

We derive (46) from the above inequality and (33), (45). $\square$

We turn to a convergence analysis of the Rothe method. Define a continuous piecewise linear function $\mathbf{u}_k: [0, T] \rightarrow \mathbf{V}$ and three piecewise constant functions $\bar{\mathbf{u}}_k: [0, T] \rightarrow \mathbf{V}$, $\bar{p}_k: [0, T] \rightarrow \mathcal{Q}$, and $\bar{\mathbf{f}}_k: [0, T] \rightarrow \mathbf{V}^*$ by

$$\mathbf{u}_k(t) = \mathbf{u}_n^k + \left(\frac{t}{k} - n \right) (\mathbf{u}_n^k - \mathbf{u}_{n-1}^k), \quad \bar{\mathbf{u}}_k(t) = \mathbf{u}_n^k, \quad \bar{p}_k(t) = p_n^k, \quad \bar{\mathbf{f}}_k(t) = \mathbf{f}_n^k$$

for $t \in (t_{n-1}, t_n]$, $1 \leq n \leq N$, and

$$\mathbf{u}_k(0) = \mathbf{u}_0, \quad \bar{\mathbf{u}}_k(0) = \mathbf{u}_0, \quad \bar{\mathbf{f}}_k(0) = \mathbf{f}(0).$$

Rewrite (26) and (27) as, a.e. $t \in (0, T)$,

$$(\mathbf{u}'_k(t), \mathbf{v}) + a(\bar{\mathbf{u}}_k(t), \mathbf{v}) - b(\mathbf{v}, \bar{p}_k(t)) + I_{\Gamma_S}(\psi^0(\bar{\mathbf{u}}_k(t); \mathbf{v}_\tau)) \geq \langle \bar{\mathbf{f}}_k(t), \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}, \quad (47)$$

$$b(\bar{\mathbf{u}}_k(t), q) = 0 \quad \forall q \in \mathcal{Q}. \quad (48)$$

Define Nemytskii operators $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{V}^*$ and $\mathcal{B}: \mathcal{V} \rightarrow \mathcal{Q}^*$ by

$$\begin{aligned} \langle \mathcal{A}\mathbf{u}, \mathbf{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} &= \int_0^T a(\mathbf{u}(t), \mathbf{v}(t)) dt \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}, \\ \langle \mathcal{B}\mathbf{v}, q \rangle_{\mathcal{Q}^* \times \mathcal{Q}} &= \int_0^T b(\mathbf{v}(t), q(t)) dt \quad \forall \mathbf{v} \in \mathcal{V}, q \in \mathcal{Q}. \end{aligned}$$

Then, $\mathcal{A} \in \mathcal{L}(\mathcal{V}; \mathcal{V}^*)$, $\mathcal{B} \in \mathcal{L}(\mathcal{V}; \mathcal{Q}^*)$. We derive from equations (47)–(48) that

$$\begin{aligned} &(\mathbf{u}'_k, \mathbf{v})_{\mathcal{H}} + \langle \mathcal{A}\bar{\mathbf{u}}_k, \mathbf{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} - \langle \mathcal{B}\mathbf{v}, \bar{p}_k \rangle_{\mathcal{Q}^* \times \mathcal{Q}} \\ &\quad + I_{(0,T) \times \Gamma_S}(\psi^0(\bar{\mathbf{u}}_{k,\tau}; \mathbf{v}_\tau)) \geq \langle \bar{\mathbf{f}}_k, \mathbf{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} \quad \forall \mathbf{v} \in \mathcal{V}, \end{aligned} \quad (49)$$

$$\langle \mathcal{B}\bar{\mathbf{u}}_k, q \rangle_{\mathcal{Q}^* \times \mathcal{Q}} = 0 \quad \forall q \in \mathcal{Q}. \quad (50)$$

Let us establish some bounds on $\{\bar{\mathbf{u}}_k\}$, $\{\mathbf{u}_k\}$, and $\{\bar{\mathbf{p}}_k\}$ next.

Lemma 4.5. Assume $H(\psi)$, (H_0) and (29). There exists a constant $c > 0$ such that

$$\|\bar{\mathbf{u}}_k\|_{L^\infty(0,T;\mathbf{H})} \leq c, \quad (51)$$

$$\|\bar{\mathbf{u}}_k\|_{\mathcal{V}} \leq c, \quad (52)$$

$$\|\bar{\mathbf{u}}_k\|_{BV(0,T;\mathbf{H})} \leq c, \quad (53)$$

$$\|\mathbf{u}_k\|_{C([0,T];\mathbf{H})} \leq c, \quad (54)$$

$$\|\mathbf{u}_k\|_{\mathcal{V}} \leq c, \quad (55)$$

$$\|\mathbf{u}'_k\|_{\mathcal{V}} \leq c, \quad (56)$$

$$\|\bar{\mathbf{p}}_k\|_{L^\infty(0,T;Q)} \leq c. \quad (57)$$

Proof. The bound (51) is a direct consequence of (43). The bound (52) follows from equation (45):

$$\|\bar{\mathbf{u}}_k\|_{\mathcal{V}}^2 = \sum_{n=1}^N \|\mathbf{u}_n^k\|_{\mathcal{V}}^2 k \leq c.$$

The bound (53) is derived from (44) as follows. By definition,

$$\|\bar{\mathbf{u}}_k\|_{BV(0,T;\mathbf{H})} = \sup_{0=s_0 < s_1 < \dots < s_{i_0}=T} \sum_{i=1}^{i_0} \|\bar{\mathbf{u}}_k(s_i) - \bar{\mathbf{u}}_k(s_{i-1})\|_{\mathbf{H}},$$

where the supremum is taken with respect to all partitions of $[0, T]$. For a particular partition $0 = s_0 < s_1 < \dots < s_{i_0} = T$, we have positive integers $m_1 \leq m_2 \leq \dots \leq m_{i_0} = N$ such that $s_i \in ((m_i - 1)k, m_i k]$, $1 \leq i \leq i_0$. Then,

$$\bar{\mathbf{u}}_k(s_i) - \bar{\mathbf{u}}_k(s_{i-1}) = \mathbf{u}_{m_i}^k - \mathbf{u}_{m_{i-1}}^k$$

and

$$\sum_{i=1}^{i_0} \|\bar{\mathbf{u}}_k(s_i) - \bar{\mathbf{u}}_k(s_{i-1})\|_{\mathbf{H}} = \sum_{i=1}^{i_0} \|\mathbf{u}_{m_i}^k - \mathbf{u}_{m_{i-1}}^k\|_{\mathbf{H}} \leq \sum_{i=1}^{i_0} \sum_{j=m_{i-1}+1}^{m_i} \|\mathbf{u}_j^k - \mathbf{u}_{j-1}^k\|_{\mathbf{H}} = \sum_{n=1}^N \|\mathbf{u}_n^k - \mathbf{u}_{n-1}^k\|_{\mathbf{H}}$$

which is uniformly bounded with respect to k by (44). Thus, (53) holds.

For any $t \in (t_{n-1}, t_n]$, $1 \leq n \leq N$, by (43),

$$\|\mathbf{u}_k(t)\|_{\mathbf{H}} \leq \max \left\{ \|\mathbf{u}_n^k\|_{\mathbf{H}}, \|\mathbf{u}_{n-1}^k\|_{\mathbf{H}} \right\} \leq c.$$

Notice that $\mathbf{u}_k(t)$ is continuous in t . So (54) holds. Next,

$$\|\mathbf{u}_k\|_{\mathcal{V}}^2 = \int_0^T \|\mathbf{u}_k(t)\|_{\mathcal{V}}^2 dt = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left\| \mathbf{u}_n^k + \left(\frac{t}{k} - n \right) (\mathbf{u}_n^k - \mathbf{u}_{n-1}^k) \right\|_{\mathcal{V}}^2 dt \leq c \sum_{n=0}^N k \|\mathbf{u}_n^k\|_{\mathcal{V}}^2$$

which is uniformly bounded with respect to k by (45). So (55) holds. From the formula

$$\mathbf{u}'_k(t) = \frac{\mathbf{u}_n^k - \mathbf{u}_{n-1}^k}{k}, \quad t \in (t_{n-1}, t_n], \quad 1 \leq n \leq N,$$

we find

$$\|u'_k\|_{\mathcal{V}}^2 = \int_0^T \|u'_k(t)\|_{\mathcal{V}}^2 dt = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left\| \frac{u_n^k - u_{n-1}^k}{k} \right\|_{\mathcal{V}}^2 dt = k \sum_{n=1}^N \left\| \frac{u_n^k - u_{n-1}^k}{k} \right\|_{\mathcal{V}}^2$$

which is uniformly bounded with respect to k by (34). So (56) holds.

Finally, (57) follows from (46). $\square$

Lemma 4.6. Assume $H(\psi)$, (H_0) and (29). For a subsequence $\{k\}$, we have, as $k \rightarrow 0$,

$$\bar{u}_k \rightharpoonup u \quad \text{in } \mathcal{V}, \quad (58)$$

$$\bar{u}_k \rightarrow u \quad \text{in } \mathcal{H}, \quad (59)$$

$$\bar{u}_{k,\tau} \rightarrow u_\tau \quad \text{in } L^2(0, T; L^2(\Gamma_S)^d), \quad (60)$$

$$u_k \rightharpoonup u \quad \text{in } \mathcal{V}, \quad (61)$$

$$u'_k \rightharpoonup u' \quad \text{in } \mathcal{V} \text{ and in } \mathcal{H}, \quad (62)$$

$$\bar{p}_k \rightarrow p \quad \text{in } \mathcal{Q}, \quad (63)$$

and (u, p) solves Problem 3.3.

Proof. Since $\{\|\bar{u}_k\|_{\mathcal{V}}\}$ is bounded by (52) and $\mathcal{V}$ is reflexive, we have the existence of a subsequence, still denoted as $\{\bar{u}_k\}$, and an element $u \in \mathcal{V}$ such that (58) holds.

By (52) and (53), $\{\|\bar{u}_k\|_{M^{2,1}(0,T;\mathcal{V},\mathbf{H})}\}$ is bounded. Recall Theorem 2.2 and apply it with the compact embedding

$$M^{2,1}(0, T; \mathcal{V}, \mathbf{H}) \hookrightarrow L^2(0, T; (H^{3/5}(\Omega))^d) \subset \mathcal{H}.$$

Hence, we have a further subsequence, still denoted as $\{\bar{u}_k\}$, such that (59) and (60) are valid. Note that the tangential component trace operator $v \mapsto v_\tau$ is continuous from $(H^{3/5}(\Omega))^d$ to $L^2(\Gamma_S)^d$ as a consequence of the continuity of the trace operator from $H^{3/5}(\Omega)$ to $L^2(\partial\Omega)$ (cf. [27, Theorem 3.37]).

Similarly, by the boundedness of $\{\|u_k\|_{\mathcal{V}}\}$ and $\{\|u'_k\|_{\mathcal{V}}\}$, we have a yet further subsequence $\{u_k\}$ and an element $w \in \mathcal{V}$ with $w' \in \mathcal{V}$ such that

$$u_k \rightharpoonup w \quad \text{in } \mathcal{V}, \quad u'_k \rightharpoonup w' \quad \text{in } \mathcal{V}.$$

Now that

$$\|\bar{u}_k - u_k\|_{\mathcal{V}}^2 = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (t - t_n)^2 \left\| \frac{u_n^k - u_{n-1}^k}{k} \right\|_{\mathcal{V}}^2 dt = \frac{k^3}{3} \sum_{n=1}^N \left\| \frac{u_n^k - u_{n-1}^k}{k} \right\|_{\mathcal{V}}^2 = \frac{k^2}{3} \|u'_k\|_{\mathcal{V}}^2$$

tending to 0 as $k \rightarrow 0$, $w = u$ and we conclude (61) and (62).

From (57), $\{\|\bar{p}_k\|_{\mathcal{Q}}\}$ is bounded. Due to the reflexivity of the space $\mathcal{Q}$, we have a still further subsequence $\{\bar{p}_k\}$ and an element $p \in \mathcal{Q}$ such that (63) holds.

Now we consider the limits of each term in (49) and (50) as $k \rightarrow 0$ along the still further subsequence. Fix an arbitrary element $v \in \mathcal{V}$. By (62),

$$(u'_k, v)_{\mathcal{H}} \rightarrow (u', v)_{\mathcal{H}}.$$

From the continuity of $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{V}^*$ and (58),

$$\langle \mathcal{A}\bar{u}_k, v \rangle_{\mathcal{V}^* \times \mathcal{V}} \rightarrow \langle \mathcal{A}u, v \rangle_{\mathcal{V}^* \times \mathcal{V}}.$$

By (63),

$$\langle \mathcal{B}v, \bar{p}_k \rangle_{\mathcal{Q}^* \times \mathcal{Q}} \rightarrow \langle \mathcal{B}v, p \rangle_{\mathcal{Q}^* \times \mathcal{Q}}.$$

From (60), resorting to a further subsequence if necessary, we have

$$\bar{\mathbf{u}}_{k,\tau} \rightarrow \mathbf{u}_\tau \quad \text{a.e. in } (0, T) \times \Gamma_S.$$

Then, by Proposition 2.1 (iv),

$$\psi^0(\mathbf{u}_\tau; \mathbf{v}_\tau) \geq \limsup \psi^0(\bar{\mathbf{u}}_{k,\tau}; \mathbf{v}_\tau) \quad \text{a.e. in } (0, T) \times \Gamma_S.$$

Hence, by applying Fatou's lemma,

$$I_{(0,T) \times \Gamma_S}(\psi^0(\mathbf{u}_\tau; \mathbf{v}_\tau)) \geq I_{(0,T) \times \Gamma_S}(\limsup \psi^0(\bar{\mathbf{u}}_{k,\tau}; \mathbf{v}_\tau)) \geq \limsup I_{(0,T) \times \Gamma_S}(\psi^0(\bar{\mathbf{u}}_{k,\tau}; \mathbf{v}_\tau)).$$

By [28, Lemma 3.3],

$$\bar{\mathbf{f}}_k \rightarrow \mathbf{f} \quad \text{in } \mathcal{V}^*.$$

Hence,

$$\langle \bar{\mathbf{f}}_k, \mathbf{v} \rangle_{\mathcal{V}^* \times \mathcal{V}} \rightarrow \langle \mathbf{f}, \mathbf{v} \rangle_{\mathcal{V}^* \times \mathcal{V}}.$$

By (58),

$$\langle \mathcal{B}\bar{\mathbf{u}}_k, q \rangle_{\mathcal{Q}^* \times \mathcal{Q}} = \langle \mathcal{B}^* q, \bar{\mathbf{u}}_k \rangle_{\mathcal{V}^* \times \mathcal{V}} \rightarrow \langle \mathcal{B}^* q, \mathbf{u} \rangle_{\mathcal{V}^* \times \mathcal{V}} = \langle \mathcal{B}\mathbf{u}, q \rangle_{\mathcal{Q}^* \times \mathcal{Q}}.$$

In conclusion, by taking the upper limit of both sides of (49), we obtain (22), and by taking the limit of both sides of (50), we obtain (23). From (61) and (62), we have $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $H^1(0, T; \mathbf{V})$. Then, $\mathbf{u}_k(t) \rightharpoonup \mathbf{u}(t)$ in $\mathbf{V}$ for all $t \in [0, T]$ (cf. [29, Lemma 4 (a)]). In particular, since $\mathbf{u}_k(0) = \mathbf{u}_0$ for all k , we have $\mathbf{u}(0) = \mathbf{u}_0$. Therefore, $(\mathbf{u}, p)$ is a solution of Problem 3.3. $\square$

Theorem 4.7. Assume $H(\psi)$, (H_0) and (29). Then, Problem 3.2 has a unique solution.

Proof. Since Problem 3.2 and Problem 3.3 are equivalent by Proposition 3.4, existence of a solution to Problem 3.2 follows from Lemma 4.6. Let us show the uniqueness of the solution. Suppose both $(\mathbf{u}_1, p_1), (\mathbf{u}_2, p_2) \in \mathcal{W} \times \mathcal{Q}$ are solutions of Problem 3.2. Then for a.e. $t \in (0, T)$,

$$\langle \mathbf{u}'_1(t), \mathbf{v} \rangle + a(\mathbf{u}_1(t), \mathbf{v}) - b(\mathbf{v}, p_1(t)) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{1,\tau}(t); \mathbf{v}_\tau)) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}, \quad (64)$$

$$b(\mathbf{u}_1(t), q) = 0 \quad \forall q \in \mathcal{Q}, \quad (65)$$

$$\langle \mathbf{u}'_2(t), \mathbf{v} \rangle + a(\mathbf{u}_2(t), \mathbf{v}) - b(\mathbf{v}, p_2(t)) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{2,\tau}(t); \mathbf{v}_\tau)) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}, \quad (66)$$

$$b(\mathbf{u}_2(t), q) = 0 \quad \forall q \in \mathcal{Q}, \quad (67)$$

and

$$\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0. \quad (68)$$

Take $\mathbf{v} = \mathbf{u}_2(t) - \mathbf{u}_1(t)$ in (64), $\mathbf{v} = \mathbf{u}_1(t) - \mathbf{u}_2(t)$ in (66), and add the two inequalities to obtain

$$\begin{aligned} & -\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}}^2 - a(\mathbf{u}_1(t) - \mathbf{u}_2(t), \mathbf{u}_1(t) - \mathbf{u}_2(t)) + b(\mathbf{u}_1(t) - \mathbf{u}_2(t), p_1(t) - p_2(t)) \\ & + I_{\Gamma_S}(\psi^0(\mathbf{u}_{1,\tau}(t); \mathbf{u}_{2,\tau}(t) - \mathbf{u}_{1,\tau}(t)) + \psi^0(\mathbf{u}_{2,\tau}(t); \mathbf{u}_{1,\tau}(t) - \mathbf{u}_{2,\tau}(t))) \geq 0. \end{aligned}$$

By (65) and (67),

$$b(\mathbf{u}_1(t) - \mathbf{u}_2(t), p_1(t) - p_2(t)) = 0.$$

By Lemma 3.1,

$$I_{\Gamma_S}(\psi^0(\mathbf{u}_{1,\tau}(t); \mathbf{u}_{2,\tau}(t) - \mathbf{u}_{1,\tau}(t)) + \psi^0(\mathbf{u}_{2,\tau}(t); \mathbf{u}_{1,\tau}(t) - \mathbf{u}_{2,\tau}(t))) \leq \alpha_\psi \lambda_\tau^{-1} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{V}}^2.$$

Hence,

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}}^2 + m \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{V}}^2 \leq 0.$$

We integrate the inequality from 0 to $t \in (0, T)$ and note that $\mathbf{u}_1(0) - \mathbf{u}_2(0) = \mathbf{0}$ from equation (68),

$$\frac{1}{2} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}}^2 + m \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_{\mathbf{V}}^2 ds \leq 0.$$

Therefore, $\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{H}} = 0$ for a.e. $t \in (0, T)$. Thanks to the embedding $\mathcal{W} \subset C([0, T]; \mathbf{H})$, we know that $\mathbf{u}_1, \mathbf{u}_2 \in C([0, T]; \mathbf{H})$. Therefore, $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ for all $t \in [0, T]$.

Regarding the uniqueness of p , we derive from (64) and (66) that

$$\begin{aligned} \langle \mathbf{u}'(t), \mathbf{v} \rangle + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p_1(t)) &= \langle \mathbf{f}(t), \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}_0, \\ \langle \mathbf{u}'(t), \mathbf{v} \rangle + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p_2(t)) &= \langle \mathbf{f}(t), \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}_0. \end{aligned}$$

Subtract the two equalities to obtain

$$b(\mathbf{v}, p_1(t) - p_2(t)) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_0.$$

Apply the inf-sup condition (17),

$$c \|p_1(t) - p_2(t)\|_Q \leq \sup_{\mathbf{v} \in \mathbf{V}_0} \frac{b(\mathbf{v}, p_1(t) - p_2(t))}{\|\mathbf{v}\|_{\mathbf{V}}} = 0.$$

Therefore, $p_1(t) = p_2(t)$ a.e. $t \in (0, T)$. □

5. Analysis of a fully discrete method

In this section, we analyze a fully discrete method for solving Problem 3.2. Throughout the section, we assume $H(\psi)$, (H_0) and (29) so that Problem 3.2 is guaranteed to have a unique solution. We use backward Euler scheme to approximate the time derivative and use a mixed finite element method for the spatial discretization. As in Section 4, we use a uniform partition of the time interval $[0, T]$ with the time step-size $k = T/N$ for a positive integer N . A consequence of (H_0) and the Sobolev embedding $H^1(0, T) \hookrightarrow C([0, T])$ is the continuity of $\mathbf{f}$: $\mathbf{f} \in C([0, T]; \mathbf{V}^*)$. We write $\mathbf{f}_n = \mathbf{f}(t_n)$, $0 \leq n \leq N$. For the spatial discretization, we assume Ω is a polygonal/polyhedral domain and introduce a regular family of finite element partitions $\{\mathcal{T}^h\}_h$ of $\bar{\Omega}$ into triangular/tetrahedral elements. The partitions are compatible with the boundary splitting $\partial\Omega = \Gamma_D \cup \Gamma_S$ in the sense that each face of any element on the boundary lies entirely in Γ_D or Γ_S . Corresponding to a finite element partition $\mathcal{T}^h$, we use a stable pair of finite element spaces $\mathbf{V}^h \subset \mathbf{V}$ and $Q^h \subset Q$: for a constant $\beta_b > 0$ independent of h , the discrete inf-sup condition holds,

$$\sup_{\mathbf{0} \neq \mathbf{v}^h \in \mathbf{V}_0^h} \frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_{\mathbf{V}}} \geq \beta_b \|q^h\|_Q \quad \forall q^h \in Q^h, \quad (69)$$

where $\mathbf{V}_0^h = \mathbf{V}_0 \cap \mathbf{V}^h$. Let $\mathbf{u}_0^h \in \mathbf{V}^h$ be an approximation of $\mathbf{u}_0$. Then, the numerical method is stated as follows.

Problem 5.1. Find $\{\mathbf{u}_n^{hk}\}_{n=0}^N \subset \mathbf{V}^h$ and $\{p_n^{hk}\}_{n=1}^N \subset Q^h$ such that for $1 \leq n \leq N$,

$$\left(\frac{\mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}}{k}, \mathbf{v}^h \right)_{\mathbf{H}} + a(\mathbf{u}_n^{hk}, \mathbf{v}^h) - b(\mathbf{v}^h, p_n^{hk}) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^{hk}, \mathbf{v}_\tau^h)) \geq \langle \mathbf{f}_n, \mathbf{v}^h \rangle \quad \forall \mathbf{v}^h \in \mathbf{V}^h, \quad (70)$$

$$b(\mathbf{u}_n^{hk}, q^h) = 0 \quad \forall q^h \in Q^h, \quad (71)$$

and

$$\mathbf{u}_0^{hk} = \mathbf{u}_0^h. \quad (72)$$

Similar to Proposition 4.2, we can show that under the assumptions $H(\psi)$, (H_0) and (29), Problem 5.1 admits a unique solution $\{\mathbf{u}_n^{hk}\}_{n=0}^N \subset \mathbf{V}^h$ and $\{p_n^{hk}\}_{n=1}^N \subset \mathcal{Q}^h$. We provide a bound on the numerical solution in the next result.

Lemma 5.2. *There exists a constant $c > 0$ such that*

$$\max_{1 \leq n \leq N} \|\mathbf{u}_n^{hk}\|_{\mathbf{H}}^2 + k \sum_{n=1}^N \|\mathbf{u}_n^{hk}\|_{\mathbf{V}}^2 \leq \|\mathbf{u}_0^h\|_{\mathbf{H}}^2 + c \left(1 + \|\mathbf{f}\|_{C([0,T];\mathbf{V}^*)}^2 \right). \quad (73)$$

Proof. We take $\mathbf{v}^h = -\mathbf{u}_n^{hk}$ in (70) and make use of (71) to obtain

$$\frac{1}{k} \left(\mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}, \mathbf{u}_n^{hk} \right)_{\mathbf{H}} + a(\mathbf{u}_n^{hk}, \mathbf{u}_n^{hk}) \leq I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^{hk}; -\mathbf{u}_{n,\tau}^{hk})) + \langle \mathbf{f}_n, \mathbf{u}_n^{hk} \rangle.$$

As

$$\begin{aligned} I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^{hk}; -\mathbf{u}_{n,\tau}^{hk})) &= I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^{hk}; -\mathbf{u}_{n,\tau}^{hk}) + \psi^0(0; \mathbf{u}_{n,\tau}^{hk})) - I_{\Gamma_S}(\psi^0(0; \mathbf{u}_{n,\tau}^{hk})) \\ &\leq \alpha_{\psi} \lambda_{\tau}^{-1} \|\mathbf{u}_n^{hk}\|_{\mathbf{V}}^2 + c \|\mathbf{u}_n^{hk}\|_{\mathbf{V}}, \end{aligned}$$

we have

$$\frac{1}{2k} \left(\|\mathbf{u}_n^{hk}\|_{\mathbf{H}}^2 - \|\mathbf{u}_{n-1}^{hk}\|_{\mathbf{H}}^2 \right) + 2\mu \|\mathbf{u}_n^{hk}\|_{\mathbf{V}}^2 \leq \alpha_{\psi} \lambda_{\tau}^{-1} \|\mathbf{u}_n^{hk}\|_{\mathbf{V}}^2 + c (1 + \|\mathbf{f}_n\|_{\mathbf{V}^*}) \|\mathbf{u}_n^{hk}\|_{\mathbf{V}}.$$

Apply the modified Cauchy–Schwarz inequality (1) with $\delta = m/2$ on the last term of the above inequality,

$$c (1 + \|\mathbf{f}_n\|_{\mathbf{V}^*}) \|\mathbf{u}_n^{hk}\|_{\mathbf{V}} \leq (m/2) \|\mathbf{u}_n^{hk}\|_{\mathbf{V}}^2 + c \left(1 + \|\mathbf{f}_n\|_{\mathbf{V}^*}^2 \right).$$

Thus, for a constant c ,

$$\frac{1}{2k} \left(\|\mathbf{u}_n^{hk}\|_{\mathbf{H}}^2 - \|\mathbf{u}_{n-1}^{hk}\|_{\mathbf{H}}^2 \right) + \frac{m}{2} \|\mathbf{u}_n^{hk}\|_{\mathbf{V}}^2 \leq c \left(1 + \|\mathbf{f}_n\|_{\mathbf{V}^*}^2 \right).$$

We then derive from the above inequality that

$$\|\mathbf{u}_n^{hk}\|_{\mathbf{H}}^2 - \|\mathbf{u}_{n-1}^{hk}\|_{\mathbf{H}}^2 + m k \|\mathbf{u}_n^{hk}\|_{\mathbf{V}}^2 \leq c k \left(1 + \|\mathbf{f}_n\|_{\mathbf{V}^*}^2 \right).$$

We replace n by j in the above inequality and add the inequalities from $j = 1$ to $j = n$,

$$\|\mathbf{u}_n^{hk}\|_{\mathbf{H}}^2 + m k \sum_{j=1}^n \|\mathbf{u}_j^{hk}\|_{\mathbf{V}}^2 \leq \|\mathbf{u}_0^h\|_{\mathbf{H}}^2 + c \left(1 + k \sum_{j=1}^n \|\mathbf{f}_j\|_{\mathbf{V}^*}^2 \right),$$

from which, we conclude (73). $\square$

In the rest of the section, we focus on an error analysis. For this purpose, we will make regularity assumptions:

$$\mathbf{f} \in C([0, T]; \mathbf{H}), \quad \mathbf{u} \in C([0, T]; H^2(\Omega)^d) \cap C^1([0, T]; \mathbf{H}), \quad p \in C([0, T]; H^1(\Omega)). \quad (74)$$

Write $\mathbf{u}_n = \mathbf{u}(t_n)$ etc., and denote the errors

$$\mathbf{e}_n := \mathbf{u}_n - \mathbf{u}_n^{hk}, \quad 0 \leq n \leq N. \quad (75)$$

Note that the regularity assumptions (74) imply that for all $\mathbf{v} \in \mathbf{V}$,

$$\langle \mathbf{f}_n, \mathbf{v} \rangle = (\mathbf{f}_n, \mathbf{v})_H, \quad \langle \mathbf{u}'_n, \mathbf{v} \rangle = (\mathbf{u}'_n, \mathbf{v})_H.$$

Also, we can show that the solution of Problem 3.2 satisfies the differential equation (2) pointwise almost everywhere.

Lemma 5.3. *Assume (74). Then the solution $(\mathbf{u}, p)$ of Problem 3.2 satisfies, for a.e. $t \in (0, T)$ and a.e. $\mathbf{x} \in \Omega$,*

$$\mathbf{u}'(\mathbf{x}, t) - 2\mu \operatorname{div} \varepsilon(\mathbf{u}(\mathbf{x}, t)) + \nabla p(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t).$$

Proof. We derive from (19) that for a.e. $t \in (0, T)$,

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p(t)) = \langle \mathbf{f}(t), \mathbf{v} \rangle \quad \forall \mathbf{v} \in C_0^\infty(\Omega)^d.$$

Perform an integration by parts to obtain

$$\int_{\Omega} [\mathbf{u}'(t) - 2\mu \operatorname{div} \varepsilon(\mathbf{u}(t)) + \nabla p(t) - \mathbf{f}(t)] \cdot \mathbf{v} \, dx = 0 \quad \forall \mathbf{v} \in C_0^\infty(\Omega)^d,$$

where for simplicity, we suppress the $\mathbf{x}$ -dependence in expressions of various functions. By the density of $C_0^\infty(\Omega)^d$ in $L^2(\Omega)^d$, we obtain

$$\int_{\Omega} [\mathbf{u}'(t) - 2\mu \operatorname{div} \varepsilon(\mathbf{u}(t)) + \nabla p(t) - \mathbf{f}(t)] \cdot \mathbf{v} \, dx = 0 \quad \forall \mathbf{v} \in L^2(\Omega)^d.$$

Taking $\mathbf{v} = \mathbf{u}'(t) - 2\mu \operatorname{div} \varepsilon(\mathbf{u}(t)) + \nabla p(t) - \mathbf{f}(t)$ in the above equality, we obtain the required pointwise equation. $\square$

For $(\mathbf{w}, r) \in \mathbf{V} \times Q$, define its Stokes projection $(\Pi^h \mathbf{w}, \Pi^h r) \in \mathbf{V}^h \times Q^h$ as the unique solution of the problem

$$a(\Pi^h \mathbf{w}, \mathbf{v}^h) - b(\mathbf{v}^h, \Pi^h r) = a(\mathbf{w}, \mathbf{v}^h) - b(\mathbf{v}^h, r) \quad \forall \mathbf{v}^h \in \mathbf{V}^h, \quad (76)$$

$$b(\Pi^h \mathbf{w}, q^h) = 0 \quad \forall q^h \in Q^h. \quad (77)$$

Then, we have the following Céa's inequality ([24, Lemma 4.43]):

$$\|\mathbf{w} - \Pi^h \mathbf{w}\|_{\mathbf{V}} + \|r - \Pi^h r\|_Q \leq c \inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\mathbf{w} - \mathbf{v}^h\|_{\mathbf{V}} + c \inf_{q^h \in Q^h} \|r - q^h\|_Q. \quad (78)$$

Note that in John [24, Lemma 4.43], this inequality is proved for the case where the spaces $\mathbf{V}$ and $\mathbf{V}^h$ are $\mathbf{V}_0$ and $\mathbf{V}_0^h$, respectively. Thanks to the inf-sup condition (69), the same proof also applies to the case where $\mathbf{V}$ is defined in (7) and $\mathbf{V}^h \subset \mathbf{V}$. Moreover, for time-dependent problems, $(\Pi^h \partial_t \mathbf{w}, \Pi^h \partial_t r) = (\partial_t \Pi^h \mathbf{w}, \partial_t \Pi^h r)$ ([24, Remark 4.44]).

Denote

$$\boldsymbol{\epsilon}(t) := \mathbf{u}(t) - \Pi^h \mathbf{u}(t), \quad t \in [0, T], \quad \text{and} \quad \boldsymbol{\epsilon}_n := \boldsymbol{\epsilon}(t_n), \quad 1 \leq n \leq N. \quad (79)$$

From equation (20) at $t = t_n$ and (71),

$$b(\mathbf{u}_n, q^h) = b(\mathbf{u}_n^{hk}, q^h) = 0 \quad \forall q^h \in Q^h. \quad (80)$$

We replace $\mathbf{v}^h$ by $\Pi^h \mathbf{u}_n - \mathbf{u}_n^{hk}$ in (70) and note that $b(\Pi^h \mathbf{u}_n - \mathbf{u}_n^{hk}, p_n^{hk}) = 0$ by (77) and (71),

$$\left(\frac{\mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}}{k}, \Pi^h \mathbf{u}_n - \mathbf{u}_n^{hk} \right)_H + a(\mathbf{u}_n^{hk}, \Pi^h \mathbf{u}_n - \mathbf{u}_n^{hk}) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^{hk}; \Pi^h \mathbf{u}_{n,\tau} - \mathbf{u}_{n,\tau}^{hk})) \geq \langle \mathbf{f}_n, \Pi^h \mathbf{u}_n - \mathbf{u}_n^{hk} \rangle.$$

Take $\mathbf{v} = \mathbf{u}_n^{hk} - \mathbf{u}_n$ in (19) at $t = t_n$,

$$\langle \mathbf{u}'_n, \mathbf{u}_n^{hk} - \mathbf{u}_n \rangle + a(\mathbf{u}_n, \mathbf{u}_n^{hk} - \mathbf{u}_n) - b(\mathbf{u}_n^{hk} - \mathbf{u}_n, p_n) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}; \mathbf{u}_{n,\tau}^{hk} - \mathbf{u}_{n,\tau})) \geq \langle \mathbf{f}_n, \mathbf{u}_n^{hk} - \mathbf{u}_n \rangle.$$

Add the two inequalities and make use of (80), after a rearrangement of the terms,

$$\begin{aligned} a(\mathbf{e}_n, \mathbf{e}_n) &\leq \langle \mathbf{E}_n, \mathbf{e}_n \rangle - \left(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k}, \mathbf{e}_n \right)_H + \left(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k}, \mathbf{e}_n \right)_H - \left(\frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{k}, \mathbf{e}_n \right)_H \\ &\quad + a(\mathbf{e}_n, \mathbf{e}_n) - a(\mathbf{u}_n, \mathbf{e}_n) + b(\mathbf{e}_n, p_n - q_n^h) \\ &\quad + I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^{hk}; \Pi^h \mathbf{u}_{n,\tau} - \mathbf{u}_{n,\tau}^{hk}) + \psi^0(\mathbf{u}_{n,\tau}; \mathbf{u}_{n,\tau}^{hk} - \mathbf{u}_{n,\tau})) + \langle \mathbf{f}_n, \mathbf{e}_n \rangle, \end{aligned}$$

where

$$\mathbf{E}_n := \frac{\mathbf{u}_n - \mathbf{u}_{n-1}}{k} - \mathbf{u}'_n, \quad 1 \leq n \leq N. \quad (81)$$

By the sub-additivity property, cf. Proposition 2.1 (ii),

$$\psi^0(\mathbf{u}_{n,\tau}^{hk}; \Pi^h \mathbf{u}_{n,\tau} - \mathbf{u}_{n,\tau}^{hk}) \leq \psi^0(\mathbf{u}_{n,\tau}^{hk}; \Pi^h \mathbf{u}_{n,\tau} - \mathbf{u}_{n,\tau}) + \psi^0(\mathbf{u}_{n,\tau}^{hk}; \mathbf{u}_{n,\tau} - \mathbf{u}_{n,\tau}^{hk}).$$

Then, applying Lemma 3.1,

$$I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^{hk}; \Pi^h \mathbf{u}_{n,\tau} - \mathbf{u}_{n,\tau}^{hk}) + \psi^0(\mathbf{u}_{n,\tau}; \mathbf{u}_{n,\tau}^{hk} - \mathbf{u}_{n,\tau})) \leq \alpha_\psi \lambda_\tau^{-1} \|\mathbf{e}_n\|_{\mathbf{V}}^2 + I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^{hk}; -\mathbf{e}_{n,\tau})).$$

Hence,

$$\begin{aligned} \left(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k}, \mathbf{e}_n \right)_H + a(\mathbf{e}_n, \mathbf{e}_n) &\leq \langle \mathbf{E}_n, \mathbf{e}_n \rangle + \left(\frac{\mathbf{e}_n - \mathbf{e}_{n-1}}{k}, \mathbf{e}_n \right)_H - \langle \mathbf{E}_n, \mathbf{e}_n \rangle \\ &\quad + a(\mathbf{e}_n, \mathbf{e}_n) + b(\mathbf{e}_n, p_n - q_n^h) - b(\mathbf{e}_n, p_n - q_n^h) \\ &\quad - R_n(\mathbf{e}_n) + \alpha_\psi \lambda_\tau^{-1} \|\mathbf{e}_n\|_{\mathbf{V}}^2 + I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^{hk}; -\mathbf{e}_{n,\tau})), \end{aligned}$$

where

$$R_n(\mathbf{v}) := \langle \mathbf{u}'_n, \mathbf{v} \rangle + a(\mathbf{u}_n, \mathbf{v}) - b(\mathbf{v}, p_n) - \langle \mathbf{f}_n, \mathbf{v} \rangle. \quad (82)$$

By $H(\psi)$ (iii),

$$I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^{hk}; -\mathbf{e}_{n,\tau})) \leq c \left(1 + \|\mathbf{u}_n^{hk}\|_{\mathbf{V}} \right) \|\mathbf{e}_{n,\tau}\|_{L^2(\Gamma_S)},$$

and then

$$\begin{aligned} (\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{e}_n)_H + m k \|\mathbf{e}_n\|_{\mathbf{V}}^2 &\leq k \langle \mathbf{E}_n, \mathbf{e}_n \rangle + k a(\mathbf{e}_n, \mathbf{e}_n) + (\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{e}_n)_H \\ &\quad - k \langle \mathbf{E}_n, \mathbf{e}_n \rangle + k b(\mathbf{e}_n, p_n - q_n^h) - k b(\mathbf{e}_n, p_n - q_n^h) \\ &\quad - k R_n(\mathbf{e}_n) + c k \left(1 + \|\mathbf{u}_n^{hk}\|_{\mathbf{V}} \right) \|\mathbf{e}_{n,\tau}\|_{L^2(\Gamma_S)}. \end{aligned} \quad (83)$$

Note that

$$(\mathbf{e}_n - \mathbf{e}_{n-1}, \mathbf{e}_n)_H \geq \frac{1}{2} \left(\|\mathbf{e}_n\|_H^2 - \|\mathbf{e}_{n-1}\|_H^2 \right).$$

Then, we rewrite (83) as

$$\begin{aligned} \|e_n\|_H^2 - \|e_{n-1}\|_H^2 + 2mk\|e_n\|_V^2 &\leq 2k\langle E_n, e_n \rangle + 4\mu k\|e_n\|_V\|\epsilon_n\|_V + 2(e_n - e_{n-1}, \epsilon_n)_H \\ &\quad - 2k\langle E_n, \epsilon_n \rangle + ck\|e_n\|_V\|p_n - q_n^h\|_Q + ck\|\epsilon_n\|_V\|p_n - q_n^h\|_Q \\ &\quad - 2kR_n(\epsilon_n) + ck\left(1 + \|u_n^{hk}\|_V\right)\|\epsilon_{n,\tau}\|_{L^2(\Gamma_S)^d}. \end{aligned}$$

Apply the Cauchy–Schwarz inequality or its modified version (1),

$$\begin{aligned} \langle E_n, e_n \rangle &\leq \|E_n\|_{V^*}\|e_n\|_V \leq (m/6)\|e_n\|_V^2 + c\|E_n\|_{V^*}^2, \\ 2\mu\|e_n\|_V\|\epsilon_n\|_V &\leq (m/6)\|e_n\|_V^2 + c\|\epsilon_n\|_V^2, \\ -\langle E_n, \epsilon_n \rangle &\leq \|E_n\|_{V^*}\|\epsilon_n\|_V \leq (1/2)\|E_n\|_{V^*}^2 + (1/2)\|\epsilon_n\|_V^2, \\ c\|e_n\|_V\|p_n - q_n^h\|_Q &\leq (m/6)\|e_n\|_V^2 + c\|p_n - q_n^h\|_Q^2 \\ ck\|\epsilon_n\|_V\|p_n - q_n^h\|_Q &\leq c\left(\|\epsilon_n\|_V^2 + \|p_n - q_n^h\|_Q^2\right). \end{aligned}$$

Hence,

$$\begin{aligned} \|e_n\|_H^2 - \|e_{n-1}\|_H^2 + mk\|e_n\|_V^2 &\leq 2(e_n - e_{n-1}, \epsilon_n)_H - 2kR_n(\epsilon_n) \\ &\quad + ck\left(\|E_n\|_{V^*}^2 + \|\epsilon_n\|_H^2 + \|p_n - q_n^h\|_Q^2\right) \\ &\quad + ck\left(1 + \|u_n^{hk}\|_V\right)\|\epsilon_{n,\tau}\|_{L^2(\Gamma_S)^d}. \end{aligned} \quad (84)$$

Change the index n to j , and add the modified (84) for $j = 1, \dots, n$ to obtain

$$\begin{aligned} \|e_n\|_H^2 + mk\sum_{j=1}^n\|e_j\|_V^2 &\leq \|e_0\|_H^2 + 2\sum_{j=1}^n(e_j - e_{j-1}, \epsilon_j)_H - 2k\sum_{j=1}^nR_j(\epsilon_j) \\ &\quad + ck\sum_{j=1}^n\left(\|E_j\|_{V^*}^2 + \|\epsilon_j\|_H^2 + \|p_j - q_j^h\|_Q^2\right) \\ &\quad + ck\sum_{j=1}^n\left(1 + \|u_j^{hk}\|_V\right)\|\epsilon_{j,\tau}\|_{L^2(\Gamma_S)^d}. \end{aligned} \quad (85)$$

Now,

$$\begin{aligned} \sum_{j=1}^n(e_j - e_{j-1}, \epsilon_j)_H &= \sum_{j=1}^n(e_j, \epsilon_j)_H - \sum_{j=0}^{n-1}(e_j, \epsilon_{j+1})_H \\ &= (e_n, \epsilon_n)_H + \sum_{j=1}^{n-1}(e_j, \epsilon_j - \epsilon_{j+1})_H - (e_0, \epsilon_1)_H. \end{aligned}$$

Apply the modified Cauchy–Schwarz inequality (1) to obtain

$$\begin{aligned}
 2 \sum_{j=1}^n (\mathbf{e}_j - \mathbf{e}_{j-1}, \boldsymbol{\epsilon}_j)_{\mathbf{H}} &\leq \frac{1}{2} \|\mathbf{e}_n\|_{\mathbf{H}}^2 + c \|\boldsymbol{\epsilon}_n\|_{\mathbf{H}}^2 + \|\mathbf{e}_0\|_{\mathbf{H}}^2 + \|\boldsymbol{\epsilon}_1\|_{\mathbf{H}}^2 + 2 \sum_{j=1}^{n-1} \|\mathbf{e}_j\|_{\mathbf{H}} \|\boldsymbol{\epsilon}_j - \boldsymbol{\epsilon}_{j+1}\|_{\mathbf{H}} \\
 &\leq \frac{1}{2} \|\mathbf{e}_n\|_{\mathbf{H}}^2 + c \|\boldsymbol{\epsilon}_n\|_{\mathbf{H}}^2 + \|\mathbf{e}_0\|_{\mathbf{H}}^2 + \|\boldsymbol{\epsilon}_1\|_{\mathbf{H}}^2 \\
 &\quad + k \sum_{j=1}^{n-1} \|\mathbf{e}_j\|_{\mathbf{H}}^2 + k \sum_{j=1}^{n-1} \left\| \frac{\boldsymbol{\epsilon}_j - \boldsymbol{\epsilon}_{j+1}}{k} \right\|_{\mathbf{H}}^2
 \end{aligned} \tag{86}$$

The regularity assumptions (74) imply $\boldsymbol{\sigma} \in C([0, T]; H^1(\Omega)^d)$ and $\boldsymbol{\sigma}_\tau \in C([0, T]; L^2(\Gamma_S)^d)$. We apply Lemma 5.3 to derive

$$\begin{aligned}
 R_n(\mathbf{v}) &= \int_{\Omega} [\mathbf{u}'_n \cdot \mathbf{v} + 2\mu \boldsymbol{\varepsilon}(\mathbf{u}_n) : \boldsymbol{\varepsilon}(\mathbf{v}) - p_n \operatorname{div} \mathbf{v} - \mathbf{f}_n \cdot \mathbf{v}] \, dx \\
 &= \int_{\Gamma} [2\mu \boldsymbol{\varepsilon}(\mathbf{u}_n) \boldsymbol{\nu} - p_n \boldsymbol{\nu}] \cdot \mathbf{v} \, ds + \int_{\Omega} (\mathbf{u}'_n - \operatorname{div} \boldsymbol{\sigma}_n - \mathbf{f}_n) \cdot \mathbf{v} \, dx \\
 &= \int_{\Gamma_S} \boldsymbol{\sigma}_{n,\tau} \cdot \mathbf{v}_\tau \, ds.
 \end{aligned}$$

Then, for some constant $c > 0$ depending only on $\|\boldsymbol{\sigma}_\tau\|_{C([0,T];L^2(\Gamma_S)^d)}$,

$$|R_n(\mathbf{v})| \leq c \|\mathbf{v}_\tau\|_{L^2(\Gamma_S)^d} \quad \forall \mathbf{v} \in \mathbf{V}, \quad 1 \leq n \leq N. \tag{87}$$

The last term in (85) is bounded as follows:

$$c k \sum_{j=1}^n \left(1 + \|\mathbf{u}_j^{hk}\|_{\mathbf{V}}\right) \|\boldsymbol{\epsilon}_{j,\tau}\|_{L^2(\Gamma_S)^d} \leq c \left(k \sum_{j=1}^n \left(1 + \|\mathbf{u}_j^{hk}\|_{\mathbf{V}}^2\right)\right)^{1/2} \left(k \sum_{j=1}^n \|\boldsymbol{\epsilon}_{j,\tau}\|_{L^2(\Gamma_S)^d}^2\right)^{1/2},$$

which is further bounded by

$$c \left(k \sum_{j=1}^n \|\boldsymbol{\epsilon}_{j,\tau}\|_{L^2(\Gamma_S)^d}^2\right)^{1/2},$$

thanks to Lemma 5.2. Thus, from (85),

$$\begin{aligned}
 \frac{1}{2} \|\mathbf{e}_n\|_{\mathbf{H}}^2 + m k \sum_{j=1}^n \|\mathbf{e}_j\|_{\mathbf{V}}^2 &\leq c \left(\|\mathbf{e}_0\|_{\mathbf{H}}^2 + \|\boldsymbol{\epsilon}_n\|_{\mathbf{H}}^2 + \|\boldsymbol{\epsilon}_1\|_{\mathbf{H}}^2\right) + k \sum_{j=1}^{n-1} \|\mathbf{e}_j\|_{\mathbf{H}}^2 + k \sum_{j=1}^{n-1} \left\| \frac{\boldsymbol{\epsilon}_j - \boldsymbol{\epsilon}_{j+1}}{k} \right\|_{\mathbf{H}}^2 \\
 &\quad + c k \sum_{j=1}^n \left(\|\mathbf{E}_j\|_{\mathbf{V}^*}^2 + \|\boldsymbol{\epsilon}_j\|_{\mathbf{H}}^2 + \|p_j - q_j^h\|_Q^2\right) + c \left(k \sum_{j=1}^n \|\boldsymbol{\epsilon}_{j,\tau}\|_{L^2(\Gamma_S)^d}^2\right)^{1/2}.
 \end{aligned} \tag{88}$$

Applying Gronwall's inequality (cf. [30, Lemma 7.25]) for $\|e_n\|_{\mathbf{H}}^2$, and then bounding $k \sum_{j=1}^n \|e_j\|_{\mathbf{V}}^2$, we derive from (88) that

$$\begin{aligned} \max_{1 \leq n \leq N} \|e_n\|_{\mathbf{H}}^2 + k \sum_{n=1}^N \|e_n\|_{\mathbf{V}}^2 &\leq c \|e_0\|_{\mathbf{H}}^2 + c \max_{1 \leq n \leq N} \|\epsilon_n\|_{\mathbf{H}}^2 + c k \sum_{n=1}^{N-1} \left\| \frac{\epsilon_n - \epsilon_{n+1}}{k} \right\|_{\mathbf{H}}^2 \\ &\quad + c k \sum_{n=1}^N \left(\|E_n\|_{\mathbf{V}^*}^2 + \|\epsilon_n\|_{\mathbf{H}}^2 + \|p_n - q_n^h\|_Q^2 \right) \\ &\quad + c \left(k \sum_{n=1}^N \|\epsilon_{n,\tau}\|_{L^2(\Gamma_S)^d}^2 \right)^{1/2}. \end{aligned} \quad (89)$$

In summary, under the solution regularity assumptions (74), we have derived a Céa-type inequality (89) which is a basis for error estimation.

Next, we state a sample error estimate with the use of the P1b/P1 element pair ([31]). It is known that the inf-sup condition is satisfied.

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and let us use the P1b/P1 element pair over a regular family of finite element partitions of $\bar{\Omega}$ that are compatible to the splitting of the boundary $\partial\Omega$ into Γ_D and Γ_S . Assume $H(\psi)$, (H_0) , (29), and (74). Also, assume the discrete initial value $\mathbf{u}_0^{hk} \in \mathbf{V}^h$ is chosen such that*

$$\|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_{\mathbf{V}} \leq c h. \quad (90)$$

Then, for the fully discrete numerical solution defined by Problem 5.1, we have the error estimate

$$\max_{1 \leq n \leq N} \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_{\mathbf{H}}^2 + k \sum_{n=1}^N \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_{\mathbf{V}}^2 \leq c (h + k^2). \quad (91)$$

Proof. By Céa's inequality (78) and the theory of standard finite element interpolation error estimates (cf. [32,33]), under the stated assumptions, for a constant $c > 0$ independent of h and k , we have

$$\max_{1 \leq n \leq N} \|\epsilon_n\|_{\mathbf{V}} \leq c h$$

and we can choose $q_n^h \in Q^h$ such that

$$\|p_n - q_n^h\|_Q \leq c h.$$

Moreover,

$$k \sum_{n=1}^{N-1} \left\| \frac{\epsilon_n - \epsilon_{n+1}}{k} \right\|_{\mathbf{H}}^2 = \frac{1}{k} \sum_{n=1}^{N-1} \left\| \int_{t_n}^{t_{n+1}} (\epsilon(t))' dt \right\|_{\mathbf{H}}^2 \leq \sum_{n=1}^{N-1} \int_{t_n}^{t_{n+1}} \|(\epsilon(t))'\|_{\mathbf{H}}^2 dt \leq c h^2 \|\mathbf{u}'\|_{L^2(0,T;H^1(\Omega)^d)}^2.$$

In addition, since

$$E_n = \frac{1}{k} \int_{t_{n-1}}^{t_n} \int_{t_n}^t \mathbf{u}''(s) ds dt,$$

we have

$$k \sum_{n=1}^N \|E_n\|_{\mathbf{V}^*}^2 \leq c k^2 \|\mathbf{u}''\|_{H^2(0,T;\mathbf{V}^*)}^2.$$

Summarizing, we have the error bound (91) from (89). $\square$

Remark 5.5. Since $\mathbf{u}_0 \in H^2(\Omega)^2$ as a consequence of (74), we can choose $\mathbf{u}_0^{hk} \in \mathbf{V}^h$ to be the continuous piecewise interpolant of $\mathbf{u}_0$ and the error bound (90) holds by the standard finite element interpolation theory (cf. [32,33]). The error bound (91) is of optimal order with respect to the time-step k . For the error bound to be of optimal order with respect to the mesh-size h , the term h in the estimate (91) needs to be improved to h^2 . The numerical results reported in Section 6 support this. The topic is worth further investigation. Also, it will be interesting to derive an error bound, especially one of optimal order, for the pressure approximation.

6. Numerical examples

In the numerical examples, we choose the potential function

$$\psi(\mathbf{z}) = \int_0^{|\mathbf{z}|} \omega(t) dt, \quad \mathbf{z} \in \mathbb{R}^d, \quad \omega(t) = (a-b)e^{-\beta t} + b, \quad (92)$$

where a, b, β are all positive constants, and $a > b$. Since ω is decreasing on $[0, +\infty)$, ψ is not convex. Note that the function $\psi(\mathbf{z})$ is Lipschitz continuous but is not differentiable in the classical sense. The function (92) satisfies the hypothesis $H(\psi)$ stated in Section 3, cf. [34, Lemma 3.2]. Moreover, it is easy to find that $\alpha_\psi = \beta(a-b)$. The slip boundary condition in (5) is equivalent to

$$|\boldsymbol{\sigma}_\tau| \leq \omega(0) \text{ if } \mathbf{u}_\tau = \mathbf{0}, \quad -\boldsymbol{\sigma}_\tau = \omega(|\mathbf{u}_\tau|) \frac{\mathbf{u}_\tau}{|\mathbf{u}_\tau|} \text{ if } \mathbf{u}_\tau \neq \mathbf{0}.$$

This is verified as follows. When $\mathbf{z} \neq \mathbf{0}$, the function $\psi(\mathbf{z})$ is differentiable in the classical sense, leading to the second part of the above formula for $\boldsymbol{\sigma}_\tau$. When $\mathbf{u}_\tau = \mathbf{0}$, the result $|\boldsymbol{\sigma}_\tau| \leq \omega(0)$ follows from an application of a formula for the Clarke subdifferential of locally Lipschitz continuous functions of one or several real variables, cf. [20, Theorem 2.5.1].

Define a Lagrangian multiplier $\boldsymbol{\lambda} = -\boldsymbol{\sigma}_\tau / \omega(|\mathbf{u}_\tau|)$, an element of the set

$$\Lambda = \{\boldsymbol{\lambda} \in L^2(\Gamma_S; \mathbb{R}^d) \mid |\boldsymbol{\lambda}| \leq 1 \text{ a.e. on } \Gamma_S\}.$$

and introduce the following form of the weak formulation of the problem.

Problem 6.1. Find $(\mathbf{u}, p, \boldsymbol{\lambda}) \in \mathcal{W} \times \mathcal{Q} \times L^2(0, T; \Lambda)$ such that $\mathbf{u}(0) = \mathbf{u}_0$ and for a.e. $t \in (0, T)$,

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p(t)) + \int_{\Gamma_S} \omega(|\mathbf{u}_\tau(t)|) \boldsymbol{\lambda}(t) \cdot \mathbf{v}_\tau ds = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V},$$

$$b(\mathbf{u}(t), q) = 0 \quad \forall q \in \mathcal{Q}.$$

$$\boldsymbol{\lambda}(t) \cdot \mathbf{u}_\tau(t) = |\mathbf{u}_\tau(t)| \quad \text{a.e. on } \Gamma_S.$$

For a full discretization, we use the backward Euler divided difference for the time derivative and a stable mixed finite element method (the P1b/P1 pair) for the spatial discretization. For a fixed positive integer N , define the time step-size $k = T/N$. Let $\mathbf{P}: \mathbb{R}^d \rightarrow B(\mathbf{0}; 1)$ be the orthogonal projection operator, $B(\mathbf{0}; 1)$ being the closed unit ball in $\mathbb{R}^d$ centered at the origin. We implement the following Uzawa algorithm for the numerical solution of Problem 6.1.

Initialization. Choose a parameter $\rho > 0$, a maximal iteration number l_0 and an error tolerance $\varepsilon > 0$, determine the approximate initial velocity $\mathbf{u}_0^{hk}$ from $\mathbf{u}_0$, and set $\boldsymbol{\lambda}_0^{hk} = \mathbf{0}$.

Iteration. For $n = 1, \dots, N$, let $\mathbf{u}_{0,n}^{hk} = \mathbf{u}_{n-1}^{hk}$, $\boldsymbol{\lambda}_{0,n}^{hk} = \boldsymbol{\lambda}_{n-1}^{hk}$;

for $l \geq 1$, find $(\mathbf{u}_{l,n}^{hk}, p_{l,n}^{hk}) \in \mathbf{V}^h \times Q^h$ such that

$$\begin{aligned} & \frac{1}{k} (\mathbf{u}_{l,n}^{hk}, \mathbf{v}^h) + a(\mathbf{u}_{l,n}^{hk}, \mathbf{v}^h) - b(\mathbf{v}^h, p_{l,n}^{hk}) \\ &= \langle \mathbf{f}_n, \mathbf{v}^h \rangle - \int_{\Gamma_S} \omega(|\mathbf{u}_{l-1,n,\tau}^{hk}|) \boldsymbol{\lambda}_{l-1,n}^{hk} \cdot \mathbf{v}_\tau^h ds + \frac{1}{k} (\mathbf{u}_{n-1}^{hk}, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathbf{V}^h, \\ & b(\mathbf{u}_{l,n}^{hk}, q^h) = 0 \quad \forall q^h \in Q^h \end{aligned}$$

and update

$$\boldsymbol{\lambda}_{l,n}^{hk} = \mathbf{P}(\boldsymbol{\lambda}_{l-1,n}^{hk} + \rho \mathbf{u}_{l,n,\tau}^{hk}) \quad \text{on } \Gamma_S$$

until $l = l_0$ or $\|\mathbf{u}_{l,n}^{hk} - \mathbf{u}_{l-1,n}^{hk}\|_{L^2(\Omega)^d} / \|\mathbf{u}_{l,n}^{hk}\|_{L^2(\Omega)^d} < \varepsilon$;

set $\mathbf{u}_n^{hk} = \mathbf{u}_{l,n}^{hk}$, $p_n^{hk} = p_{l,n}^{hk}$, and $\boldsymbol{\lambda}_n^{hk} = \boldsymbol{\lambda}_{l,n}^{hk}$.

The algorithm is implemented in Python with the FEniCS package. We let $\mu = 1$ for the viscosity coefficient, $d = 2$ for the spatial dimension, $T = 1$, $\rho = 10$. For the function $\omega(t)$, use $\beta = 2$, $a = 0.255$, $b = 0.25$. The source function is defined by

$$f(x, y, t) = 2t \begin{pmatrix} -10(1 - 6x + 6x^2)y(1 - y)(1 - 2y) + 30x^2(1 - x)^2(1 - 2y) - (1 - 2y) \\ 10(1 - 6y + 6y^2)x(1 - x)(1 - 2x) - 30y^2(1 - y)^2(1 - 2x) - (1 - 2x) \end{pmatrix}.$$

Let the initial velocity $\mathbf{u}_0 = \mathbf{0}$, and let $\mathbf{u}_0^{hk} = \mathbf{0}$. For the stopping criteria, let $l_0 = 100$ and $\varepsilon = 10^{-6}$. In both examples below, the Uzawa iterates meet the error tolerance condition before the maximum number of $l_0 = 100$ iterations is reached in any single time loop.

6.1. Example 6.1

In the first example, the domain is the unit square $\Omega = (0, 1) \times (0, 1)$. Its boundary is split into two parts Γ_D and Γ_S ; Γ_D is the union of the left, top, and right sides, and Γ_S is the bottom side.

We use uniform triangulations of the domain $\Omega = (0, 1) \times (0, 1)$. The unit interval $[0, 1]$ is split into $1/h$ ($h = 1/8, 1/16, \dots$) equal length sub-intervals. The triangulation of the unit square with $h = 1/16$ is shown in Figure 1.

We solve the problems over the time interval $[0, 1]$ and report numerical results at the final time $T = 1$. To calculate numerical errors, we use the numerical solution with $h = 1/256$ and $k = 1/250$, denoted by $(\mathbf{u}^*, p^*, \boldsymbol{\lambda}^*)$, as the reference solution. We report numerical convergence orders of $\mathbf{u}^{hk} - \mathbf{u}^*$ in the $(L^2(\Omega))^2$ -norm and the $(H^1(\Omega))^2$ -norm, that of $p^{hk} - p^*$ in the $L^2(\Omega)$ -norm, and that of $\boldsymbol{\lambda}^{hk} - \boldsymbol{\lambda}^*$ in the $(L^2(\Gamma_S))^2$ -norm. For the numerical convergence orders with respect to h , we fix the time step-size at $k = 1/250$. For the numerical convergence orders with respect to k , we fix the mesh-size at $h = 1/256$. We also provide the total number of Uzawa iterations required to get the numerical solution at $T = 1$.

Tables 1 and 2 show the numerical convergence orders with respect to h and k , respectively. Values of the $L^2(\Omega)^2$ -norm and $H^1(\Omega)^2$ -norm of $\mathbf{u}^*$, the $L^2(\Omega)$ -norm of p^* , and the $L^2(\Gamma_S)^2$ -norm of $\boldsymbol{\lambda}^*$ for the reference solution at $T = 1$ are

$$\begin{aligned} \|\mathbf{u}^*\|_{L^2(\Omega)^2} &\doteq 4.138 \times 10^{-2}, & \|\mathbf{u}^*\|_{H^1(\Omega)^2} &\doteq 2.885 \times 10^{-1}, \\ \|p^*\|_{L^2(\Omega)} &\doteq 2.892 \times 10^{-1}, & \|\boldsymbol{\lambda}^*\|_{L^2(\Gamma_S)^2} &\doteq 8.954 \times 10^{-1}. \end{aligned}$$

The numerical results at $T = 1$ for $h = 1/16$, $k = 1/250$ are shown in Figures 2–4. To illustrate the friction effect, in Figures 5 and 6, we show $\mathbf{u}_\tau$ and $\boldsymbol{\sigma}_\tau$ along the slip boundary at $T = 1$ for the reference solution.

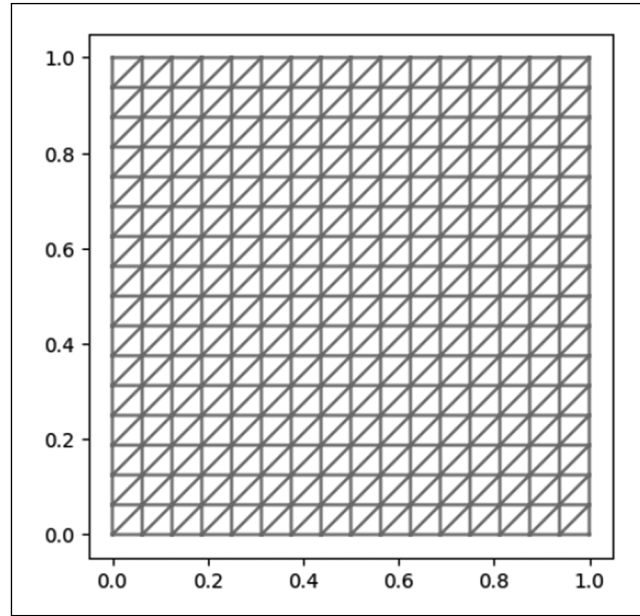


Figure 1. Triangulation for Example 6.1.

Table 1. Numerical convergence orders w.r.t. h at $t = 1$ for Example 6.1, $k = 1/250$.

h	$\ u^h - u^*\ _{L^2}$	Order	$\ u^h - u^*\ _{H^1}$	Order	$\ p^h - p^*\ _{L^2}$	Order	$\ \lambda^h - \lambda^*\ _{L^2}$	Order	Iter. No.
1/8	5.361e-03	-	9.380e-02	-	7.281e-02	-	3.469e-01	-	11207
1/16	1.524e-03	1.81	4.647e-02	1.01	2.336e-02	1.64	1.388e-01	1.32	10680
1/32	3.928e-04	1.96	2.281e-02	1.03	7.230e-03	1.69	5.297e-02	1.39	10373
1/64	9.481e-05	2.05	1.117e-02	1.03	2.076e-03	1.80	1.937e-02	1.45	10273

Table 2. Numerical convergence orders w.r.t. k at $t = 1$ for Example 6.1, $h = 1/256$.

k	$\ u^k - u^*\ _{L^2}$	Order	$\ u^k - u^*\ _{H^1}$	Order	$\ p^k - p^*\ _{L^2}$	Order	$\ \lambda^k - \lambda^*\ _{L^2}$	Order	Iter. No.
1/5	9.220e-03	-	6.237e-02	-	5.103e-02	-	1.296e-01	-	292
1/10	4.533e-03	1.02	3.065e-02	1.03	2.487e-02	1.04	6.581e-02	0.98	630
1/20	2.175e-03	1.06	1.470e-02	1.06	1.189e-02	1.06	3.302e-02	1.00	1269
1/40	9.935e-04	1.13	6.714e-03	1.13	5.420e-03	1.13	1.621e-02	1.03	2286

The bottom side of the domain is straight and is parallel to the x -axis. Therefore, the tangential velocity u_τ on the side can be equated with its first component, and so can the friction force σ_τ on the side be.

We observe that the total number of Uzawa iterations required to reach the numerical solution at the final time $T = 1$ is pretty stable with respect to the mesh-size h of the spatial discretization, for a fixed time-step k . As h gets smaller, there is a slight decrease for the total number of Uzawa iterations; this is likely because the initial guess $\lambda_{0,n}^{hk} = \lambda_{n-1}^{hk}$ at n -th iteration provides a somewhat more accurate approximation to λ for a smaller h .

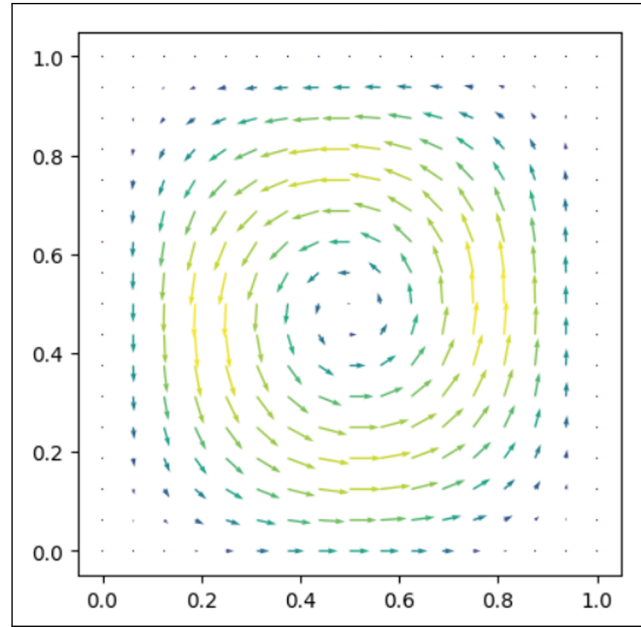


Figure 2. Velocity field for Example 6.1.

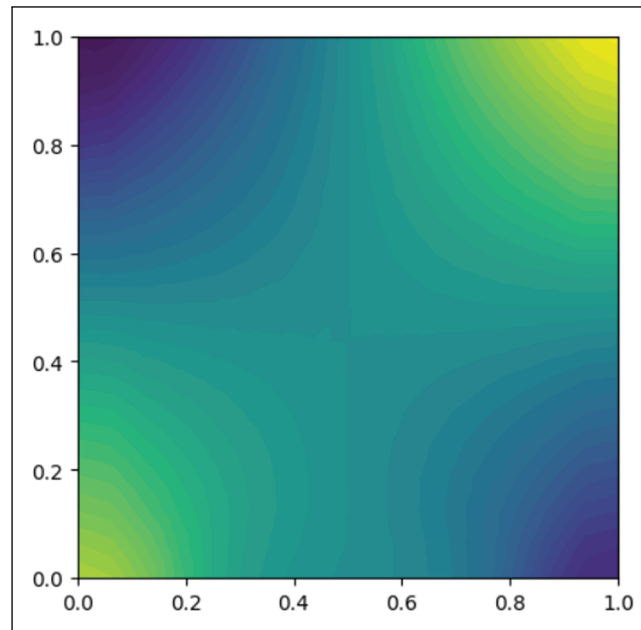


Figure 3. Pressure isobars for Example 6.1.

6.2. Example 6.2

In the second example, the domain is a curved variant of the unit square, with the bottom side replaced by a curve defined by

$$y = \frac{1}{10} \sin(2\pi x), \quad x \in [0, 1].$$

We start with the uniform triangulations of the unit square as in Example 6.1. Then, to account for the curved shape, we deform the mesh by adjusting the coordinates of the nodes along the bottom edge. The FEniCS package automatically incorporates the resulting curved geometry into the later finite element computations. Figure 7 shows such a triangulation with $h = 1/16$.

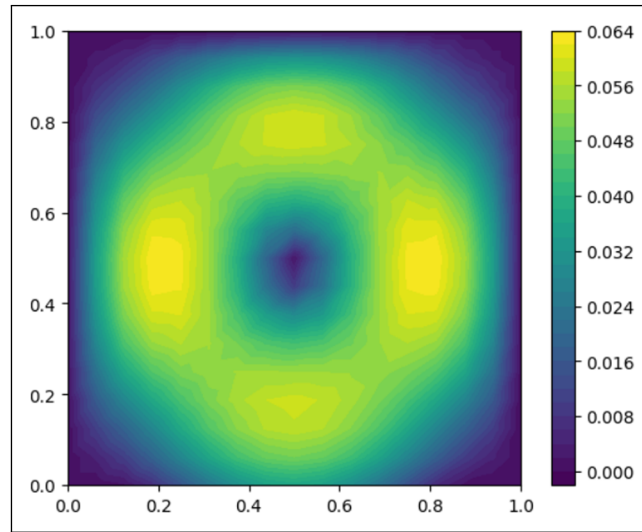


Figure 4. Magnitude of the velocity $|u^{hk}|$ for Example 6.1.

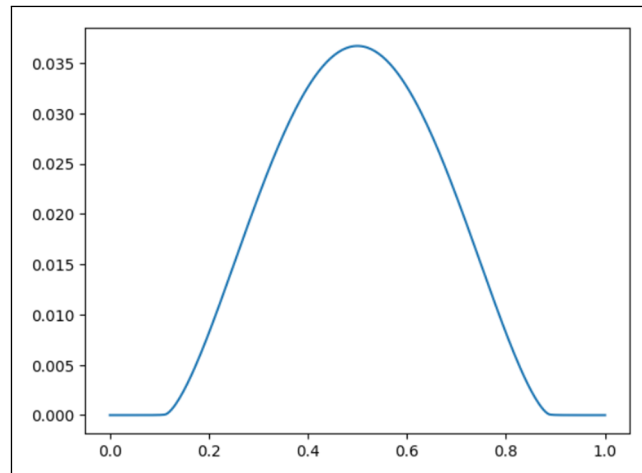


Figure 5. u_{τ} along the slip boundary for Example 6.1.

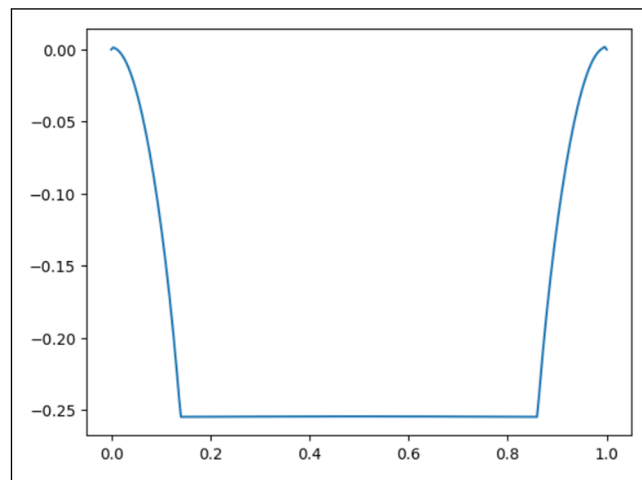


Figure 6. σ_{τ} along the slip boundary for Example 6.1.

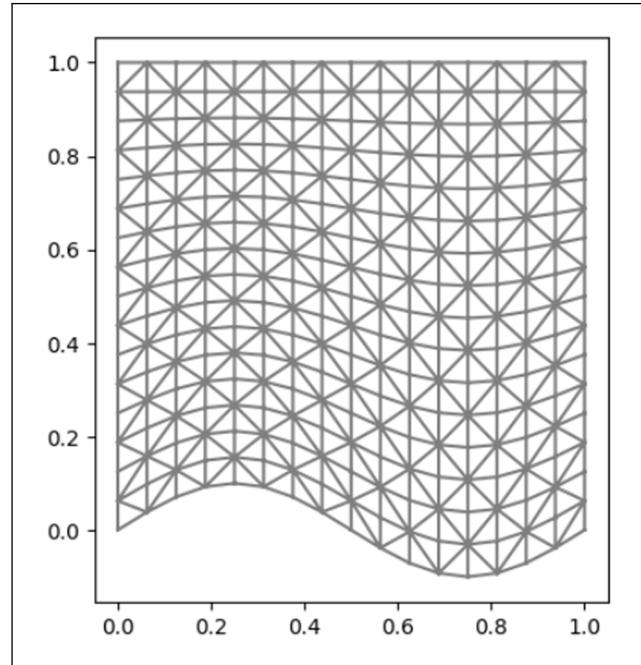


Figure 7. Triangulation for Example 6.2.

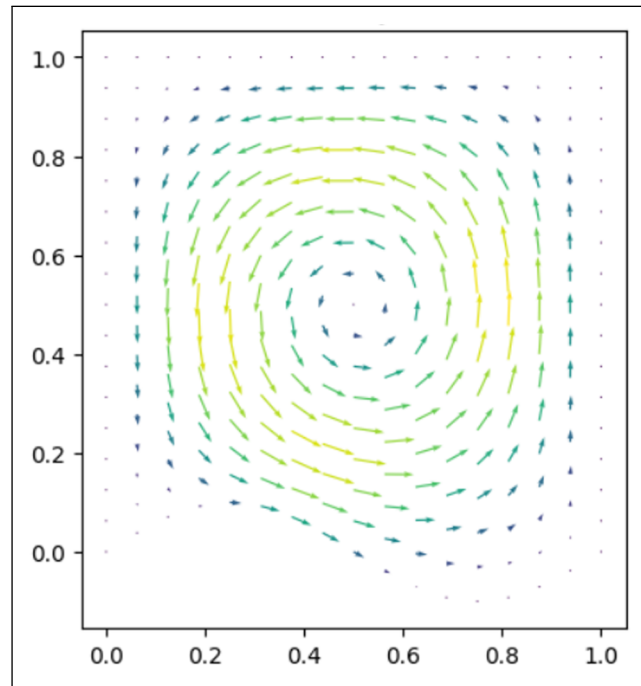


Figure 8. Velocity field for Example 6.2.

As in Example 6.1, we use the numerical solution with $h=1/256$ and $k=1/250$, denoted by $(\mathbf{u}^*, p^*, \boldsymbol{\lambda}^*)$, as the reference solution in computing errors of the numerical solutions on coarse meshes and/or with bigger time step-sizes. Tables 3 and 4 show the numerical convergence orders as h decreases for a fixed time step $k=1/250$, and the numerical convergence orders as k decreases for a fixed mesh size $h=1/256$, respectively. Values of the $L^2(\Omega)^2$ -norm and $H^1(\Omega)^2$ -norm of $\mathbf{u}^*$, the $L^2(\Omega)$ -norm of p^* ,

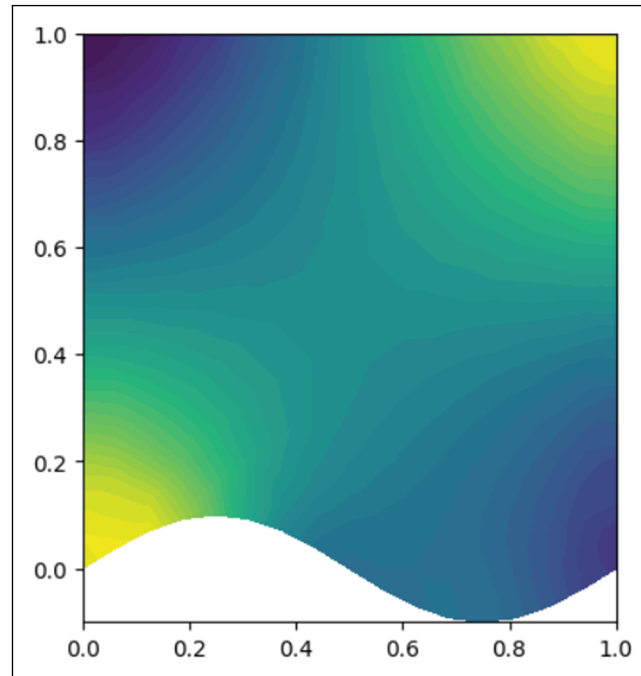


Figure 9. Pressure isobars for Example 6.2.

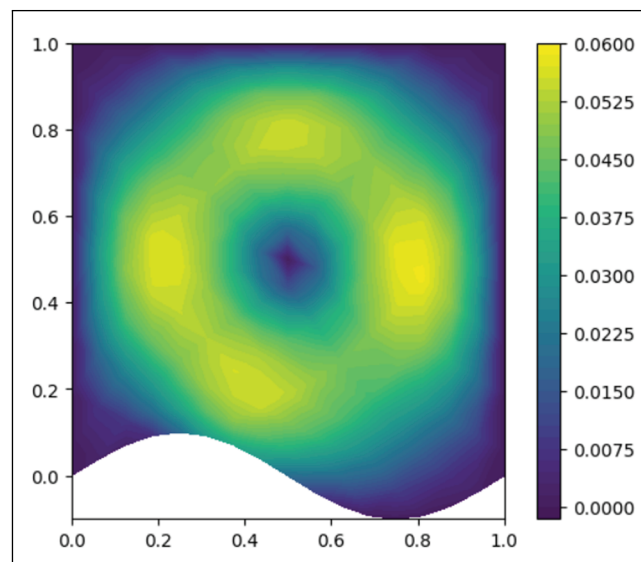


Figure 10. Magnitude of the velocity $|u^{hk}|$ for Example 6.2.

Table 3. Numerical convergence orders w.r.t. h for Example 6.2, $k = 1/250$.

h	$\ u^h - u^*\ _{L^2}$	Order	$\ u^h - u^*\ _{H^1}$	Order	$\ p^h - p^*\ _{L^2}$	Order	$\ \lambda^h - \lambda^*\ _{L^2}$	Order	Iter. No.
1/8	6.029e-03	-	9.243e-02	-	1.008e-01	-	5.375e-01	-	12810
1/16	2.060e-03	1.55	4.644e-02	0.99	4.542e-02	1.15	3.669e-01	0.55	12142
1/32	7.202e-04	1.52	2.306e-02	1.01	2.243e-02	1.02	2.096e-01	0.81	11730
1/64	2.599e-04	1.47	1.178e-02	0.97	1.086e-02	1.05	1.064e-01	0.98	11776

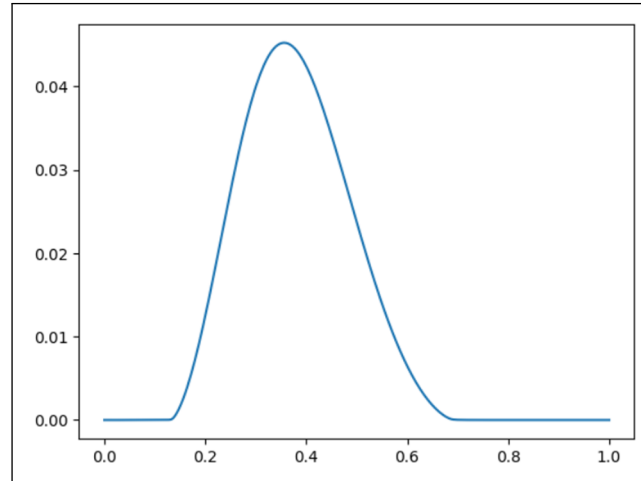


Figure 11. $|u_\tau|$ along the slip boundary for Example 6.2.

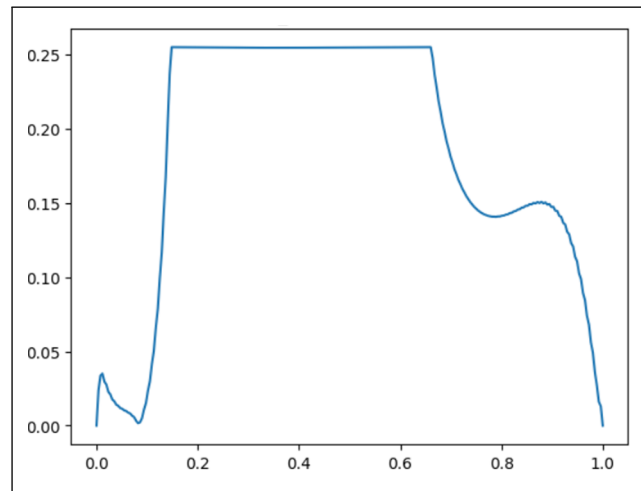


Figure 12. $|\sigma_\tau|$ along the slip boundary for Example 6.2.

Table 4. Numerical convergence orders w.r.t. k for Example 6.2, $h = 1/256$.

k	$\ u^k - u^*\ _{L^2}$	Order	$\ u^k - u^*\ _{H^1}$	Order	$\ p^k - p^*\ _{L^2}$	Order	$\ \lambda^k - \lambda^*\ _{L^2}$	Order	Iter. No.
1/5	8.556e-03	-	5.979e-02	-	5.772e-02	-	1.461e-01	-	387
1/10	4.203e-03	1.02	2.935e-02	1.02	2.825e-02	1.03	7.527e-02	0.96	832
1/20	2.016e-03	1.06	1.407e-02	1.06	1.353e-02	1.06	3.784e-02	0.99	1645
1/40	9.208e-04	1.13	6.426e-03	1.13	6.176e-03	1.13	1.907e-02	0.99	3000

and the $L^2(\Gamma_S)^2$ -norm of λ^* for the reference solution at the final time $T = 1$:

$$\begin{aligned} \|u^*\|_{L^2(\Omega)^2} &\doteq 3.947 \times 10^{-2}, & \|u^*\|_{H^1(\Omega)^2} &\doteq 2.807 \times 10^{-1}, \\ \|p^*\|_{L^2(\Omega)} &\doteq 3.162 \times 10^{-1}, & \|\lambda^*\|_{L^2(\Gamma_S)^2} &\doteq 8.523 \times 10^{-1}. \end{aligned}$$

The numerical results at the final time when $h = 1/16$, $k = 1/250$ are shown in Figures 8–10. The magnitudes of the tangential velocity and the friction force on the slip boundary for the reference solution are shown in Figures 11 and 12.

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ORCID iDs

Weimin Han  <https://orcid.org/0000-0002-0960-2281>

Yuan Yao  <https://orcid.org/0009-0008-0612-9709>

Shengda Zeng  <https://orcid.org/0000-0003-1818-842X>

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