

A second-order scheme for a nonstationary Stokes hemivariational inequality

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Abstract. This paper presents a temporal semi-discrete scheme for a nonstationary Stokes hemivariational inequality. To approximate the time derivative, a mid-point discretization is used in the first time step, and the BDF2 for the other time steps. Under appropriate solution regularity assumptions, the scheme is shown to be of second order with respect to the time step-size.

Keywords. Nonstationary Stokes hemivariational inequality, second-order scheme, error estimate.

1 Introduction

Through the formulation of hemivariational inequalities (HVIs), problems involving nonsmooth, nonmonotone, multivalued relations can be successfully treated, mathematically and numerically. One can consult [3, 5, 13, 14, 15] for mathematical theories and applications of HVIs. In particular, the book [5] offers a more accessible coverage of the mathematical theory without the need of the notion and results of pseudomonotone operators, commonly adopted in the majority of the references in the area. The reader is referred to the survey papers [8, 6] and the references therein for recent advances on numerical methods for solving HVIs.

Existence of a solution to a dynamic HVI can be shown through a limiting process on a temporally semi-discretization of the time derivative. For parabolic HVIs, several references can be consulted, cf. [11, 1, 2]. The backward Euler scheme (called the Rothe method) is used in [11, 1] for this purpose, whereas the BDF2 scheme is used in [2]. A nonstationary Stokes HVI is studied in [9, 10], including well-posedness and numerical analysis. The numerical method in [9] is based on a backward Euler difference for approximation of the time derivative, and a first-order error estimate with respect to the time-step is derived under appropriate solution regularity assumptions. *The goal of this paper is the development of a second-order temporal semi-discrete scheme for the nonstationary Stokes HVI. In the existing publications on the numerical solution of nonstationary Stokes HVIs, only first-order schemes on temporal discretizations can be found.*

We need notions of the generalized directional derivative and the generalized subdifferential in describing HVIs. For a locally Lipschitz continuous function $\Psi: V \rightarrow \mathbb{R}$ on a Banach space V , its generalized directional derivative at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u; v) = \limsup_{w \rightarrow u, \lambda \rightarrow 0} \frac{\Psi(w + \lambda v) - \Psi(w)}{\lambda},$$

and its generalized gradient or subdifferential at u is defined by

$$\partial\Psi(u) = \{\zeta \in V^* \mid \Psi^0(u; v) \geq \langle \zeta, v \rangle_{V^* \times V} \ \forall v \in V\}.$$

For properties and detailed discussions of the generalized directional derivative and the generalized subdifferential of a locally Lipschitz function, the reader is referred to [4] or [13, Section 3.2].

On several occasions, we will use the following elementary inequality, with a properly chosen $\epsilon > 0$,

$$ab \leq \epsilon a^2 + cb^2, \ c = 1/(4\epsilon) \quad \forall a, b \in \mathbb{R}. \quad (1.1)$$

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The rest of this paper is organized as follows. In Section 2, we describe the Stokes HVI and recall a well-posedness result. In Section 3, we present and analyze a semi-discrete scheme to solve the Stokes HVI. The time derivative is discretized with the mid-point rule in the first time step and with the backward difference formula of order two (BDF2) for other time steps. We derive an error bound for the semi-discrete solution under certain solution regularity assumptions; the error estimate is of optimal second-order with respect to the time-step for the semi-discrete scheme. Finally, we report simulation results on a numerical example, illustrating the second-order convergence in time.

2 A nonstationary Stokes hemivariational inequality

The model problem is discussed in [9, 10]. For convenience, we describe the problem in this section. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, the dimension $d \leq 3$ for applications. The boundary Γ of the domain is split into two non-trivial disjoint parts: $\Gamma = \Gamma_D \cup \Gamma_S$, Γ_D being relatively closed and Γ_S being relatively open. The unit outward normal vector on the boundary is denoted by $\boldsymbol{\nu}$. We consider the motion of an incompressible viscous fluid flow in Ω . We use $\mathbf{u}$ and p for the velocity and pressure fields of the fluid. Denote by $\mu > 0$ the viscosity of the fluid. We use the Stokes equations to model the flow. Let $\mathbf{f}$ be the density of the volume force applied to the fluid and let $\mathbf{u}_0: \Omega \rightarrow \mathbb{R}^d$ be the initial velocity. Denote the stress tensor $\boldsymbol{\sigma} = 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) - p\mathbf{I}$, where $\boldsymbol{\varepsilon}(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$ and $\mathbf{I}$ is the identity tensor. The normal and tangential components of $\mathbf{u}$ on the boundary are $u_\nu = \mathbf{u} \cdot \boldsymbol{\nu}$ and $\mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}$, and those of the stress are $\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. We impose the homogeneous Dirichlet boundary condition on Γ_D and a non-smooth, possibly non-monotone, friction condition with a potential function ψ on Γ_S . The classical pointwise formulation of the problem is

$$\mathbf{u}' - 2\mu \operatorname{div} \boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } (0, T) \times \Omega, \quad (2.1)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } (0, T) \times \Omega, \quad (2.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } (0, T) \times \Gamma_D, \quad (2.3)$$

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi(\mathbf{u}_\tau) \quad \text{on } (0, T) \times \Gamma_S. \quad (2.4)$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega, \quad (2.5)$$

Here, (2.4) describes the non-leak slip boundary condition of the friction type where the potential function ψ is assumed to be locally Lipschitz continuous. Such a boundary condition can be used to model stick-slip behavior at fluid-solid interfaces where the transition between the stick status and the slip status may exhibit hysteresis or abrupt jumps, leading to multivalued and nonmonotone friction laws. Some applications include lubricated mechanical contacts, microfluidic channels with engineered surfaces, polymer melts flowing along walls.

For simplicity in writing, we use $I_{\Gamma_S}(z)$ for the integral of a function z over Γ_S . Introduce function spaces

$$\begin{aligned} \mathbf{V} &= \left\{ \mathbf{v} \in H^1(\Omega)^d \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D, v_\nu = 0 \text{ on } \Gamma_S \right\}, \quad \mathbf{Q} = \{q \in L^2(\Omega) \mid (q, 1)_{L^2(\Omega)} = 0\}, \\ \mathbf{H} &= L^2(\Omega)^d, \quad \mathcal{V} = L^2(0, T; \mathbf{V}), \quad \mathcal{W} = \{\mathbf{v} \in \mathcal{V} \mid \mathbf{v}' \in \mathcal{V}^*\}, \quad \mathcal{Q} = L^2(0, T; \mathbf{Q}). \end{aligned}$$

Thanks to Korn's inequality, $\mathbf{V}$ is a Hilbert space with the inner product

$$(\mathbf{u}, \mathbf{v})_{\mathbf{V}} := \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V},$$

and the induced norm $\|\mathbf{v}\|_{\mathbf{V}} = (\mathbf{v}, \mathbf{v})_{\mathbf{V}}^{1/2}$ is equivalent to the standard $H^1(\Omega)^d$ -norm over $\mathbf{V}$. The spaces $\mathbf{Q}$, $\mathbf{H}$, $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Q}$ are all Hilbert spaces with their standard inner products. We have the trace inequality

$$\|\mathbf{v}_\tau\|_{L^2(\Gamma_S)^d} \leq \lambda_\tau^{-1/2} \|\mathbf{v}\|_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}, \quad (2.6)$$

where $\lambda_\tau > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\mathbf{u} \in \mathbf{V}, \quad (\mathbf{u}, \mathbf{v})_{\mathbf{V}} = \lambda (\mathbf{u}_\tau, \mathbf{v}_\tau)_{L^2(\Gamma_S)^d} \quad \forall \mathbf{v} \in \mathbf{V}.$$

Define bilinear forms

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) dx, \quad b(\mathbf{v}, q) = \int_{\Omega} q \operatorname{div} \mathbf{v} dx \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}, q \in Q.$$

Concerning the data, we assume

$$\mathbf{f} \in \mathcal{V}^*, \quad \mathbf{u}_0 \in \mathbf{H}, \quad (2.7)$$

and

$H(\psi)$: $\psi: \mathbb{R}^d \rightarrow \mathbb{R}$ is locally Lipschitz continuous on $\mathbb{R}^d$ and for two constants $c_0, \alpha_\psi > 0$,

(i) $|\boldsymbol{\eta}| \leq c_0(1 + |\boldsymbol{\xi}|)$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, $\boldsymbol{\eta} \in \partial\psi(\boldsymbol{\xi})$;

(ii) $(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq -\alpha_\psi |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2$ for all $\boldsymbol{\eta}_i \in \partial\psi(\boldsymbol{\xi}_i)$, $\boldsymbol{\xi}_i \in \mathbb{R}^d$, $i = 1, 2$.

Similar to [5, Lemma 5.17], we have the following inequality needed in the next section:

$$I_{\Gamma_S}(\psi^0(\mathbf{u}_\tau; \mathbf{v}_\tau - \mathbf{u}_\tau) + \psi^0(\mathbf{v}_\tau; \mathbf{u}_\tau - \mathbf{v}_\tau)) \leq \alpha_\psi \lambda_\tau^{-1} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{V}}^2 \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}. \quad (2.8)$$

The problem (2.1)–(2.5) is studied through its weak formulation.

Problem 2.1 Find $\mathbf{u} \in \mathcal{W}$ and $p \in \mathcal{Q}$ such that $\mathbf{u}(0) = \mathbf{u}_0$ and for a.e. $t \in (0, T)$,

$$\langle \mathbf{u}'(t), \mathbf{v} \rangle + a(\mathbf{u}(t), \mathbf{v}) - b(\mathbf{v}, p(t)) + I_{\Gamma_S}(\psi^0(\mathbf{u}_\tau(t); \mathbf{v}_\tau)) \geq \langle \mathbf{f}(t), \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}, \quad (2.9)$$

$$b(\mathbf{u}(t), q) = 0 \quad \forall q \in Q. \quad (2.10)$$

Theorem 2.2 ([10]) Assume (2.7), $H(\psi)$, and

$$m_0 := 2\mu - \alpha_\psi \lambda_\tau^{-1} > 0. \quad (2.11)$$

Then, Problem 2.1 admits a unique solution.

An assumption of the form (2.11) is known as a smallness condition in the literature. The condition requires the strength of the non-convexity of ψ , reflected by the value α_ψ , to be relative small compared to the viscosity of the fluid. This assumption is needed in proving the existence and uniqueness of Problem 2.1.

3 Analysis of a second-order semi-discretization scheme

In this section, we consider a temporal semi-discrete scheme to solve Problem 2.1. For a positive integer N , we split the time interval into N sub-intervals of equal length. The time step-size is $k = T/N$, and the nodal points are $t_n = nk$, $0 \leq n \leq N$. Denote $t_{1/2} = k/2$. Assume

$$\mathbf{f} \in C([0, T]; \mathbf{V}^*), \quad \mathbf{u}_0 \in \mathbf{V}. \quad (3.1)$$

For a continuous function $g(t)$, we denote $g_n = g(t_n)$, $0 \leq n \leq N$, and $g_{1/2} = g(t_{1/2})$. We use the backward difference formula (BDF2) to discretize the time derivative. The BDF2 scheme is strongly A-stable and is popularly used. In [12], a semi-discrete BDF2 scheme for a dynamic hemivariational inequality containing a history-dependent operator is discussed and an optimal second-order error bound in the time step-size is derived under appropriate solution regularity assumptions. Since the BDF2 scheme involves the numerical solution at three time-levels, in the first time step, we need a scheme involving two time-levels. In [12], the first time step is done through solving two hemivariational inequalities. In our method, the first step is based on a mid-point discretization, and consequently, only one hemivariational inequality needs to be solved. As we will show, the overall method is of second-order in time.

Problem 3.1 Find $\mathbf{u}^k := \{\mathbf{u}_n^k\}_{n=1}^N \subset \mathbf{V}$ and $p_{1/2}^k \in Q$, $\{p_n^k\}_{n=2}^N \subset Q$ such that with $\mathbf{u}_0^k = \mathbf{u}_0$,

$$\frac{1}{k} \left(\mathbf{u}_1^k - \mathbf{u}_0, \mathbf{v} \right)_{\mathbf{H}} + a(\mathbf{u}_{1/2}^k, \mathbf{v}) - b(\mathbf{v}, p_{1/2}^k) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{1/2, \tau}^k; \mathbf{v}_\tau)) \geq \langle \mathbf{f}_{1/2}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3.2)$$

$$b(\mathbf{u}_{1/2}^k, q) = 0 \quad \forall q \in Q, \quad (3.3)$$

where

$$\mathbf{u}_{1/2}^k = (\mathbf{u}_1^k + \mathbf{u}_0) / 2, \quad (3.4)$$

and for $n = 2, \dots, N$,

$$\frac{1}{2k} (3\mathbf{u}_n^k - 4\mathbf{u}_{n-1}^k + \mathbf{u}_{n-2}^k, \mathbf{v})_{\mathbf{H}} + a(\mathbf{u}_n^k, \mathbf{v}) - b(\mathbf{v}, p_n^k) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^k; \mathbf{v}_\tau)) \geq \langle \mathbf{f}_n, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3.5)$$

$$b(\mathbf{u}_n^k, q) = 0 \quad \forall q \in Q. \quad (3.6)$$

Proposition 3.2 Assume (3.1), $H(\psi)$, and (2.11). Then, Problem 3.1 has a unique solution.

Proof. First, we show the existence and uniqueness of $\mathbf{u}_{1/2}^k \in \mathbf{V}$. From (3.4), $\mathbf{u}_1^k = 2\mathbf{u}_{1/2}^k - \mathbf{u}_0$. Thus, (3.2) can be expressed in terms of the unknown $\mathbf{u}_{1/2}^k \in \mathbf{V}$ as

$$\frac{2}{k} (\mathbf{u}_{1/2}^k - \mathbf{u}_0, \mathbf{v})_{\mathbf{H}} + a(\mathbf{u}_{1/2}^k, \mathbf{v}) - b(\mathbf{v}, p_{1/2}^k) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{1/2,\tau}^k; \mathbf{v}_\tau)) \geq \langle \mathbf{f}_{1/2}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}. \quad (3.7)$$

i.e.,

$$(\mathbf{u}_{1/2}^k, \mathbf{v})_{\mathbf{H}} + \frac{k}{2} a(\mathbf{u}_{1/2}^k, \mathbf{v}) - \frac{k}{2} b(\mathbf{v}, p_{1/2}^k) + \frac{k}{2} I_{\Gamma_S}(\psi^0(\mathbf{u}_{1/2,\tau}^k; \mathbf{v}_\tau)) \geq (\mathbf{u}_0, \mathbf{v})_{\mathbf{H}} + \frac{k}{2} \langle \mathbf{f}_{1/2}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}. \quad (3.8)$$

By an analogue of [5, Theorem 8.22], we know that the problem (3.8) and (3.3) has a unique solution $(\mathbf{u}_{1/2}^k, p_{1/2}^k) \in \mathbf{V} \times Q$. Then we obtain a unique element $\mathbf{u}_1^k = 2\mathbf{u}_{1/2}^k - \mathbf{u}_0 \in \mathbf{V}$.

Consider the mixed HVI (3.5)–(3.6) for $(\mathbf{u}_n^k, p_n^k) \in \mathbf{V} \times Q$, $2 \leq n \leq N$. Rewrite the inequality (3.5) as

$$(\mathbf{u}_n^k, \mathbf{v})_{\mathbf{H}} + \frac{2k}{3} a(\mathbf{u}_n^k, \mathbf{v}) - \frac{2k}{3} b(\mathbf{v}, p_n^k) + \frac{2k}{3} I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}^k; \mathbf{v}_\tau)) \geq \frac{1}{3} (4\mathbf{u}_{n-1}^k - \mathbf{u}_{n-2}^k, \mathbf{v})_{\mathbf{H}} + \frac{2k}{3} \langle \mathbf{f}_n, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}.$$

Again, by an analogue of [5, Theorem 8.22], we have the existence of a unique solution $(\mathbf{u}_n^k, p_n^k) \in \mathbf{V} \times Q$ to the mixed HVI (3.5)–(3.6). ■

We turn to an error estimation for Problem 3.1. For this purpose, assume

$$\mathbf{u}'' \in \mathcal{V}, \quad \mathbf{u}^{(3)} \in \mathcal{V}^*. \quad (3.9)$$

This implies $\mathbf{u}'' \in C([0, T]; \mathbf{H})$, and (2.9)–(2.10) hold for all $t \in (0, T)$. We introduce some auxiliary bounds.

Lemma 3.3 Assume (3.9). Define

$$\begin{aligned} E_{1/2} &= \frac{\mathbf{u}_1 - \mathbf{u}_0}{k} - \mathbf{u}'_{1/2}, \quad F_{1/2} = \mathbf{u}_{1/2} - \frac{\mathbf{u}_1 + \mathbf{u}_0}{2}, \\ E_n &= \frac{1}{2k} (3\mathbf{u}_n - 4\mathbf{u}_{n-1} + \mathbf{u}_{n-2}) - \mathbf{u}'_n, \quad n = 2, \dots, N. \end{aligned}$$

Then, for some constant $c > 0$,

$$\|E_{1/2}\|_{\mathbf{V}^*}^2 \leq c k^3 \|\mathbf{u}^{(3)}\|_{L^2(t_0, t_1; \mathbf{V}^*)}^2, \quad (3.10)$$

$$\|F_{1/2}\|_{\mathbf{H}} \leq c k^2 \|\mathbf{u}''\|_{L^\infty(\mathbf{H})}, \quad (3.11)$$

$$\|F_{1/2}\|_{\mathbf{V}}^2 \leq c k^3 \|\mathbf{u}''\|_{L^2(t_0, t_1; \mathbf{V})}^2, \quad (3.12)$$

$$\|E_n\|_{\mathbf{V}^*}^2 \leq c k^3 \|\mathbf{u}^{(3)}\|_{L^2(t_{n-2}, t_n; \mathbf{V}^*)}^2, \quad 2 \leq n \leq N, \quad (3.13)$$

$$k \sum_{n=2}^N \|E_n\|_{\mathbf{V}^*}^2 \leq c k^4 \|\mathbf{u}^{(3)}\|_{L^2(\mathbf{V}^*)}^2. \quad (3.14)$$

Proof. By [7, Lemma 11.5], $\|E_{1/2}\|_{\mathbf{V}^*} \leq ck \|\mathbf{u}^{(3)}\|_{L^1(t_0, t_1; \mathbf{V}^*)}$ which implies (3.10). From [7, Lemma 11.2], we have the bound (3.11) and $\|F_{1/2}\|_{\mathbf{V}} \leq ck \|\mathbf{u}''\|_{L^1(t_0, t_1; \mathbf{V})}$; (3.12) follows from the latter. A proof of bound (3.13) can be found in [12]. The inequality (3.14) follows easily from (3.13). $\blacksquare$

Denote the error

$$\mathbf{e}_{1/2} := \mathbf{u}_{1/2} - \mathbf{u}_{1/2}^k, \quad \mathbf{e}_n := \mathbf{u}_n - \mathbf{u}_n^k, \quad 0 \leq n \leq N. \quad (3.15)$$

First, we derive an error bound on $\mathbf{e}_1 = \mathbf{u}_1 - \mathbf{u}_1^k$. Take $\mathbf{v} = \mathbf{u}_{1/2}^k - \mathbf{u}_{1/2} = -\mathbf{e}_{1/2}$ in (2.9) at $t = t_{1/2}$:

$$-\langle \mathbf{u}'_{1/2}, \mathbf{e}_{1/2} \rangle - a(\mathbf{u}_{1/2}, \mathbf{e}_{1/2}) + b(\mathbf{e}_{1/2}, p_{1/2}) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{1/2, \tau}; -\mathbf{e}_{1/2, \tau})) \geq -\langle \mathbf{f}_{1/2}, \mathbf{e}_{1/2} \rangle.$$

Take $\mathbf{v} = \mathbf{u}_{1/2} - \mathbf{u}_{1/2}^k = \mathbf{e}_{1/2}$ in (3.7):

$$\frac{2}{k} \left(\mathbf{u}_{1/2}^k - \mathbf{u}_0, \mathbf{e}_{1/2} \right)_{\mathbf{H}} + a(\mathbf{u}_{1/2}^k, \mathbf{e}_{1/2}) - b(\mathbf{e}_{1/2}, p_{1/2}^k) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{1/2, \tau}^k; \mathbf{e}_{1/2, \tau})) \geq \langle \mathbf{f}_{1/2}, \mathbf{e}_{1/2} \rangle.$$

We add the two inequalities to obtain

$$\begin{aligned} & -\langle \mathbf{u}'_{1/2}, \mathbf{e}_{1/2} \rangle + \frac{2}{k} \left(\mathbf{u}_{1/2}^k - \mathbf{u}_0, \mathbf{e}_{1/2} \right)_{\mathbf{H}} - a(\mathbf{e}_{1/2}, \mathbf{e}_{1/2}) + b(\mathbf{e}_{1/2}, p_{1/2} - p_{1/2}^k) \\ & + I_{\Gamma_S}(\psi^0(\mathbf{u}_{1/2, \tau}; -\mathbf{e}_{1/2, \tau}) + \psi^0(\mathbf{u}_{1/2, \tau}^k; \mathbf{e}_{1/2, \tau})) \geq 0, \end{aligned}$$

which is rewritten as, noting that $b(\mathbf{e}_{1/2}, p_{1/2} - p_{1/2}^k) = 0$ by (2.10) and (3.3),

$$\frac{2}{k} \|\mathbf{e}_{1/2}\|_{\mathbf{H}}^2 + 2\mu \|\mathbf{e}_{1/2}\|_{\mathbf{V}}^2 \leq \langle E_{1/2}, \mathbf{e}_{1/2} \rangle + \frac{2}{k} (F_{1/2}, \mathbf{e}_{1/2})_{\mathbf{H}} + I_{\Gamma_S}(\psi^0(\mathbf{u}_{1/2, \tau}; -\mathbf{e}_{1/2, \tau}) + \psi^0(\mathbf{u}_{1/2, \tau}^k; \mathbf{e}_{1/2, \tau})).$$

Multiply both sides of the inequality by k and use (2.8) to obtain

$$2 \|\mathbf{e}_{1/2}\|_{\mathbf{H}}^2 + 2\mu k \|\mathbf{e}_{1/2}\|_{\mathbf{V}}^2 \leq k \langle E_{1/2}, \mathbf{e}_{1/2} \rangle + 2 (F_{1/2}, \mathbf{e}_{1/2})_{\mathbf{H}} + \alpha_\psi \lambda_\tau^{-1} k \|\mathbf{e}_{1/2}\|_{\mathbf{V}}^2.$$

From this inequality, we derive that, recalling the definition (2.11) for m_0 ,

$$2 \|\mathbf{e}_{1/2}\|_{\mathbf{H}}^2 + m_0 k \|\mathbf{e}_{1/2}\|_{\mathbf{V}}^2 \leq k \|E_{1/2}\|_{\mathbf{V}^*} \|\mathbf{e}_{1/2}\|_{\mathbf{V}} + 2 \|F_{1/2}\|_{\mathbf{H}} \|\mathbf{e}_{1/2}\|_{\mathbf{H}}. \quad (3.16)$$

Applying the inequality (1.1) with $\epsilon = \min(1, m_0/2)$,

$$k \|\mathbf{e}_{1/2}\|_{\mathbf{V}} \|E_{1/2}\|_{\mathbf{V}^*} \leq \epsilon k \|\mathbf{e}_{1/2}\|_{\mathbf{V}}^2 + ck \|E_{1/2}\|_{\mathbf{V}^*}^2, \quad 2 \|\mathbf{e}_{1/2}\|_{\mathbf{H}} \|F_{1/2}\|_{\mathbf{H}} \leq \epsilon \|\mathbf{e}_{1/2}\|_{\mathbf{H}}^2 + c \|F_{1/2}\|_{\mathbf{H}}^2,$$

we derive from (3.16) that

$$\|\mathbf{e}_{1/2}\|_{\mathbf{H}}^2 + k \|\mathbf{e}_{1/2}\|_{\mathbf{V}}^2 \leq ck \|E_{1/2}\|_{\mathbf{V}^*}^2 + c \|F_{1/2}\|_{\mathbf{H}}^2.$$

Since $\mathbf{e}_1 := \mathbf{u}_1 - \mathbf{u}_1^k = 2\mathbf{e}_{1/2} - 2F_{1/2}$, we find from the above inequality that

$$\|\mathbf{e}_1\|_{\mathbf{H}}^2 + k \|\mathbf{e}_1\|_{\mathbf{V}}^2 \leq c (k \|E_{1/2}\|_{\mathbf{V}^*}^2 + \|F_{1/2}\|_{\mathbf{H}}^2 + k \|F_{1/2}\|_{\mathbf{V}}^2).$$

Applying (3.10), (3.11) and (3.12), get the bound

$$\|\mathbf{e}_1\|_{\mathbf{H}}^2 + k \|\mathbf{e}_1\|_{\mathbf{V}}^2 \leq ck^4 \left(\|\mathbf{u}^{(3)}\|_{L^2(t_0, t_1; \mathbf{V}^*)}^2 + \|\mathbf{u}''\|_{L^\infty(\mathbf{H})}^2 + \|\mathbf{u}''\|_{L^2(t_0, t_1; \mathbf{V})}^2 \right). \quad (3.17)$$

For $2 \leq n \leq N$, take $\mathbf{v} = \mathbf{u}_n^k - \mathbf{u}_n = -\mathbf{e}_n$ in (2.9) at $t = t_n$:

$$-\langle \mathbf{u}'_n, \mathbf{e}_n \rangle - a(\mathbf{u}_n, \mathbf{e}_n) + b(\mathbf{e}_n, p_n) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{n, \tau}; -\mathbf{e}_{n, \tau})) \geq -\langle \mathbf{f}_n, \mathbf{e}_n \rangle.$$

Take $\mathbf{v} = \mathbf{u}_n - \mathbf{u}_n^k = \mathbf{e}_n$ in (3.5):

$$\frac{1}{2k} \left(3\mathbf{u}_n^k - 4\mathbf{u}_{n-1}^k + \mathbf{u}_{n-2}^k, \mathbf{e}_n \right)_{\mathbf{H}} + a(\mathbf{u}_n^k, \mathbf{e}_n) - b(\mathbf{e}_n, p_n^k) + I_{\Gamma_S}(\psi^0(\mathbf{u}_{n, \tau}^k; \mathbf{e}_{n, \tau})) \geq \langle \mathbf{f}_n, \mathbf{e}_n \rangle.$$

Add the above two inequalities, noting that $b(\mathbf{e}_n, p_n) = 0$ by (2.10) and $b(\mathbf{e}_n, p_n^k) = 0$ by (3.3),

$$\frac{1}{2k} (3\mathbf{e}_n - 4\mathbf{e}_{n-1} + \mathbf{e}_{n-2}, \mathbf{e}_n)_{\mathbf{H}} + a(\mathbf{e}_n, \mathbf{e}_n) \leq \langle E_n, \mathbf{e}_n \rangle + I_{\Gamma_S}(\psi^0(\mathbf{u}_{n,\tau}; -\mathbf{e}_{n,\tau}) + \psi^0(\mathbf{u}_{n,\tau}^k; \mathbf{e}_{n,\tau})).$$

Use the identity

$$(3\mathbf{u} - 4\mathbf{v} + \mathbf{w}, \mathbf{u})_{\mathbf{H}} = \frac{1}{2} (\|\mathbf{u}\|_{\mathbf{H}}^2 - \|\mathbf{v}\|_{\mathbf{H}}^2 + \|2\mathbf{u} - \mathbf{v}\|_{\mathbf{H}}^2 - \|2\mathbf{v} - \mathbf{w}\|_{\mathbf{H}}^2 + \|\mathbf{u} - 2\mathbf{v} + \mathbf{w}\|_{\mathbf{H}}^2),$$

and (2.8) to obtain

$$\|\mathbf{e}_n\|_{\mathbf{H}}^2 - \|\mathbf{e}_{n-1}\|_{\mathbf{H}}^2 + \|2\mathbf{e}_n - \mathbf{e}_{n-1}\|_{\mathbf{H}}^2 - \|2\mathbf{e}_{n-1} - \mathbf{e}_{n-2}\|_{\mathbf{H}}^2 + 4m_0k \|\mathbf{e}_n\|_{\mathbf{V}}^2 \leq 4k \|E_n\|_{\mathbf{V}^*} \|\mathbf{e}_n\|_{\mathbf{V}}.$$

Apply the inequality (1.1) with a proper choice of ϵ ,

$$4k \|E_n\|_{\mathbf{V}^*} \|\mathbf{e}_n\|_{\mathbf{V}} \leq 2m_0k \|\mathbf{e}_n\|_{\mathbf{V}}^2 + ck \|E_n\|_{\mathbf{V}^*}^2.$$

Then, for $2 \leq n \leq N$,

$$\|\mathbf{e}_n\|_{\mathbf{H}}^2 - \|\mathbf{e}_{n-1}\|_{\mathbf{H}}^2 + \|2\mathbf{e}_n - \mathbf{e}_{n-1}\|_{\mathbf{H}}^2 - \|2\mathbf{e}_{n-1} - \mathbf{e}_{n-2}\|_{\mathbf{H}}^2 + 2m_0k \|\mathbf{e}_n\|_{\mathbf{V}}^2 \leq ck \|E_n\|_{\mathbf{V}^*}^2. \quad (3.18)$$

Replace n by j in (3.18) and add the inequalities from $j = 2$ to $j = n$:

$$\|\mathbf{e}_n\|_{\mathbf{H}}^2 - \|\mathbf{e}_1\|_{\mathbf{H}}^2 + \|2\mathbf{e}_n - \mathbf{e}_{n-1}\|_{\mathbf{H}}^2 - \|2\mathbf{e}_1 - \mathbf{e}_0\|_{\mathbf{H}}^2 + 2m_0k \sum_{j=2}^n \|\mathbf{e}_j\|_{\mathbf{V}}^2 \leq ck \sum_{j=2}^n \|E_j\|_{\mathbf{V}^*}^2.$$

Then,

$$\|\mathbf{e}_n\|_{\mathbf{H}}^2 + 2m_0k \sum_{j=2}^n \|\mathbf{e}_j\|_{\mathbf{V}}^2 \leq c (\|\mathbf{e}_1\|_{\mathbf{H}}^2 + \|\mathbf{e}_0\|_{\mathbf{H}}^2) + 4k \sum_{j=2}^n \|E_j\|_{\mathbf{V}^*}^2. \quad (3.19)$$

We derive from (3.19) that

$$\max_{2 \leq n \leq N} \|\mathbf{e}_n\|_{\mathbf{H}}^2 + k \sum_{n=2}^N \|\mathbf{e}_n\|_{\mathbf{V}}^2 \leq c (\|\mathbf{e}_1\|_{\mathbf{H}}^2 + \|\mathbf{e}_0\|_{\mathbf{H}}^2) + 4k \sum_{n=2}^N \|E_n\|_{\mathbf{V}^*}^2. \quad (3.20)$$

Note that $\mathbf{e}_0 = \mathbf{0}$. We combine (3.17) and (3.20), and apply (3.14) to obtain the following result.

Theorem 3.4 *Assume $H(\psi)$, (2.11), (3.1) and (3.9). Then, for the semi-discrete solution defined by Problem 3.1, we have the optimal second-order error bound*

$$\max_{0 \leq n \leq N} \|\mathbf{u}_n - \mathbf{u}_n^k\|_{\mathbf{H}}^2 + k \sum_{n=0}^N \|\mathbf{u}_n - \mathbf{u}_n^k\|_{\mathbf{V}}^2 \leq ck^4 \left(\|\mathbf{u}^{(3)}\|_{L^2(\mathbf{V}^*)}^2 + \|\mathbf{u}''\|_{L^\infty(\mathbf{H})}^2 + \|\mathbf{u}''\|_{L^2(\mathbf{V})}^2 \right).$$

4 A numerical example

We take $\Omega = (0, 1) \times (0, 1)$, $\mu = 1$, $\psi(\mathbf{z}) = \int_0^{|\mathbf{z}|} \omega(t) dt$ with $\omega(t) = 0.005e^{-t} + 0.25$, $\mathbf{u}_0 = \mathbf{0}$, and the source function

$$\left(10x^2(1-x)^2y(1-y)(1-2y) + 2t^2(-10(1-6x+6x^2)y(1-y)(1-2y) + 30x^2(1-x)^2(1-2y) - (1-2x)) \right. \\ \left. - 10y^2(1-y)^2x(1-x)(1-2x) + 2t^2(10(1-6y+6y^2)x(1-x)(1-2x) - 30y^2(1-y)^2(1-2x) - (1-2y)) \right).$$

For spatial discretization, we use the stable MINI finite elements corresponding to a uniform triangulation of Ω based on partitioning the four sides into 160 equal size subintervals. Then, we use the numerical solution for the step-size $k = 1/120$ as the reference solution to compute the errors of the numerical solutions for larger time steps. The $L^2(\Omega)$ -norm errors and numerical convergence orders at $t = 1$ are as given in the following table for time-steps $k = 1/5, 1/10, 1/20$ and $1/40$. We observe that the numerical convergence orders are around 2 with respect to the time step k , matching the theoretical prediction in Theorem 3.4.

k	1/5	1/10	1/20	1/40
$L^2(\Omega)$ -error	4.226e-5	1.351e-5	2.067e-6	4.043e-7
Order	-	1.65	2.71	2.35

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