



Analysis of a family of Navier-Stokes-type variational-hemivariational inequalities with applications

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ABSTRACT

The paper provides a well-posedness analysis for a family of stationary Navier-Stokes-type variational-hemivariational inequalities, motivated by applications in fluid mechanics. The family contains various mixed variational equations, mixed variational inequalities and mixed hemivariational inequalities found in the literature as special cases. The main features of the paper are that the existence and uniqueness of both the velocity field and pressure field are established, and the results are proved in an accessible fashion without the need of knowledge of abstract surjectivity results for pseudomonotone operators as required in many references on variational-hemivariational inequalities. The results are applied to the study of a variational-hemivariational inequality of the Navier-Stokes equations for incompressible fluid flows subject to slip conditions of frictional type, both monotone and non-monotone.

1. Introduction

Hemivariational inequalities and more generally, variational-hemivariational inequalities, are useful for applications involving non-smooth, non-monotone and set-valued relations among physical quantities. Since the pioneering work of Panagiotopoulos [1], modeling, analysis, numerical solution and applications of variational-hemivariational inequalities have attracted more and more attention. On the rapidly expanding list of references, a handful of books can be found for comprehensive coverage of the area, cf. [2–8] in early years, and [9–12] in recent years. Note that in most references on mathematical theory of variational-hemivariational inequalities, abstract surjectivity results on pseudomonotone operators are applied to show the existence of a solution. An alternative, accessible approach for the study of variational-hemivariational inequalities, started in the references [13,14], is systematically documented in the book [10] that does not require knowledge on abstract mathematical theory of pseudomonotone operators.

This paper is devoted to a well-posedness analysis of a family of mixed variational-hemivariational inequalities of Navier-Stokes-type, motivated by applications in fluid mechanics modeled with the Navier-Stokes equations. The problems considered are stationary/time-independent. On an abstract level, the problem is of the following form.

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Problem 1.1. Find $(u, p) \in K_V \times Q$ such that

$$a(u, v - u) + d(u, u, v - u) + b(v - u, p) + \Phi(v) - \Phi(u) + \Psi^0(u; v - u) \geq \langle f, v - u \rangle \quad \forall v \in K_V, \quad (1.1)$$

$$b(u, q) = 0 \quad \forall q \in Q. \quad (1.2)$$

In [Problem 1.1](#), K_V is a subset of a real Hilbert space V , Q is another real Hilbert space; $a: V \times V \rightarrow \mathbb{R}$ and $b: V \times Q \rightarrow \mathbb{R}$ are two bilinear forms; $d: V \times V \times V \rightarrow \mathbb{R}$ is a trilinear form; $\Phi: V \rightarrow \mathbb{R}$ and $\Psi: V \rightarrow \mathbb{R}$ are two functionals; and f is a linear functional on V . We use the standard notation $(\cdot, \cdot)$ for the inner product in V and $\langle \cdot, \cdot \rangle$ for the duality pairing on $V^* \times V$. The functional $\Psi: V \rightarrow \mathbb{R}$ is assumed to be locally Lipschitz continuous, and $\Psi^0(u; v)$ is the generalized (Clarke) directional derivative at u in the direction v :

$$\Psi^0(u; v) := \limsup_{w \rightarrow u, \lambda \downarrow 0} \frac{\Psi(w + \lambda v) - \Psi(w)}{\lambda}. \quad (1.3)$$

For application problems, the term $\Psi^0(u; v - u)$ in [Problem 1.1](#) is usually replaced by an integral of the generalized derivative of a locally Lipschitz continuous function $\psi: I_\Delta(\psi^0(\gamma u; \gamma v - \gamma u))$. Here, I_Δ stands for an integration over a spatial set Δ ; for a positive integer m , $\psi: \mathbb{R}^m \rightarrow \mathbb{R}$ is locally Lipschitz continuous and for a function space V_ψ on Δ , the operator $\gamma: V \rightarrow V_\psi$ is such that $\psi^0(\gamma u; \gamma v) \in L^1(\Delta)$ for all $u, v \in V$. In other words, the following variant of [Problem 1.1](#) is usually the form of mixed variational-hemivariational inequalities directly modeling the application problem.

Problem 1.2. Find $(u, p) \in K_V \times Q$ such that

$$a(u, v - u) + d(u, u, v - u) + b(v - u, p) + \Phi(v) - \Phi(u) + I_\Delta(\psi^0(\gamma u; \gamma v - \gamma u)) \geq \langle f, v - u \rangle \quad \forall v \in K_V, \quad (1.4)$$

$$b(u, q) = 0 \quad \forall q \in Q. \quad (1.5)$$

In the context of fluid mechanics, the unknowns u and p in these problems represent the velocity field and pressure field of the fluid flow. Following the nomenclature used in [\[10\]](#), [Problems 1.1](#) and [1.2](#) are mixed variational-hemivariational inequalities of rank $(1, 1)$ since the functional Φ depends on one variable and the functional Ψ or the function ψ depends on one variable.

In their general forms, the variational-hemivariational inequality contains both a nonsmooth convex term Φ and a nonsmooth nonconvex term, Ψ for [Problem 1.1](#) and ψ for [Problem 1.2](#). When the nonsmooth convex term Φ is absent and $K_V = V$, we have a commonly seen “pure” Navier-Stokes hemivariational inequality: find $(u, p) \in V \times Q$ such that

$$a(u, v) + d(u, u, v) + b(v, p) + \Psi^0(u; v) \geq \langle f, v \rangle \quad \forall v \in V, \quad (1.6)$$

$$b(u, q) = 0 \quad \forall q \in Q, \quad (1.7)$$

or

$$a(u, v) + d(u, u, v) + b(v, p) + I_\Delta(\psi^0(\gamma u; \gamma v)) \geq \langle f, v \rangle \quad \forall v \in V, \quad (1.8)$$

$$b(u, q) = 0 \quad \forall q \in Q. \quad (1.9)$$

With the use of a constrained subspace $\tilde{V}$ of V defined by

$$\tilde{V} = \{v \in V \mid b(v, q) = 0 \quad \forall q \in Q\},$$

a reduced form of the Navier-Stokes hemivariational inequality [\(1.6\)–\(1.7\)](#) or [\(1.8\)–\(1.9\)](#) can be introduced in which the pressure unknown p is eliminated: find $u \in \tilde{V}$ such that

$$a(u, v) + d(u, u, v) + \Psi^0(u; v) \geq \langle f, v \rangle \quad \forall v \in \tilde{V}, \quad (1.10)$$

or

$$a(u, v) + d(u, u, v) + I_\Delta(\psi^0(\gamma u; \gamma v)) \geq \langle f, v \rangle \quad \forall v \in \tilde{V}. \quad (1.11)$$

An early reference [\[15\]](#) provides an existence and uniqueness result for a velocity field of the problem [\(1.10\)](#) by applying a surjectivity result for pseudomonotone operators. In [\[16\]](#), well-posedness of the problem [\(1.11\)](#) is shown using the technique of a combination of minimization principles and fixed-point argument developed in [\[13\]](#) and [\[14\]](#).

In [\[17\]](#), the complete form of the Navier-Stokes hemivariational inequality [\(1.6\)–\(1.7\)](#) or [\(1.8\)–\(1.9\)](#) is studied and the pressure field is recovered starting with the reduced form of the problem by making use of the inf-sup property of the bilinear form $b(\cdot, \cdot)$. Then, the finite element method is applied and analyzed for the numerical solution of the problem [\(1.8\)–\(1.9\)](#).

In this paper, we consider the well-posedness of the Navier-Stokes variational-hemivariational inequality in the form of [Problem 1.1](#) and its variant in the form of [Problem 1.2](#). The proofs are based on results presented in [\[18\]](#) for a family of mixed variational-hemivariational inequalities of Stokes type (rephrased as [Theorem 2.2](#) below), together with a repeated application of the Banach fixed-point argument initially developed in [\[19\]](#) and further described in [\[10,20\]](#) (cf. [Problems 2.7](#) and [2.10](#)). In other words, for both [Problems 1.1](#) and [1.2](#), we provide existence and uniqueness results for both the velocity and pressure fields simultaneously, and moreover, the argument does not rely on surjectivity results for pseudomonotone operators. In the context of the application problem studied in [Section 4](#), we study a mixed variational-hemivariational inequality for the Navier-Stokes equations subject to a non-leak monotone slip boundary condition of friction type on a part of the boundary, $\Gamma_{S,1}$, and to a non-leak non-monotone slip boundary condition of friction type on another part of the boundary, $\Gamma_{S,2}$.

Classical boundary value problems of the Navier-Stokes equations correspond to the special case where $K_V = V$, $\Phi \equiv 0$, $\Psi \equiv 0$ or $\psi \equiv 0$, and (1.1) and (1.4) take the form of an equation:

$$a(u, v) + d(u, u, v) + b(v, p) = \langle f, v \rangle \quad \forall v \in V.$$

A large number of references is available for analysis and numerical solution of classical boundary value problems of the Navier-Stokes equations, and some representative comprehensive references are [21–25]. When the trilinear form $d(\cdot, \cdot, \cdot)$ is dropped, Problems 1.1 and 1.2 are reduced to a Stokes variational-hemivariational inequality. Well-posedness and numerical analysis of a “pure” Stokes hemivariational inequality can be found in [26]. Well-posedness of mixed variational-hemivariational inequalities applicable to studies of Stokes variational-hemivariational inequalities is developed in [27–30] by applying complicated fixed-point principles for set-valued mappings, in [31,32] by applying the standard saddle-point theory combined with the Banach fixed-point argument, and in [33] by applying a projection iteration technique for a mixed hemivariational inequality which is extended to a general mixed variational-hemivariational inequality in [18]. We also notice that the framework of Stokes-type variational-hemivariational inequalities include as a special case the Stokes-type variational inequalities found in the literature, e.g., [34–37].

Closely related to the notion of the generalized (Clarke) directional derivative $\Psi^0(u; v)$ defined in (1.3), a companion concept is the generalized subdifferential of the locally Lipschitz continuous functional Ψ at $u \in V$:

$$\partial\Psi(u) := \{\eta \in V^* \mid \Psi^0(u; v) \geq \langle \eta, v \rangle \quad \forall v \in V\}.$$

Detailed discussions of the generalized directional derivative and the generalized subdifferential for locally Lipschitz continuous functionals, including their properties, can be found in several references, e.g. [11,38].

The rest of the paper is organized as follows. In Section 2, we address the well-posedness of Problem 1.1. In Section 3, we address the well-posedness of Problem 1.2. In Section 4, we apply the results presented in previous sections in the study of a mixed variational-hemivariational inequality for the Navier-Stokes equations subject to non-leak slip boundary conditions of friction type.

2. Well-posedness of Problem 1.1

In this section, we present a well-posedness result on Problem 1.1. First, we list conditions on the problem data.

- $H(K_V)$ V is a real Hilbert space, $K_V \subset V$ is non-empty, closed and convex, and it contains a subspace $V_0 \subset V$.
- $H(Q)$ Q is a real Hilbert space.
- $H(a)$ $a : V \times V \rightarrow \mathbb{R}$ is bilinear, bounded and V -elliptic: for two constants $M_a \geq m_a > 0$,

$$|a(u, v)| \leq M_a \|u\|_V \|v\|_V \quad \forall u, v \in V, \quad (2.1)$$

$$a(v, v) \geq m_a \|v\|_V^2 \quad \forall v \in V. \quad (2.2)$$

- $H(b)$ $b : V \times Q \rightarrow \mathbb{R}$ is bilinear and bounded: for a constant $M_b > 0$,

$$|b(v, q)| \leq M_b \|v\|_V \|q\|_Q \quad \forall v \in V, q \in Q. \quad (2.3)$$

Moreover, the inf-sup condition holds:

$$\alpha_b \|q\|_Q \leq \sup_{w \in V_0} \frac{b(w, q)}{\|w\|_V} \quad \forall q \in Q. \quad (2.4)$$

- $H(d)$ $d : V \times V \times V \rightarrow \mathbb{R}$ is trilinear and bounded: for a constant $c_d > 0$,

$$|d(u, v, w)| \leq c_d \|u\|_V \|v\|_V \|w\|_V \quad \forall u, v, w \in V. \quad (2.5)$$

Moreover,

$$d(u, v, v) = 0 \quad \forall u \in \tilde{V}, v \in V, \quad (2.6)$$

where

$$\tilde{V} = \{v \in V \mid b(v, q) = 0 \quad \forall q \in Q\}. \quad (2.7)$$

- $H(\Phi)$ $\Phi : V \rightarrow \mathbb{R}$ is convex and bounded above on a non-empty open set, and

$$\Phi(v + w) = \Phi(v) \quad \forall v \in K_V, w \in V_0 \quad (2.8)$$

- $H(\Psi)$ $\Psi : V \rightarrow \mathbb{R}$ is locally Lipschitz continuous, and there exists a constant $\alpha_\Psi \geq 0$ such that

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall v_1, v_2 \in V. \quad (2.9)$$

Moreover,

$$\Psi^0(v; w) = 0 \quad \forall v \in K_V, w \in V_0. \quad (2.10)$$

- $H(s)$

$$m_a - \alpha_\Psi > 0, \quad (m_a - \alpha_\Psi)^2 > c_d c_\Psi, \quad (2.11)$$

where c_Ψ is a local Lipschitz constant of the functional Ψ at 0.

Under the assumption $H(K_V)$, it can be proved that $v + w \in K_V$ for any $v \in K_V$ and any $w \in V_0$ ([37, Lemma 2.1]).

Assumption $H(b)$ allows us to define an operator $B \in \mathcal{L}(V; Q)$ by the relation

$$(Bv, q)_Q = b(v, q) \quad \forall v \in V, q \in Q.$$

The inequality (2.3) is equivalent to

$$\|B\| \leq M_b.$$

Note that the assumption $\Phi : V \rightarrow \mathbb{R}$ being convex and bounded above on a non-empty open set implies that Φ is continuous on V (cf. [39]). The convex function $\Phi : V \rightarrow \mathbb{R}$ is assumed to be continuous, instead of l.s.c. As is explained in [13,14] or [10, Section 5.1], there is no loss of generality with the stronger assumption of continuity for a vast majority of applications. The constant $\alpha_\Psi \geq 0$ in (2.9) measures the strength of non-convexity of Ψ : the smaller the value of α_Ψ , the weaker the non-convexity of Ψ . In particular, in the degenerate case where Ψ is convex, (2.9) holds with $\alpha_\Psi = 0$. It is known that (2.9) if and only if $\Psi(v) + (\alpha_\Psi/2) \|v\|_V^2$ is convex (cf. [10, Theorem 2.43]). We also notice that (2.8) and (2.10) are valid for Problem 4.1 considered in Section 4 where functions in V_0 vanish on the boundary of the spatial domain and both Φ and Ψ solely depend on traces of their arguments on the boundary.

The assumption $H(s)$ consists of two parts. The first part is a smallness condition on the strength of the non-convexity of Ψ , relative to the V -ellipticity of $a(\cdot, \cdot)$: $\alpha_\Psi < m_a$. The second part $(m_a - \alpha_\Psi)^2 > c_d c_\Psi$ will allow us to impose a boundedness condition on f . We will fix a constant c_0 in the following range:

$$0 < c_0 < \frac{(m_a - \alpha_\Psi)^2}{c_d} - c_\Psi, \quad (2.12)$$

and denote

$$M_0 := \frac{c_\Psi + c_0}{m_a - \alpha_\Psi}. \quad (2.13)$$

It is easy to verify that

$$c_d M_0 < m_a - \alpha_\Psi. \quad (2.14)$$

Define a subset of V^* :

$$V_{c_0}^* = \{f \in V^* \mid \|f\|_{V^*} \leq c_0\}. \quad (2.15)$$

An existence and uniqueness result for Problem 1.1 is the following theorem.

Theorem 2.1. Assume $H(K_V)$, $H(Q)$, $H(a)$, $H(b)$, $H(d)$, $H(\Phi)$, $H(\Psi)$, and $H(s)$. Further assume $\Phi(v)$ achieves its minimum at $v = 0$. For any $f \in V_{c_0}^*$, Problem 1.1 has a unique solution $(u, p) \in K_V \times Q$.

This result will be proved in several steps. We will apply a result that follows from [18, Theorem 2.3].

Theorem 2.2. Assume $H(K_V)$, $H(Q)$, $H(a)$, $H(b)$, $H(\Phi)$, $H(\Psi)$, and $\alpha_\Psi < m_a$. Then, for any $f \in V^*$, the problem

$$a(u, v - u) + b(v - u, p) + \Phi(v) - \Phi(u) + \Psi^0(u; v - u) \geq \langle f, v - u \rangle \quad \forall v \in K_V, \quad (2.16)$$

$$b(u, q) = 0 \quad \forall q \in Q \quad (2.17)$$

has a unique solution $(u, p) \in K_V \times Q$.

In the rest of the section, we keep the assumptions stated in Theorem 2.1. Now we present several lemmas. The first result is mentioned in some references. For completeness, we include a proof of the result here.

Lemma 2.3. If $\Psi : V \rightarrow \mathbb{R}$ is locally Lipschitz continuous with a local Lipschitz constant c_Ψ at 0, then

$$|\Psi^0(0; w)| \leq c_\Psi \|w\|_V \quad \forall w \in V. \quad (2.18)$$

Proof. By definition,

$$\Psi^0(0; w) = \limsup_{v \rightarrow 0, \lambda \downarrow 0} \frac{\Psi(v + \lambda w) - \Psi(v)}{\lambda}.$$

Evidently, $\Psi^0(0; w) \leq c_\Psi \|w\|_V$.

Similarly,

$$-\Psi^0(0; w) = -\limsup_{v \rightarrow 0, \lambda \downarrow 0} \frac{\Psi(v + \lambda w) - \Psi(v)}{\lambda} = \liminf_{v \rightarrow 0, \lambda \downarrow 0} \frac{\Psi(v) - \Psi(v + \lambda w)}{\lambda}$$

and we obtain $-\Psi^0(0; w) \leq c_\Psi \|w\|_V$.

Therefore, (2.18) holds. $\square$

The second result is also regularly used in references, and we list it here for convenience.

Lemma 2.4. If $\Psi : V \rightarrow \mathbb{R}$ is locally Lipschitz continuous with a local Lipschitz constant c_Ψ at 0 and has the property (2.9), then

$$\Psi^0(w; -w) \leq \alpha_\Psi \|w\|_V^2 + c_\Psi \|w\|_V \quad \forall w \in V. \quad (2.19)$$

Proof. It follows from (2.9) that

$$\Psi^0(w; -w) \leq \alpha_\Psi \|w\|_V^2 - \Psi^0(0; w).$$

Then, we apply (2.18) to get (2.19). $\square$

Introduce a subset of K_V :

$$K_{M_0} := \{v \in K_V \cap \tilde{V} \mid \|v\|_V \leq M_0\}, \quad (2.20)$$

where $\tilde{V}$ is defined by (2.7), M_0 is defined in (2.13). Note that $K_{M_0} \neq \emptyset$ since obviously, $0 \in K_{M_0}$.

Lemma 2.5. Under the assumptions stated in Theorem 2.1, for any solution $(u, p) \in K_V \times Q$ of Problem 1.1 with $f \in V_{c_0}^*$, $u \in K_{M_0}$.

Proof. Note that $u \in K_V \cap \tilde{V}$. So we only need to derive the bound

$$\|u\|_V \leq M_0. \quad (2.21)$$

Take $v = 0$ in (1.1) to obtain

$$a(u, u) \leq -d(u, u, u) - b(u, p) + \Phi(0) - \Phi(u) + \Psi^0(u; -u) + \langle f, u \rangle.$$

We have $d(u, u, u) = 0$ from (2.6), $b(u, p) = 0$ from (1.2), and $\Phi(0) - \Phi(u) \leq 0$ by assumption. Thus,

$$a(u, u) \leq \Psi^0(u; -u) + \langle f, u \rangle.$$

By the V -ellipticity of $a(\cdot, \cdot)$ and Lemma 2.4, we derive from the above inequality that

$$m_a \|u\|_V^2 \leq \alpha_\Psi \|u\|_V^2 + (c_\Psi + \|f\|_{V^*}) \|u\|_V,$$

from which, the bound (2.21) follows. $\square$

Remark 2.6. The assumption that Φ achieves its minimum at 0 is used in proving the bound (2.21). If, instead of this assumption, we assume Φ is Lipschitz continuous with a Lipschitz constant L_Φ , then we still have the bound (2.21) and all other related results in the rest of the section after redefining M_0 as

$$M_0 := \frac{L_\Phi + c_\Psi + c_0}{m_a - \alpha_\Psi}.$$

In the special case where $K_V = V$, the assumption that Φ achieves its minimum at 0 can be replaced by a more general assumption that Φ achieves its minimum in V . Since in this special case, we can formally write $\Phi(v)$ as $\bar{\Phi}(v + v_0)$ in Problems 1.1 and 1.2, v_0 being a minimizer of Φ . Then, $\bar{\Phi}$ satisfies the same conditions stated in $H(\Phi)$ for $K_V = V$ and it has its minimum at 0.

We will consider two auxiliary problems. Given $f \in V^*$, $w_1 \in \tilde{V}$, $w_2 \in K_{M_0}$, and $\theta > 0$, the first auxiliary problem is the following.

Problem 2.7. Find $(u, p) \in K_V \times Q$ such that

$$\begin{aligned} (u, v - u) + \theta b(v - u, p) + \theta [\Phi(v) - \Phi(u) + \Psi^0(u; v - u)] \\ \geq (w_1, v - u) - \theta [a(w_1, v - u) + d(w_2, w_1, v - u) - \langle f, v - u \rangle] \quad \forall v \in K_V, \end{aligned} \quad (2.22)$$

$$b(u, q) = 0 \quad \forall q \in Q. \quad (2.23)$$

Lemma 2.8. Under the assumptions stated in Theorem 2.1, if $\theta < \alpha_\Psi^{-1}$, Problem 2.7 has a unique solution $(u, p) \in K_V \times Q$ for any $f \in V^*$.

Proof. Applying Theorem 2.2, we obtain the existence and uniqueness of a solution to Problem 2.7 for $\theta \alpha_\Psi < 1$. $\square$

For the solution (u, p) of Problem 2.7, evidently, $u \in \tilde{V}$. Thus, for a fixed element $w_2 \in K_{M_0}$ and $\theta \in (0, 1/\alpha_\Psi)$, Lemma 2.8 allows us to define an operator $P_1: \tilde{V} \rightarrow \tilde{V}$ by

$$P_1(w_1) = u,$$

where (u, p) is the solution of Problem 2.7 for some element $p \in Q$.

Lemma 2.9. Under the assumptions stated in Theorem 2.1, for any $f \in V^*$, if $\theta > 0$ is sufficiently small, the operator $P_1: \tilde{V} \rightarrow \tilde{V}$ has a unique fixed-point $u \in \tilde{V}$.

Proof. Let us show that the operator P_1 is a contraction over the space $\tilde{V}$. For this purpose, let $\bar{w}_1 \in \tilde{V}$ and denote by $(\bar{u}, \bar{p}) \in K_V \times Q$ the corresponding solution of Problem 2.7 with w_1 replaced by $\bar{w}_1$:

$$\begin{aligned} (\bar{u}, v - \bar{u}) + \theta b(v - \bar{u}, \bar{p}) + \theta [\Phi(v) - \Phi(\bar{u}) + \Psi^0(\bar{u}; v - \bar{u})] \\ \geq (\bar{w}_1, v - \bar{u}) - \theta [a(\bar{w}_1, v - \bar{u}) + d(w_2, \bar{w}_1, v - \bar{u}) - \langle f, v - \bar{u} \rangle] \quad \forall v \in K_V, \end{aligned} \quad (2.24)$$

$$b(\bar{u}, q) = 0 \quad \forall q \in Q. \quad (2.25)$$

Take $v = \bar{u}$ in (2.22), $v = u$ in (2.24), add the two inequalities and use (2.23) and (2.25) to obtain

$$\|u - \bar{u}\|_V^2 \leq I_1 + I_2 + I_3, \quad (2.26)$$

where

$$\begin{aligned} I_1 &= (w_1 - \bar{w}_1, u - \bar{u}) - \theta a(w_1 - \bar{w}_1, u - \bar{u}), \\ I_2 &= \theta [\Psi^0(u; \bar{u} - u) + \Psi^0(\bar{u}; u - \bar{u})], \\ I_3 &= \theta d(w_2, w_1 - \bar{w}_1, \bar{u} - u). \end{aligned}$$

By [10, Lemma 5.3],

$$I_1 \leq (1 - 2m_a\theta + M_a^2\theta^2)^{1/2} \|w_1 - \bar{w}_1\|_V \|u - \bar{u}\|_V.$$

Applying the assumption $H(\Psi)$,

$$I_2 \leq \theta \alpha_\Psi \|u - \bar{u}\|_V^2.$$

By the continuity of the trilinear form $d(\cdot, \cdot, \cdot)$ and the condition $w_2 \in K_{M_0}$,

$$I_3 \leq \theta c_d \|w_2\|_V \|w_1 - \bar{w}_1\|_V \|u - \bar{u}\|_V \leq \theta c_d M_0 \|w_1 - \bar{w}_1\|_V \|u - \bar{u}\|_V.$$

Combining the above bounds with (2.26) to obtain

$$\|u - \bar{u}\|_V \leq \kappa_1 \|w_1 - \bar{w}_1\|_V, \quad \kappa_1 := \frac{(1 - 2m_a\theta + M_a^2\theta^2)^{1/2} + c_d M_0 \theta}{1 - \alpha_\Psi \theta}.$$

For $\theta > 0$ small,

$$\kappa_1 = \frac{1 - (m_a - c_d M_0)\theta + O(\theta^2)}{1 - \alpha_\Psi \theta} = 1 - (m_a - c_d M_0 - \alpha_\Psi)\theta + O(\theta^2).$$

From (2.14),

$$m_a - c_d M_0 - \alpha_\Psi > 0.$$

Hence, if $\theta > 0$ is sufficiently small, $\kappa_1 < 1$ and the mapping $P_1 : \tilde{V} \rightarrow \tilde{V}$ is a contraction. By the Banach fixed-point theorem, P_1 has a unique fixed-point $u \in \tilde{V}$. $\square$

From the definition of the operator P_1 , associated with its unique fixed-point $u \in K_V \cap \tilde{V}$, we know that for some element $p \in Q$, the pair (u, p) is a solution of the following second auxiliary problem and u is unique.

Problem 2.10. Find $(u, p) \in K_V \times Q$ such that

$$a(u, v - u) + d(w_2, u, v - u) + b(v - u, p) + \Phi(v) - \Phi(u) + \Psi^0(u; v - u) \geq \langle f, v - u \rangle \quad \forall v \in K_V, \quad (2.27)$$

$$b(u, q) = 0 \quad \forall q \in Q. \quad (2.28)$$

With the same argument in the proof of Lemma 2.5, we can show that $\|u\|_V \leq M_0$ if $f \in V_{c_0}^*$. We summarize the above results in the form of a lemma.

Lemma 2.11. Under the assumptions stated in Theorem 2.1, for any $f \in V_{c_0}^*$ and any $w_2 \in K_{M_0}$, Problem 2.10 has a solution $(u, p) \in K_V \times Q$, u is unique and $u \in K_{M_0}$.

Actually, similar to the later proof of the uniqueness of p for Theorem 2.1, it can be shown that the solution component p of Problem 2.10 is also unique; yet, such a uniqueness result is not needed at this stage. Lemma 2.11 allows us to define an operator $P_2 : K_{M_0} \rightarrow K_{M_0}$ by

$$P_2(w_2) = u$$

in relation to Problem 2.10.

Lemma 2.12. Under the assumptions stated in Theorem 2.1, if $f \in V_{c_0}^*$, the operator $P_2 : K_{M_0} \rightarrow K_{M_0}$ has a unique fixed-point $u \in K_{M_0}$.

Proof. We will prove the result through verifying the contractivity of the operator P_2 . Let $\bar{w}_2 \in K_{M_0}$ and denote by $(\bar{u}, \bar{p}) \in K_V \times Q$ a corresponding solution of Problem 2.10 with w_2 replaced by $\bar{w}_2$:

$$a(\bar{u}, v - \bar{u}) + d(\bar{w}_2, \bar{u}, v - \bar{u}) + b(v - \bar{u}, \bar{p}) + \Phi(v) - \Phi(\bar{u}) + \Psi^0(\bar{u}; v - \bar{u}) \geq \langle f, v - \bar{u} \rangle \quad \forall v \in K_V, \quad (2.29)$$

$$b(\bar{u}, q) = 0 \quad \forall q \in Q. \quad (2.30)$$

Take $v = \bar{u}$ in (2.27), $v = u$ in (2.29), add the two inequalities and use (2.28) and (2.30) to obtain

$$m_a \|u - \bar{u}\|_V^2 \leq I_1 + I_2, \quad (2.31)$$

where

$$\begin{aligned} I_1 &= \Psi^0(u; \bar{u} - u) + \Psi^0(\bar{u}; u - \bar{u}), \\ I_2 &= d(w_2, u, \bar{u} - u) + d(\bar{w}_2, \bar{u}, u - \bar{u}). \end{aligned}$$

Applying the assumption $H(\Psi)$,

$$I_1 \leq \alpha_\Psi \|u - \bar{u}\|_V^2.$$

To bound I_2 , write

$$I_2 = d(w_2, u - \bar{u}, \bar{u} - u) + d(w_2 - \bar{w}_2, \bar{u}, \bar{u} - u).$$

Observe that $d(w_2, u - \bar{u}, \bar{u} - u) = 0$ by (2.6). Apply the condition (2.5) and the bound $\|\bar{u}\|_V \leq M_0$ to get

$$I_2 \leq c_d M_0 \|w_2 - \bar{w}_2\|_V \|u - \bar{u}\|_V.$$

Use these relations in (2.31) to obtain

$$m_d \|u - \bar{u}\|_V^2 \leq \alpha_\Psi \|u - \bar{u}\|_V^2 + c_d M_0 \|w_2 - \bar{w}_2\|_V \|u - \bar{u}\|_V,$$

i.e.,

$$\|u - \bar{u}\|_V \leq \kappa_2 \|w_2 - \bar{w}_2\|_V, \quad \kappa_2 := \frac{c_d M_0}{m_d - \alpha_\Psi}.$$

From (2.14), we know that $\kappa_2 < 1$. Therefore, the operator $P_2 : K_{M_0} \rightarrow K_{M_0}$ is a contraction and has a unique fixed-point $u \in K_{M_0}$ by the Banach fixed-point theorem. $\square$

Now we are in a position to complete a proof of Theorem 2.1.

Proof of Theorem 2.1. By the definition of P_2 , associated with the unique fixed-point $u \in K_{M_0}$ of P_2 , we have an element $p \in Q$ such that (u, p) is a solution of Problem 1.1 and u is unique.

To prove the uniqueness of p , let $(u, p_1) \in K_V \times Q$ and $(u, p_2) \in K_V \times Q$ be two solutions of Problem 1.1. Then, for $i = 1, 2$,

$$a(u, v - u) + b(v - u, p_i) + d(u, u, v - u) + \Phi(v) - \Phi(u) + \Psi^0(u; v - u) \geq \langle f, v - u \rangle \quad \forall v \in K_V, \quad (2.32)$$

$$b(u, q) = 0 \quad \forall q \in Q. \quad (2.33)$$

Using the assumptions (2.8) and (2.10), we derive from (2.32) that for $i = 1, 2$,

$$a(u, v) + b(v, p_i) + d(u, u, v) = \langle f, v \rangle \quad \forall v \in V_0. \quad (2.34)$$

Subtract the two equalities for $i = 1$ and $i = 2$ to obtain

$$b(v, p_1 - p_2) = 0 \quad \forall v \in V_0.$$

An application of the inf-sup condition (2.4) leads to the conclusion $p_1 - p_2 = 0$, i.e., p is also unique. $\square$

We conclude the section with a Lipschitz continuous dependence property of the solution on the right side.

Theorem 2.13. Keep the assumptions stated in Theorem 2.1. Then, there exists a constant $c > 0$ such that given any $f_1, f_2 \in V_{c_0}^*$, for the corresponding solutions $(u_1, p_1), (u_2, p_2) \in K_V \times Q$ of Problem 1.1 with $f = f_1, f_2$, the following inequality holds:

$$\|u_1 - u_2\|_V + \|p_1 - p_2\|_Q \leq c \|f_1 - f_2\|_{V^*}. \quad (2.35)$$

Proof. From the definition of Problem 1.1, we have

$$a(u_1, v - u_1) + d(u_1, u_1, v - u_1) + b(v - u_1, p_1) + \Phi(v) - \Phi(u_1) + \Psi^0(u_1; v - u_1) \geq \langle f_1, v - u_1 \rangle \quad \forall v \in K_V, \quad (2.36)$$

$$b(u_1, q) = 0 \quad \forall q \in Q, \quad (2.37)$$

and

$$a(u_2, v - u_2) + d(u_2, u_2, v - u_2) + b(v - u_2, p_2) + \Phi(v) - \Phi(u_2) + \Psi^0(u_2; v - u_2) \geq \langle f_2, v - u_2 \rangle \quad \forall v \in K_V, \quad (2.38)$$

$$b(u_2, q) = 0 \quad \forall q \in Q. \quad (2.39)$$

Take $v = u_2$ in (2.36), take $v = u_1$ in (2.38), add the two inequalities and use (2.37) and (2.39) to get

$$a(u_1 - u_2, u_1 - u_2) \leq I_\Psi + I_d + I_f, \quad (2.40)$$

where

$$I_\Psi = \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2),$$

$$I_d = d(u_1, u_1, u_2 - u_1) + d(u_2, u_2, u_1 - u_2),$$

$$I_f = \langle f_1 - f_2, u_1 - u_2 \rangle.$$

By the assumption $H(\Psi)$,

$$I_\Psi \leq \alpha_\Psi \|u_1 - u_2\|_V^2. \quad (2.41)$$

Write

$$I_d = d(u_1, u_1 - u_2, u_2 - u_1) + d(u_1, u_2, u_2 - u_1) + d(u_2, u_2, u_1 - u_2).$$

Since $d(u_1, u_1 - u_2, u_2 - u_1) = 0$ by (2.6), we have

$$I_d = d(u_1 - u_2, u_2, u_2 - u_1).$$

Then, use the continuity of $d(\cdot, \cdot, \cdot)$ and the bound $\|u_2\|_V \leq M_0$ to obtain

$$|d(u_1, u_1, u_2 - u_1) + d(u_2, u_2, u_1 - u_2)| \leq c_d \|u_2\|_V \|u_1 - u_2\|_V^2 \leq c_d M_0 \|u_1 - u_2\|_V^2. \quad (2.42)$$

In addition,

$$I_f \leq \|f_1 - f_2\|_{V^*} \|u_1 - u_2\|_V.$$

Hence, from (2.40), we have

$$m_a \|u_1 - u_2\|_V^2 \leq (\alpha_\Psi + c_d M_0) \|u_1 - u_2\|_V^2 + \|f_1 - f_2\|_{V^*} \|u_1 - u_2\|_V,$$

and then

$$\|u_1 - u_2\|_V \leq \frac{1}{m_a - (\alpha_\Psi + c_d M_0)} \|f_1 - f_2\|_{V^*}. \quad (2.43)$$

Similar to (2.34), we have, for $i = 1, 2$,

$$a(u_i, v) + b(v, p_i) + d(u_i, u_i, v) = \langle f_i, v \rangle \quad \forall v \in V_0. \quad (2.44)$$

Subtract the two equations for $i = 1$ and $i = 2$ in (2.44) to get

$$-b(v, p_1 - p_2) = a(u_1 - u_2, v) + d(u_1, u_1, v) - d(u_2, u_2, v) - \langle f_1 - f_2, v \rangle \quad \forall v \in V_0.$$

Write

$$d(u_1, u_1, v) - d(u_2, u_2, v) = d(u_1, u_1 - u_2, v) + d(u_1 - u_2, u_2, v).$$

Then,

$$\begin{aligned} |d(u_1, u_1, v) - d(u_2, u_2, v)| &\leq c_d (\|u_1\|_V + \|u_2\|_V) \|u_1 - u_2\|_V \|v\|_V \\ &\leq 2 c_d M_0 \|u_1 - u_2\|_V \|v\|_V. \end{aligned}$$

Thus, applying the inf-sup condition (2.4),

$$\begin{aligned} \alpha_b \|p_1 - p_2\|_Q &\leq \sup_{v \in V_0} \frac{b(v, p_1 - p_2)}{\|v\|_V} \\ &\leq (M_a + 2 c_d M_0) \|u_1 - u_2\|_V + \|f_1 - f_2\|_{V^*}. \end{aligned}$$

This inequality and (2.43) together imply (2.35). $\square$

3. Well-posedness of Problem 1.2

Now we turn to a study of Problem 1.2. Note that the weak formulation of the application problem in Section 4 is in the form of Problem 1.2. Recall that for Δ a subset of Ω or a subset of its boundary, $I_\Delta(g)$ stands for the integral of an integrable function g on Δ , and for a positive integer m , $\psi: \mathbb{R}^m \rightarrow \mathbb{R}$ is locally Lipschitz continuous and for a function space V_ψ on Δ , the operator $\gamma: V \rightarrow V_\psi$ is such that $\psi^0(\gamma u; \gamma v) \in L^1(\Delta)$ for all $u, v \in V$. For mixed variational-hemivariational inequalities in applications, associated with second-order PDEs, we let $V_\psi = L^2(\Delta)$, and denote by $c_\Delta > 0$ the smallest constant in the inequality

$$I_\Delta(|\gamma v|_{\mathbb{R}^m}^2) \leq c_\Delta^2 \|v\|_V^2 \quad \forall v \in V. \quad (3.1)$$

We replace $H(\Psi)$ by the following assumptions on ψ .

- $H(\psi)$ For an integer $m \geq 1$, $\psi: \mathbb{R}^m \rightarrow \mathbb{R}$ is locally Lipschitz continuous, and there exist constants $\bar{c} \geq 0$ and $\alpha_\psi \geq 0$ such that

$$|\xi|_{\mathbb{R}^m} \leq \bar{c} (1 + |z|_{\mathbb{R}^m}) \quad \forall z \in \mathbb{R}^m, \xi \in \partial\psi(z), \quad (3.2)$$

$$\psi^0(z_1; z_2 - z_1) + \psi^0(z_2; z_1 - z_2) \leq \alpha_\psi |z_1 - z_2|_{\mathbb{R}^m}^2 \quad \forall z_1, z_2 \in \mathbb{R}^m. \quad (3.3)$$

Define a functional $\Psi: V \rightarrow \mathbb{R}$ by the formula

$$\Psi(v) = I_\Delta(\psi(\gamma v)), \quad v \in V. \quad (3.4)$$

It is shown in [11, Section 3.3] that Ψ is well-defined and is locally Lipschitz continuous on V . Moreover,

$$\Psi^0(u; v) \leq I_\Delta(\psi^0(\gamma u; \gamma v)) \quad \forall u, v \in V, \quad (3.5)$$

and

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \alpha_\psi c_\Delta^2 \|v_1 - v_2\|_V^2 \quad \forall v_1, v_2 \in V.$$

Hence, $H(\psi)$ implies $H(\Psi)$ with $\alpha_\Psi = \alpha_\psi c_\Delta^2$. Moreover, denoting by c_ψ a local Lipschitz constant of ψ at $0 \in \mathbb{R}^m$, we can show via an elementary calculation that

$$c_\Psi = c_\psi |\Delta|^{1/2} c_\Delta$$

is a local Lipschitz constant of Ψ at $0 \in V$, where $|\Delta|$ is the measure of Δ .

We deduce from Theorems 2.1 and 2.13 the next result on Problem 1.2.

Theorem 3.1. Assume $H(K_V)$, $H(Q)$, $H(a)$, $H(b)$, $H(d)$, $H(\Phi)$, $H(\psi)$, Φ has its minimum at 0, and

$$m_a - \alpha_\psi c_\Delta^2 > 0, \quad (m_a - \alpha_\psi c_\Delta^2)^2 > c_d c_\psi |\Delta|^{1/2} c_\Delta. \quad (3.6)$$

Let c_0 be a positive number smaller than $(m_a - \alpha_\psi c_\Delta^2)^2 / c_d - c_\psi |\Delta|^{1/2} c_\Delta$. Then, if $\|f\|_{V^*} \leq c_0$, Problem 1.2 has a unique solution $(u, p) \in K_V \times Q$. Moreover, there exists a constant $c > 0$ such that for solutions $(u_1, p_1), (u_2, p_2) \in K_V \times Q$ of Problem 1.2 with $f = f_1, f_2 \in V^*$, $\max\{\|f_1\|_{V^*}, \|f_2\|_{V^*}\} \leq c_0$, the following inequality holds:

$$\|u_1 - u_2\|_V + \|p_1 - p_2\|_Q \leq c \|f_1 - f_2\|_{V^*}. \quad (3.7)$$

Proof. Consider the auxiliary problem, Problem 1.1, for Ψ defined by (3.4). From Theorems 2.1 and 2.13, we know that under the stated assumptions, the auxiliary problem has a unique solution $(u, p) \in K_V \times Q$ and the solution depends Lipschitz continuously on f . Thanks to the property (3.5), (u, p) is also a solution of Problem 1.2.

Let us show the uniqueness of the solution of Problem 1.2. Suppose $(\bar{u}, \bar{p}) \in K_V \times Q$ is also a solution of Problem 1.2:

$$a(\bar{u}, v - \bar{u}) + d(\bar{u}, \bar{u}, v - \bar{u}) + b(v - \bar{u}, \bar{p}) + \Phi(v) - \Phi(\bar{u}) + I_\Delta(\psi^0(\gamma \bar{u}; \gamma v - \gamma \bar{u})) \geq \langle f, v - \bar{u} \rangle \quad \forall v \in K_V, \quad (3.8)$$

$$b(\bar{u}, q) = 0 \quad \forall q \in Q. \quad (3.9)$$

We take $v = \bar{u}$ in (1.4), take $v = u$ in (3.8), add the two resulting inequalities and use (1.5) and (3.9) to obtain

$$a(u - \bar{u}, u - \bar{u}) \leq d(u, u, u - \bar{u}) + d(\bar{u}, \bar{u}, u - \bar{u}) + I_\Delta(\psi^0(\gamma u; \gamma \bar{u} - \gamma u) + \psi^0(\gamma \bar{u}; \gamma u - \gamma \bar{u})). \quad (3.10)$$

Similar to (2.42),

$$d(u, u, u - \bar{u}) + d(\bar{u}, \bar{u}, u - \bar{u}) \leq c_d M_1 \|u - \bar{u}\|_V^2,$$

where

$$M_1 = \frac{c_\psi |\Delta|^{1/2} c_\Delta + c_0}{m_a - \alpha_\psi c_\Delta^2},$$

and similar to (2.41),

$$I_\Delta(\psi^0(\gamma u; \gamma \bar{u} - \gamma u) + \psi^0(\gamma \bar{u}; \gamma u - \gamma \bar{u})) \leq \alpha_\psi c_\Delta^2 \|u - \bar{u}\|_V^2.$$

Hence, we derive from (3.9) that

$$m_a \|u - \bar{u}\|_V^2 \leq (c_d M_1 + \alpha_\psi c_\Delta^2) \|u - \bar{u}\|_V^2. \quad (3.11)$$

It follows from the second condition in (3.6) that

$$m_a > c_d M_1 + \alpha_\psi c_\Delta^2.$$

Therefore, we conclude from (3.11) the uniqueness of u : $\bar{u} = u$.

To prove $\bar{p} = p$, we note that from (1.4) and (3.8) (with $\bar{u} = u$),

$$a(u, v) + d(u, u, v) + b(v, p) = \langle f, v \rangle \quad \forall v \in V_0,$$

$$a(u, v) + d(u, u, v) + b(v, \bar{p}) = \langle f, v \rangle \quad \forall v \in V_0.$$

Then,

$$b(v, p - \bar{p}) = 0 \quad \forall v \in V_0.$$

Applying the inf-sup condition (2.4), we derive from the above equality that $\bar{p} = p$, and the uniqueness of p follows.

Consequently, the statements of Theorem 3.1 hold regarding Problem 1.2. $\square$

4. A Navier-Stokes Variational-hemivariational inequality

In this section, we consider a variational-hemivariational inequality of the Navier-Stokes equations for an incompressible fluid flow occupying the domain Ω subject to non-leaking slip boundary conditions of friction types on parts of the boundary. Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be an open bounded Lipschitz domain. Its boundary $\Gamma = \partial\Omega$ is split into three parts: Γ_D , $\Gamma_{S,1}$ and $\Gamma_{S,2}$. The unit outward normal vector exists a.e. on Γ and is denoted by $\mathbf{v} = (v_1, \dots, v_d)^T$. For a vector $\mathbf{v}$, its normal component and tangential component on Γ are $v_v = \mathbf{v} \cdot \mathbf{v}$ and $\mathbf{v}_\tau = \mathbf{v} - v_v \mathbf{v}$.

Let $\mu > 0$ be the viscosity of the fluid, and denote by $\mathbf{u}$ and p for the unknown velocity and pressure fields of the fluid. Denote

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$$

for the strain rate tensor and

$$\boldsymbol{\sigma} = 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) - p \mathbf{I}$$

for the stress tensor, $\mathbf{I}$ being the identity tensor. The normal component and the tangential component of the stress on the boundary are $\sigma_v = (\boldsymbol{\sigma} \mathbf{v}) \cdot \mathbf{v}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \mathbf{v} - \sigma_v \mathbf{v}$.

Given a source density function $f: \Omega \rightarrow \mathbb{R}^d$, a convex function $\phi: \Gamma_{S,1} \rightarrow \mathbb{R}$, and a non-convex locally Lipschitz continuous function $\psi: \Gamma_{S,2} \rightarrow \mathbb{R}$, the pointwise formulation of the problem is the following.

Problem 4.1. Find a velocity field $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ and a pressure field $p : \Omega \rightarrow \mathbb{R}$ such that

$$-\operatorname{div}(2\mu \varepsilon(\mathbf{u})) + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (4.1)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (4.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad (4.3)$$

$$u_\nu = 0, -\sigma_\tau \in \partial \phi(\mathbf{u}_\tau) \quad \text{on } \Gamma_{S,1}, \quad (4.4)$$

$$u_\nu = 0, -\sigma_\tau \in \partial \psi(\mathbf{u}_\tau) \quad \text{on } \Gamma_{S,2}. \quad (4.5)$$

Problem 4.1 is studied through its formulation. For this purpose, we introduce the following function spaces:

$$\mathbf{V} = \{ \mathbf{v} \in H^1(\Omega)^d \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D, v_\nu = 0 \text{ on } \Gamma_S \}, \quad (4.6)$$

$$\mathbf{V}_0 = H_0^1(\Omega)^d, \quad (4.7)$$

$$\mathcal{Q} = \{ q \in L^2(\Omega) \mid (q, 1)_{L^2(\Omega)} = 0 \}, \quad (4.8)$$

where $\Gamma_S = \Gamma_{S,1} \cup \Gamma_{S,2}$. Over the space $\mathbf{V}$, we use the norm $\|\mathbf{v}\|_{\mathbf{V}} = \|\varepsilon(\mathbf{v})\|_{L^2(\Omega)^{d \times d}}$, which is equivalent to the standard $H^1(\Omega)^d$ norm over $\mathbf{V}$ thanks to the Korn inequality (cf. e.g., [40, page 79]): there exists a constant $c > 0$ depending only on Ω and Γ_S such that

$$\|\mathbf{v}\|_{H^1(\Omega)^d} \leq c \|\varepsilon(\mathbf{v})\|_{L^2(\Omega)^{d \times d}} \quad \forall \mathbf{v} \in \mathbf{V}.$$

Assume

$$\mathbf{f} \in \mathbf{V}^* \quad (4.9)$$

and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $\mathbf{V}^*$ and $\mathbf{V}$.

Through a standard procedure, we can derive the following weak formulation of Problem 4.1.

Problem 4.2. Find $(\mathbf{u}, p) \in \mathbf{V} \times \mathcal{Q}$ such that

$$\begin{aligned} \int_{\Omega} 2\mu \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v} - \mathbf{u}) \, dx + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot (\mathbf{v} - \mathbf{u}) \, dx - \int_{\Omega} p \operatorname{div}(\mathbf{v} - \mathbf{u}) \, dx + \int_{\Gamma_{S,1}} [\phi(\mathbf{v}_\tau) - \phi(\mathbf{u}_\tau)] \, ds \\ + \int_{\Gamma_{S,2}} \psi^0(\mathbf{u}_\tau; \mathbf{v}_\tau - \mathbf{u}_\tau) \, ds \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle \quad \forall \mathbf{v} \in \mathbf{V}, \end{aligned} \quad (4.10)$$

$$\int_{\Omega} q \operatorname{div} \mathbf{u} \, dx = 0 \quad \forall q \in \mathcal{Q}. \quad (4.11)$$

We can apply Theorem 3.1 to study Problem 4.2. In this context, $K_V = \mathbf{V}$. We define for $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ and $q \in \mathcal{Q}$,

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} 2\mu \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx,$$

$$d(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \, dx,$$

$$b(\mathbf{v}, q) = - \int_{\Omega} q \operatorname{div} \mathbf{v} \, dx,$$

$$\Phi(\mathbf{v}) = \int_{\Gamma_{S,1}} \phi(\mathbf{v}_\tau) \, ds.$$

Then, the assumptions $H(K_V)$, $H(\mathcal{Q})$, $H(a)$, $H(b)$, $H(d)$ in the statement of Theorem 3.1 are valid, $M_a = m_a = 2\mu$. In particular, it is well known that the inf-sup condition (2.4) is satisfied (cf. [22,24]).

For the functions ϕ and ψ , we introduce the following assumptions.

- $H(\phi)$ $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex and Lipschitz continuous, and has its minimum at 0.
- $H(\psi)$ $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ is locally Lipschitz continuous and for some constants $\bar{c} \geq 0$ and $\alpha_\psi \geq 0$,

$$|\partial \psi(\mathbf{z})|_{\mathbb{R}^d} \leq \bar{c}(1 + |\mathbf{z}|_{\mathbb{R}^d}) \quad \forall \mathbf{z} \in \mathbb{R}^d, \quad (4.12)$$

$$\psi^0(\mathbf{z}_1; \mathbf{z}_2 - \mathbf{z}_1) + \psi^0(\mathbf{z}_2; \mathbf{z}_1 - \mathbf{z}_2) \leq \alpha_\psi |\mathbf{z}_1 - \mathbf{z}_2|_{\mathbb{R}^d}^2 \quad \forall \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^d. \quad (4.13)$$

A popular example for a function ϕ satisfying $H(\phi)$ is

$$\phi(\mathbf{z}) = g|\mathbf{z}|, \quad \mathbf{z} \in \mathbb{R}^d$$

for a positive constant or a positive bounded function g (cf. [41,42]). Some examples of a function ψ satisfying $H(\psi)$ can be found in [43, pp. 186–187].

Denote by c_ψ a local Lipschitz constant of ψ at $\mathbf{0}$, and denote by $\lambda_0 > 0$ the smallest eigenvalue of the eigenvalue problem

$$\mathbf{u} \in \mathbf{V}, \quad \int_{\Omega} 2\mu \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx = \lambda \int_{\Gamma_{S,2}} \mathbf{u}_\tau \cdot \mathbf{v}_\tau \, ds \quad \forall \mathbf{v} \in \mathbf{V}.$$

Then, the constant c_Δ^2 in (3.1) is given by $c_\Delta^2 = \lambda_0^{-1}$.

Theorem 4.3. Assume $H(\phi)$, $H(\psi)$, and

$$2\mu - \lambda_0^{-1}\alpha_\psi > 0, \quad (2\mu - \lambda_0^{-1}\alpha_\psi)^2 > \lambda_0^{-1/2}c_d c_\psi |\Delta|^{1/2}.$$

Let $c_0 > 0$ be any number smaller than $(2\mu - \lambda_0^{-1}\alpha_\psi)^2/c_d - \lambda_0^{-1/2}c_\psi |\Delta|^{1/2}$. Then for any $f \in V^*$ with $\|f\|_{V^*} \leq c_0$, Problem 4.2 admits a unique solution $(u, p) \in V \times Q$ and the solution depends Lipschitz continuously on f for $f \in V^*$ with $\|f\|_{V^*} \leq c_0$.

The stated result in Theorem 4.3 follows from an application of Theorem 3.1.

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