

Research Paper

A discontinuous Galerkin method for $H(\text{curl})$ -elliptic hemivariational inequalities

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ABSTRACT

In this paper, we develop a Discontinuous Galerkin (DG) method for solving $H(\text{curl})$ -elliptic hemivariational inequalities. By selecting an appropriate numerical flux, we construct an Interior Penalty Discontinuous Galerkin (IPDG) scheme. A comprehensive numerical analysis of the IPDG method is conducted, addressing key aspects such as consistency, boundedness, and stability of the discrete formulation, as well as the existence, uniqueness, and uniform boundedness of the numerical solutions. Building on these properties, we establish a priori error estimates, demonstrating the optimal convergence order of the numerical solutions under suitable solution regularity assumptions. Finally, a numerical example is presented to illustrate the theoretically predicted convergence order and to show the effectiveness of the proposed method.

1. Introduction

To describe the mixed state of type-II superconductors, in particular high-temperature superconductors, Bean proposed the critical-state theory [1,2]. The basic principle of this model can be stated as follows. When a superconductor is in the mixed state, the magnitude of the current density $|\mathbf{J}|$ cannot exceed a critical value g . In the regions penetrated by the magnetic field, the current density is $|\mathbf{J}| = g$, and the electric field $\mathbf{E}$ is parallel to the current. When the magnitude of the current density $\mathbf{J}$ is strictly less than the critical value g , the electric field $\mathbf{E} = \mathbf{0}$. Mathematically, this can be expressed as

$$|\mathbf{J}| \leq g; \quad |\mathbf{J}| < g \Rightarrow \mathbf{E} = \mathbf{0}; \quad |\mathbf{J}| = g \Rightarrow \mathbf{J} = \kappa \mathbf{E} \text{ for some } \kappa \geq 0,$$

One can eliminate the unknown parameter κ and derive the following equivalent expression:

$$|\mathbf{J}| \leq g, \quad \mathbf{J} \cdot \mathbf{E} = g|\mathbf{E}|. \quad (1.1)$$

With the use of the notion of convex subdifferential ∂_c , the relation can be compactly written as

$$\mathbf{J} \in \partial_c(g|\mathbf{E}|). \quad (1.2)$$

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This is a nonsmooth monotone constitutive law, and the corresponding Maxwell models naturally lead to variational inequalities.

Modeling and analysis of variational inequalities of the Maxwell equations date back to Duvaut and Lions [3] in the 1970s. In the context of superconductivity, mathematical models in the form of variational inequalities as extensions of Bean-type critical-state models were developed and studied in the early references [4,5] and [6], and more recently in [7–11]. Other references on Maxwell-type variational inequalities include [12] on a well-posedness theory for electromagnetic obstacle problems, [13] on Maxwell quasi-variational inequalities with temperature and magnetic-field-dependent critical current, [14,15] on numerical analysis of Maxwell obstacle problems in electric shielding and their eddy-current approximation, and [16] on quasilinear variational inequalities in ferromagnetic shielding.

More generally, the critical current density g may depend on temperature and magnetic field strength. It has been shown in [17,18] that its dependence on the magnetic field strength is often non-monotonic. In such cases, the resulting model leads to Maxwell quasi-variational inequalities rather than standard variational inequalities; see, for example, [13]. Another extension, motivated by non-monotonic constitutive behavior, was proposed in [19], where the convex constitutive law (1.2) is replaced by a Clarke-subdifferential relation

$$\mathbf{J} \in \partial\psi(E). \quad (1.3)$$

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume the function $\psi(x, E) : \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is locally Lipschitz continuous with respect to the variable E and this dependence is allowed to be non-convex. To simplify the notation, we write $\psi(E)$ for $\psi(x, E)$. The symbol $\partial\psi(E)$ denotes the Clarke subdifferential of ψ with respect to the variable E . Following the implicit Euler time discretization described in [19], we consider an $\mathbf{H}(\mathbf{curl})$ -elliptic hemivariational inequality: find $E \in V$ such that

$$a(E, v) + \int_{\Omega} \psi^0(E; v) \, dx \geq \langle f, v \rangle \quad \forall v \in V,$$

where

$$\begin{aligned} V &:= H_0(\mathbf{curl}, \Omega), \\ a(E, v) &:= \int_{\Omega} \epsilon E \cdot v \, dx + \int_{\Omega} \mu^{-1} (\nabla \times E) \cdot (\nabla \times v) \, dx. \end{aligned}$$

Here $f \in V^*$, ϵ and μ denote the scaled permittivity and permeability, respectively, and ψ^0 is the generalized directional derivative in the sense of Clarke. This semidiscrete stationary problem is the object of the present paper.

Since analytical solutions of hemivariational inequalities are rarely available, their numerical approximations have attracted sustained attention, cf. [20] for a summary account of the early work on the finite element solution of hemivariational inequalities. In this paper, the term “hemivariational inequalities” is used to also mean the more general variational-hemivariational inequalities. In the literature, the reference [21] is the first paper that provides an optimal first-order error estimate for linear finite element approximations of a hemivariational inequality (HVI). Representative contributions include [22–24] on finite element analyses for elliptic, parabolic, and hyperbolic HVIs, [25,26] for numerical treatments of history-dependent problems, and [27,28] for related problems in fluid models. Recent overviews of the area may be found in [29,30]. Beyond standard finite elements, virtual element methods have been developed in both conforming and nonconforming settings [31,32]. More recently, discontinuous Galerkin methods were proposed for elliptic HVIs in semi-permeable media in [33] and further extended to contact-mechanics HVIs in [34].

For the specific $\mathbf{H}(\mathbf{curl})$ -elliptic hemivariational inequality with constitutive law $\mathbf{J} \in \partial\psi(E)$, a conforming edge finite element method is analyzed in [19] where an optimal-order error estimate is proved. On the other hand, discontinuous Galerkin (DG) discretizations for Maxwell equations are by now well established; see, e.g., [35–37]. This makes it natural to ask whether interior penalty DG techniques can be adapted to the present nonconvex $\mathbf{H}(\mathbf{curl})$ setting. To the best of our knowledge, a DG discretization together with a priori error analysis for the $\mathbf{H}(\mathbf{curl})$ -elliptic hemivariational inequality governed by $\mathbf{J} \in \partial\psi(E)$ has not been reported.

In this paper, we develop an interior penalty discontinuous Galerkin (IPDG) method for the $\mathbf{H}(\mathbf{curl})$ -elliptic hemivariational inequality. The DG framework is attractive because of its flexibility in handling locally varying approximation spaces, interfaces, and boundary conditions. We prove the consistency, boundedness, and stability of the discrete formulation, establish existence and uniqueness of the discrete solution, derive a priori error estimates under suitable regularity assumptions, and present numerical experiments that confirm the predicted convergence order.

The rest of the paper is organized as follows. In Section 2, we review some definitions to be used later and recall the mathematical formulation of the $\mathbf{H}(\mathbf{curl})$ -elliptic hemivariational inequality. In Section 3, we present the DG discretization. In Section 4, we carry out an error analysis and derive a priori error estimates for the IPDG method applied to the $\mathbf{H}(\mathbf{curl})$ -elliptic hemivariational inequality. Finally, the last section reports numerical experiments that illustrate the theoretical convergence order established in this paper.

2. Preliminaries

We first recall the notions of the convex subdifferential and the Clarke (generalized) subdifferential. Let V be a Banach space and denote by V^* its dual space.

Definition 1. Assume $\varphi : V \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous function on V . The set

$$\partial_c \varphi(u) = \{\xi \in V^* : \varphi(v) - \varphi(u) \geq \langle \xi, v - u \rangle \, \forall v \in V\}$$

is called the convex subdifferential of the function φ at $u \in V$. If $\partial_c \varphi(u) \neq \emptyset$, any element $\xi \in \partial_c \varphi(u)$ is called a subgradient of φ at u .

Definition 2. Assume $\psi : V \rightarrow \mathbb{R}$ is a locally Lipschitz continuous function. The generalized directional derivative of ψ at $u \in V$ in the direction $v \in V$ is defined as

$$\psi^0(u; v) = \limsup_{w \rightarrow u, \lambda \downarrow 0} \frac{\psi(w + \lambda v) - \psi(w)}{\lambda},$$

and the Clarke subdifferential of ψ at $u \in V$ is defined as

$$\partial\psi(u) = \{\xi \in V^* : \psi^0(u; v) \geq \langle \xi, v \rangle \ \forall v \in V\}.$$

For Clarke (or generalized) subdifferentials, the following properties hold [38,39]:

(1) The generalized directional derivative can be obtained using the Clarke subdifferential:

$$\psi^0(u; v) = \max\{\langle \xi, v \rangle : \xi \in \partial\psi(u)\} \quad \forall u, v \in V. \quad (2.1)$$

(2) The generalized directional derivative is positively homogeneous and subadditive with respect to the direction variable:

$$\psi^0(u; \lambda v) = \lambda \psi^0(u; v) \quad \forall \lambda \geq 0, u, v \in V, \quad (2.2)$$

$$\psi^0(u; v_1 + v_2) \leq \psi^0(u; v_1) + \psi^0(u; v_2) \quad \forall u, v_1, v_2 \in V. \quad (2.3)$$

(3) Suppose $\psi_1, \psi_2 : V \rightarrow \mathbb{R}$ are both locally Lipschitz continuous. Then,

$$\partial(\psi_1 + \psi_2)(u) \subset \partial\psi_1(u) + \partial\psi_2(u) \quad \forall u \in V, \quad (2.4)$$

equivalently,

$$(\psi_1 + \psi_2)^0(u; v) \leq \psi_1^0(u; v) + \psi_2^0(u; v) \quad \forall u, v \in V. \quad (2.5)$$

We will use Sobolev spaces for the formulation and analysis of the problem. We refer the reader to any standard reference on Sobolev spaces for further details; see, e.g., [40–42]. Let $k \geq 0$ be an integer and $1 \leq p \leq \infty$. Given a domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ being the spatial dimension, let $W^{k,p}(\Omega)$ denote the Sobolev space of $L^p(\Omega)$ -functions whose weak derivatives of orders less than or equal to k are in $L^p(\Omega)$, equipped with the standard norm $\|\cdot\|_{W^{k,p}(\Omega)}$. When $p = 2$, we write $H^k(\Omega) = W^{k,2}(\Omega)$, with the corresponding norm denoted as $\|\cdot\|_{k,\Omega}$. In particular, for $k = 0$, $H^0(\Omega) = L^2(\Omega)$, and its norm is written as $\|\cdot\|_{0,\Omega}$. Sobolev spaces with vector-valued functions are denoted by boldface symbols, e.g., $\mathbf{L}^p(\Omega)$ and $\mathbf{H}^k(\Omega)$ are spaces of vector-valued functions with each component belonging to $L^p(\Omega)$ and $H^k(\Omega)$, respectively. In a three-dimensional domain Ω ,

$$\mathbf{L}^p(\Omega) = \{\mathbf{u} = (u_1, u_2, u_3)^\top : u_i \in L^p(\Omega), i = 1, 2, 3\},$$

$$\mathbf{H}^k(\Omega) = \{\mathbf{u} = (u_1, u_2, u_3)^\top : u_i \in H^k(\Omega), i = 1, 2, 3\}.$$

The following integration by parts formula holds for any bounded Lipschitz domain $D \subset \mathbb{R}^3$:

$$\int_D (\nabla \times \mathbf{v}) \cdot \mathbf{q} \, dx = \int_D \mathbf{v} \cdot (\nabla \times \mathbf{q}) \, dx + \int_{\partial D} (\mathbf{v} \times \mathbf{q}) \cdot \mathbf{n} \, dS \quad \forall \mathbf{v}, \mathbf{q} \in \mathbf{H}^1(D). \quad (2.6)$$

Here, $\mathbf{v} \times \mathbf{q}$ in the boundary integral term denotes the pointwise cross product of the traces of $\mathbf{v}$ and $\mathbf{q}$ on ∂D . By the identity

$$\nabla \cdot (\mathbf{v} \times \mathbf{q}) = (\nabla \times \mathbf{v}) \cdot \mathbf{q} - \mathbf{v} \cdot (\nabla \times \mathbf{q}),$$

the formula (2.6) follows for smooth vector fields from the classical Gauss theorem. The extension to $\mathbf{v}, \mathbf{q} \in \mathbf{H}^1(D)$ is obtained by density of $C^\infty(\overline{D})$ in $\mathbf{H}^1(D)$, continuity of the trace operator $\tau : \mathbf{H}^1(D) \rightarrow \mathbf{H}^{1/2}(\partial D)$, and the continuous embedding $\mathbf{H}^{1/2}(\partial D) \hookrightarrow L^2(\partial D)$.

For $\Omega \subset \mathbb{R}^3$, we define the following function spaces related to the curl operator:

$$\mathbf{H}(\mathbf{curl}, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \nabla \times \mathbf{v} \in \mathbf{L}^2(\Omega)\},$$

$$\mathbf{H}_0(\mathbf{curl}, \Omega) = \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega) : \mathbf{n} \times \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\},$$

where $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$ and the curl operator $\nabla \times$ is defined by

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right)^\top.$$

The norm associated with these spaces is given by

$$\|\mathbf{v}\|_{\mathbf{curl}, \Omega} = \left(\|\mathbf{v}\|_{0,\Omega}^2 + \|\nabla \times \mathbf{v}\|_{0,\Omega}^2 \right)^{1/2}.$$

The hemivariational inequality problem considered in this paper arises from the backward Euler semidiscretization in time of the hyperbolic Maxwell problem ([19]). Here we recall the assumptions and weak formulation of this problem; for more details we refer to [19]. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz continuous boundary Γ , and let $\mathbf{n}$ denote the unit outward normal vector on Γ , which exists a.e. Set

$$\mathbf{V} = \mathbf{H}_0(\mathbf{curl}, \Omega).$$

In a superconductor, the current density $\mathbf{J}$ and the electric field $\mathbf{E}$ are assumed to satisfy the relation

$$\mathbf{J} \in \partial\psi(\mathbf{E}). \quad (2.7)$$

Here $\psi : \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is locally Lipschitz in its second argument, and $\psi(E)$ stands for $\psi(\mathbf{x}, E)$. The notation $\partial\psi(E)$ denotes the Clarke subdifferential of ψ with respect to the second variable, and ψ^0 denotes the corresponding generalized directional derivative. We assume that ψ satisfies the following conditions:

$$\left\{ \begin{array}{l} \text{(a) } \psi(\cdot, \xi) \text{ is measurable in } \Omega \text{ for any } \xi \in \mathbb{R}^3, \text{ and } \psi(\cdot, \mathbf{0}) \in L^1(\Omega). \\ \text{(b) For a.e. } \mathbf{x} \in \Omega, \text{ the function } \psi(\mathbf{x}, \cdot) \text{ is locally Lipschitz continuous in } \mathbb{R}^3. \\ \text{(c) There exist constants } c_0, c_1 \geq 0 \text{ such that for a.e. } \mathbf{x} \in \Omega \text{ and any } \xi \in \mathbb{R}^3, \\ \quad |\eta| \leq c_0 + c_1 |\xi| \quad \forall \eta \in \partial\psi(\mathbf{x}, \xi). \\ \text{(d) There exists a constant } m \geq 0 \text{ such that for a.e. } \mathbf{x} \in \Omega, \\ \quad \psi^0(\mathbf{x}, \xi_1; \xi_2 - \xi_1) + \psi^0(\mathbf{x}, \xi_2; \xi_1 - \xi_2) \leq m |\xi_1 - \xi_2|^2 \quad \forall \xi_1, \xi_2 \in \mathbb{R}^3. \end{array} \right. \quad (2.8)$$

Let $\tilde{\epsilon}$ and $\tilde{\mu}$ denote the permittivity and permeability in the time-dependent problem, and let $k > 0$ be the time step. As in [19], we define

$$\epsilon = k^{-1} \tilde{\epsilon}, \quad \mu = k^{-1} \tilde{\mu}, \quad \epsilon_i = k^{-1} \tilde{\epsilon}_i, \quad \mu_i = k^{-1} \tilde{\mu}_i, \quad i = 0, 1. \quad (2.9)$$

Following [19], we assume that $\epsilon, \mu \in L^\infty(\Omega)$ and that

$$0 < \epsilon_0 \leq \epsilon(\mathbf{x}) \leq \epsilon_1 < \infty, \quad 0 < \mu_0 \leq \mu(\mathbf{x}) \leq \mu_1 < \infty \quad \text{for a.e. } \mathbf{x} \in \Omega.$$

For $E, v \in V$, define the bilinear form

$$a(E, v) = \int_{\Omega} \epsilon E \cdot v \, dx + \int_{\Omega} \mu^{-1} (\nabla \times E) \cdot (\nabla \times v) \, dx. \quad (2.10)$$

The hemivariational inequality under consideration is to find $E \in V$ such that

$$E \in V, \quad a(E, v) + \int_{\Omega} \psi^0(E; v) \, dx \geq \langle f, v \rangle \quad \forall v \in V. \quad (2.11)$$

where the load functional $f \in V^*$ is induced by a vector field $\tilde{l} \in L^2(\Omega)$, that is,

$$\langle f, v \rangle = \int_{\Omega} \tilde{l} \cdot v \, dx \quad \forall v \in V.$$

The existence and uniqueness of a solution to (2.11) were proved in [19, Theorem 4.3].

3. Discontinuous Galerkin discretization

In this section, we introduce an IPDG approximation of the hemivariational inequality (2.11). The construction follows the standard interior penalty DG discretization of the curl-curl Maxwell operator; see, e.g., [35–37].

Let $\mathcal{T}_h = \{K\}$ be a shape-regular partition of $\bar{\Omega}$ into tetrahedral or hexahedral elements. Denote by h_K the diameter of an element K , and let

$$h = \max_{K \in \mathcal{T}_h} h_K.$$

We use $\mathcal{F}_h^I$ to denote the set of all interior faces, $\mathcal{F}_h^B$ the set of all boundary faces, and let $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^B$.

Let $f \in \mathcal{F}_h^I$ be the common face of two tetrahedral or hexahedral elements K_1 and K_2 , and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the outward unit normal vectors on f for K_1 and K_2 , respectively, cf. Fig. 1. For a piecewise $\mathbf{H}^1$ vector-valued function v , namely,

$$v|_K \in \mathbf{H}^1(K) \quad \forall K \in \mathcal{T}_h,$$

we denote by v_1 and v_2 the traces of v on f taken from K_1 and K_2 , respectively. The tangential jump and the average of v on f are defined as

$$[[v]] = \mathbf{n}_1 \times v_1 + \mathbf{n}_2 \times v_2, \quad \{v\} = \frac{v_1 + v_2}{2}.$$

If $f \in \mathcal{F}_h^B$, we define

$$[[v]] = \mathbf{n} \times v, \quad \{v\} = v.$$

For later use, we note that for piecewise $\mathbf{H}^1$ vector fields v and q satisfying

$$v|_K, q|_K \in \mathbf{H}^1(K) \quad \forall K \in \mathcal{T}_h,$$

where $\mathbf{n}_K$ denotes the outward unit normal vector on ∂K , the following formula

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_{\partial K} (v \times q) \cdot \mathbf{n}_K \, dS &= \sum_{f \in \mathcal{F}_h^I} \int_f [[v]] \cdot \{q\} \, dS - \sum_{f \in \mathcal{F}_h} \int_f [[q]] \cdot \{v\} \, dS \\ &= \sum_{f \in \mathcal{F}_h} \int_f [[v]] \cdot \{q\} \, dS - \sum_{f \in \mathcal{F}_h^I} \int_f [[q]] \cdot \{v\} \, dS \end{aligned} \quad (3.1)$$

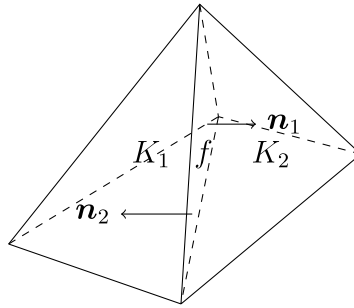


Fig. 1. Two adjacent tetrahedral elements and a common face.

holds.

Given a positive integer l , we introduce the following DG space

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathbf{v}|_K \in \mathbf{P}^l(K) \forall K \in \mathcal{T}_h\}$$

where $\mathbf{P}^l(K) = [\mathbf{P}^l(K)]^3$ and $\mathbf{P}^l(K)$ is the space of polynomials of degree at most l on K . For $\mathbf{v}_h \in \mathbf{V}_h$, the piecewise curl operator $\nabla_h \times$ is defined elementwise by

$$\nabla_h \times (\mathbf{v}_h|_K) = \nabla \times (\mathbf{v}_h|_K) \quad \forall K \in \mathcal{T}_h.$$

Numerical fluxes $\widehat{\mathbf{E}}_h$ and $\widehat{\mathbf{p}}_h$ are used to approximate the traces of $\mathbf{E}$ and $\mu^{-1} \nabla \times \mathbf{E}$ on ∂K , respectively, while $\mathbf{E}_h$ is used to approximate $\mathbf{E}$ on K . We consider the DG method: find $\mathbf{E}_h \in \mathbf{V}_h$ such that

$$a_h(\mathbf{E}_h, \mathbf{v}_h) + \int_{\Omega} \psi^0(\mathbf{E}_h; \mathbf{v}_h) dx \geq \langle \mathbf{f}, \mathbf{v}_h \rangle \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (3.2)$$

where the bilinear form $a_h(\mathbf{E}_h, \mathbf{v}_h)$ is defined as

$$\begin{aligned} a_h(\mathbf{E}_h, \mathbf{v}_h) = & \int_{\Omega} \epsilon \mathbf{E}_h \cdot \mathbf{v}_h dx + \int_{\Omega} \mu^{-1} (\nabla_h \times \mathbf{E}_h) \cdot (\nabla_h \times \mathbf{v}_h) dx \\ & + \sum_{f \in \mathcal{F}_h} \int_f \llbracket \widehat{\mathbf{E}}_h - \mathbf{E}_h \rrbracket \cdot \{\mu^{-1} \nabla_h \times \mathbf{v}_h\} dS \\ & - \sum_{f \in \mathcal{F}_h^I} \int_f \llbracket \mu^{-1} \nabla_h \times \mathbf{v}_h \rrbracket \cdot \{\widehat{\mathbf{E}}_h - \mathbf{E}_h\} dS \\ & + \sum_{f \in \mathcal{F}_h^I} \int_f \llbracket \widehat{\mathbf{p}}_h \rrbracket \cdot \{\mathbf{v}_h\} dS - \sum_{f \in \mathcal{F}_h} \int_f \llbracket \mathbf{v}_h \rrbracket \cdot \{\widehat{\mathbf{p}}_h\} dS. \end{aligned} \quad (3.3)$$

Define h_f on each $f \in \mathcal{F}_h$ by

$$h_f = \begin{cases} \min\{h_K, h_{K'}\}, & f \in \mathcal{F}_h^I, \quad f = \partial K \cap \partial K', \\ h_K, & f \in \mathcal{F}_h^B, \quad f = \partial K \cap \partial \Omega. \end{cases}$$

Let $\eta > 0$ be a constant and define α on each $f \in \mathcal{F}_h$ by

$$\alpha|_f = \eta h_f^{-1}.$$

Choose the numerical fluxes as follows:

$$\begin{cases} \widehat{\mathbf{E}}_h = \{\mathbf{E}_h\} & \text{on } \mathcal{F}_h^I, \\ \mathbf{n} \times \widehat{\mathbf{E}}_h = \mathbf{0} & \text{on } \mathcal{F}_h^B, \\ \widehat{\mathbf{p}}_h = \{\mu^{-1} \nabla_h \times \mathbf{E}_h\} - \alpha \llbracket \mathbf{E}_h \rrbracket & \text{on } \mathcal{F}_h. \end{cases}$$

Using these choices in (3.3), we obtain the following bilinear form for the IPDG scheme:

$$\begin{aligned} a_h(\mathbf{E}_h, \mathbf{v}_h) = & \int_{\Omega} \epsilon \mathbf{E}_h \cdot \mathbf{v}_h dx + \int_{\Omega} \mu^{-1} (\nabla_h \times \mathbf{E}_h) \cdot (\nabla_h \times \mathbf{v}_h) dx \\ & - \sum_{f \in \mathcal{F}_h} \int_f \llbracket \mathbf{E}_h \rrbracket \cdot \{\mu^{-1} \nabla_h \times \mathbf{v}_h\} dS - \sum_{f \in \mathcal{F}_h} \int_f \llbracket \mathbf{v}_h \rrbracket \cdot \{\mu^{-1} \nabla_h \times \mathbf{E}_h\} dS \\ & + \sum_{f \in \mathcal{F}_h} \int_f \alpha \llbracket \mathbf{E}_h \rrbracket \cdot \llbracket \mathbf{v}_h \rrbracket dS. \end{aligned} \quad (3.4)$$

With the discrete bilinear form $a_h(\cdot, \cdot)$ defined by (3.4), the IPDG scheme to solve the $\mathbf{H}(\mathbf{curl})$ -elliptic hemivariational inequality (2.11) is to find $\mathbf{E}_h \in \mathbf{V}_h$ such that

$$a_h(\mathbf{E}_h, \mathbf{v}_h) + \int_{\Omega} \psi^0(\mathbf{E}_h; \mathbf{v}_h) d\mathbf{x} \geq \langle \mathbf{f}, \mathbf{v}_h \rangle \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \quad (3.5)$$

4. Error analysis

In this section, we present some properties of the IPDG scheme and provide a priori error estimates. On several occasions, we will apply Young's inequality with an arbitrarily small parameter $\varepsilon > 0$:

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \forall a, b \in \mathbb{R}. \quad (4.1)$$

4.1. Properties of the IPDG scheme

This subsection is devoted to the consistency, stability, and boundedness of the IPDG scheme, as well as the uniform boundedness of its numerical solution.

Theorem 1 (Consistency). *Let $\mathbf{E} \in \mathbf{V}$ be a solution to the hemivariational inequality (2.11). Assume further that*

$$\mu^{-1} \nabla \times \mathbf{E} \in \mathbf{H}(\mathbf{curl}, \Omega) \quad (4.2)$$

and

$$(\mu^{-1} \nabla \times \mathbf{E})|_K \in \mathbf{H}^1(K) \quad \forall K \in \mathcal{T}_h. \quad (4.3)$$

Then the following consistency relation holds:

$$a_h(\mathbf{E}, \mathbf{v}_h) + \int_{\Omega} \psi^0(\mathbf{E}; \mathbf{v}_h) d\mathbf{x} \geq \langle \mathbf{f}, \mathbf{v}_h \rangle \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \quad (4.4)$$

Proof. The bilinear form $a_h(\mathbf{E}, \mathbf{v}_h)$ is obtained from (3.4) by replacing $\mathbf{E}_h$ with $\mathbf{E}$. Applying (3.1), we notice that the fourth term of $a_h(\mathbf{E}, \mathbf{v}_h)$ is

$$\begin{aligned} - \sum_{f \in \mathcal{F}_h} \int_f \llbracket \mathbf{v}_h \rrbracket \cdot \{\mu^{-1} \nabla \times \mathbf{E}\} dS &= - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu^{-1} (\mathbf{v}_h \times \nabla \times \mathbf{E}) \cdot \mathbf{n}_K dS \\ &\quad - \sum_{f \in \mathcal{F}_h^I} \int_f \llbracket \mu^{-1} \nabla \times \mathbf{E} \rrbracket \cdot \{\mathbf{v}_h\} dS. \end{aligned} \quad (4.5)$$

Since $\mathbf{E} \in \mathbf{V}$, the jump of $\mathbf{E}$ across $\mathcal{F}_h$ vanishes, namely,

$$\llbracket \mathbf{E} \rrbracket = \mathbf{0} \quad \text{on } \mathcal{F}_h.$$

Furthermore, in view of the regularity assumption (4.2), we also have

$$\llbracket \mu^{-1} \nabla \times \mathbf{E} \rrbracket = 0 \quad \text{on } \mathcal{F}_h^I.$$

Therefore,

$$\begin{aligned} a_h(\mathbf{E}, \mathbf{v}_h) &= \int_{\Omega} \varepsilon \mathbf{E} \cdot \mathbf{v}_h d\mathbf{x} + \int_{\Omega} \mu^{-1} (\nabla \times \mathbf{E}) \cdot (\nabla_h \times \mathbf{v}_h) d\mathbf{x} \\ &\quad - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu^{-1} (\mathbf{v}_h \times \nabla \times \mathbf{E}) \cdot \mathbf{n}_K dS. \end{aligned}$$

Since

$$(\mu^{-1} \nabla \times \mathbf{E})|_K \in \mathbf{H}^1(K) \quad \forall K \in \mathcal{T}_h,$$

we may apply the integration by parts formula (2.6) on each element $K \in \mathcal{T}_h$:

$$\begin{aligned} \int_K \mu^{-1} (\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{v}_h) d\mathbf{x} &= \int_K \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) \cdot \mathbf{v}_h d\mathbf{x} \\ &\quad - \int_{\partial K} \mu^{-1} ((\nabla \times \mathbf{E}) \times \mathbf{v}_h) \cdot \mathbf{n}_K dS. \end{aligned}$$

Thus,

$$a_h(\mathbf{E}, \mathbf{v}_h) = \int_{\Omega} \varepsilon \mathbf{E} \cdot \mathbf{v}_h d\mathbf{x} + \int_{\Omega} \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) \cdot \mathbf{v}_h d\mathbf{x}.$$

Since $\mathbf{E} \in \mathbf{V}$ and $\mu^{-1} \nabla \times \mathbf{E} \in \mathbf{H}(\mathbf{curl}, \Omega)$, we may define

$$\mathbf{g} := \varepsilon \mathbf{E} + \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) \in \mathbf{L}^2(\Omega).$$

From the previous calculations, we already know that

$$a_h(E, v_h) = \int_{\Omega} g \cdot v_h \, dx \quad \forall v_h \in V_h. \quad (4.6)$$

Now let $\phi \in C_0^\infty(\Omega) \subset V$. Since $\mu^{-1} \nabla \times E \in H(\text{curl}, \Omega)$, the integration-by-parts formula

$$\int_{\Omega} (\mu^{-1} \nabla \times E) \cdot (\nabla \times \phi) \, dx = \int_{\Omega} \nabla \times (\mu^{-1} \nabla \times E) \cdot \phi \, dx$$

holds. Therefore, by the definition of the bilinear form $a(\cdot, \cdot)$ in (2.10), we have

$$a(E, \phi) = \int_{\Omega} \epsilon E \cdot \phi \, dx + \int_{\Omega} \nabla \times (\mu^{-1} \nabla \times E) \cdot \phi \, dx = \int_{\Omega} g \cdot \phi \, dx.$$

Hence, by the hemivariational inequality (2.11),

$$\int_{\Omega} g \cdot \phi \, dx + \int_{\Omega} \psi^0(E; \phi) \, dx \geq \langle f, \phi \rangle = \int_{\Omega} \bar{l} \cdot \phi \, dx \quad \forall \phi \in C_0^\infty(\Omega).$$

Next, we show that for a.e. $x \in \Omega$ and all $z_1, z_2 \in \mathbb{R}^3$, it holds that

$$|\psi^0(x, E(x); z_1) - \psi^0(x, E(x); z_2)| \leq (c_0 + c_1 |E(x)|) |z_1 - z_2|. \quad (4.7)$$

Indeed, by the subadditivity of the Clarke generalized directional derivative with respect to the direction variable, taking

$$u = E(x), \quad v_1 = z_1 - z_2, \quad v_2 = z_2$$

in (2.3), we obtain

$$\psi^0(x, E(x); z_1) \leq \psi^0(x, E(x); z_1 - z_2) + \psi^0(x, E(x); z_2),$$

which is equivalent to

$$\psi^0(x, E(x); z_1) - \psi^0(x, E(x); z_2) \leq \psi^0(x, E(x); z_1 - z_2). \quad (4.8)$$

Similarly, we have

$$\psi^0(x, E(x); z_2) - \psi^0(x, E(x); z_1) \leq \psi^0(x, E(x); z_2 - z_1). \quad (4.9)$$

Moreover, by the characterization of the generalized directional derivative in (2.1) and assumption (2.8) (c), for any $\xi \in \mathbb{R}^3$,

$$\psi^0(x, E(x); \xi) = \max_{\eta \in \partial \psi(x, E(x))} \eta \cdot \xi \leq \max_{\eta \in \partial \psi(x, E(x))} |\eta| |\xi| \leq (c_0 + c_1 |E(x)|) |\xi|. \quad (4.10)$$

Combining (4.8) – (4.10), we immediately deduce (4.7).

Define the mapping

$$J_E(w) := \int_{\Omega} \psi^0(E; w) \, dx \quad w \in L^2(\Omega).$$

Since $E \in V$ and Ω is bounded, it follows that $c_0 + c_1 |E| \in L^2(\Omega)$. For a.e. $x \in \Omega$ and any $w \in L^2(\Omega)$, taking $z_1 = w(x)$ and $z_2 = 0$ in (4.7), we obtain

$$|\psi^0(x, E(x); w(x)) - \psi^0(x, E(x); 0)| \leq (c_0 + c_1 |E(x)|) |w(x)|.$$

Since $\psi^0(x, E(x); 0) = 0$, this reduces to

$$|\psi^0(x, E(x); w(x))| \leq (c_0 + c_1 |E(x)|) |w(x)|.$$

The right-hand side belongs to $L^1(\Omega)$. Hence, by Hölder's inequality,

$$|J_E(w)| \leq \int_{\Omega} (c_0 + c_1 |E(x)|) |w(x)| \, dx \leq \|c_0 + c_1 |E|\|_{0,\Omega} \|w\|_{0,\Omega} < \infty.$$

Therefore, $J_E(w)$ is well defined. We next show that J_E is Lipschitz continuous on $L^2(\Omega)$. For any $w_1, w_2 \in L^2(\Omega)$, we have

$$\begin{aligned} |J_E(w_1) - J_E(w_2)| &\leq \int_{\Omega} |\psi^0(E; w_1) - \psi^0(E; w_2)| \, dx \\ &\leq \int_{\Omega} (c_0 + c_1 |E|) |w_1 - w_2| \, dx \\ &\leq \|c_0 + c_1 |E|\|_{0,\Omega} \|w_1 - w_2\|_{0,\Omega}. \end{aligned}$$

This shows that J_E is Lipschitz continuous on $L^2(\Omega)$, with Lipschitz constant

$$\|c_0 + c_1 |E|\|_{0,\Omega}.$$

Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, let $\{\phi_n\} \subset C_0^\infty(\Omega)$ be such that

$$\phi_n \rightarrow w \quad \text{in } L^2(\Omega).$$

Since $g, \tilde{l} \in L^2(\Omega)$ and J_E is Lipschitz continuous on $L^2(\Omega)$, passing to the limit yields

$$\int_{\Omega} g \cdot w \, dx + \int_{\Omega} \psi^0(E; w) \, dx \geq \int_{\Omega} \tilde{l} \cdot w \, dx \quad \forall w \in L^2(\Omega).$$

Taking $w = v_h \in V_h \subset L^2(\Omega)$ and using (4.6), we obtain

$$a_h(E, v_h) + \int_{\Omega} \psi^0(E; v_h) \, dx \geq \int_{\Omega} \tilde{l} \cdot v_h \, dx = \langle f, v_h \rangle \quad \forall v_h \in V_h,$$

This completes the proof. $\square$

Let

$$V(h) = H_0(\text{curl}; \Omega) + V_h$$

with the seminorm and norm defined by

$$|v|_h^2 = \sum_{K \in \mathcal{T}_h} \|\nabla_h \times v\|_{0,K}^2 + \sum_{f \in \mathcal{F}_h} \|\alpha^{1/2} \llbracket v \rrbracket\|_{0,f}^2, \quad \|v\|_h^2 = \|v\|_{0,\Omega}^2 + |v|_h^2.$$

The bilinear form $a_h(\cdot, \cdot)$ is initially defined on $V_h \times V_h$. To extend it to $V(h) \times V(h)$, we introduce an auxiliary bilinear form as follows [37,43]

$$\tilde{a}_h(E, v) = \int_{\Omega} \epsilon E \cdot v \, dx + \tilde{b}_h(E, v), \quad (4.11)$$

where

$$\begin{aligned} \tilde{b}_h(E, v) &= \sum_{K \in \mathcal{T}_h} \int_K \mu^{-1} (\nabla_h \times E) \cdot (\nabla_h \times v) \, dx - \sum_{f \in \mathcal{F}_h} \int_f \llbracket E \rrbracket \cdot \{\mu^{-1} \Pi_h(\nabla_h \times v)\} \, dS \\ &\quad - \sum_{f \in \mathcal{F}_h} \int_f \llbracket v \rrbracket \cdot \{\mu^{-1} \Pi_h(\nabla_h \times E)\} \, dS + \sum_{f \in \mathcal{F}_h} \int_f \alpha \llbracket E \rrbracket \cdot \llbracket v \rrbracket \, dS. \end{aligned}$$

Here, Π_h is the L^2 -projection from $L^2(\Omega)$ onto V_h . Note that $\tilde{a}_h(\cdot, \cdot)$ coincides with $a_h(\cdot, \cdot)$ on $V_h \times V_h$.

Lemma 1 (Boundedness). *There is a constant $C_b > 0$ such that*

$$|\tilde{a}_h(E, v)| \leq C_b \|E\|_h \|v\|_h \quad \forall E, v \in V(h). \quad (4.12)$$

Proof. Using Hölder's inequality, we obtain

$$\begin{aligned} \int_K \mu^{-1} (\nabla_h \times E) \cdot (\nabla_h \times v) \, dx &\leq \mu_0^{-1} \int_K |\nabla_h \times E| \cdot |\nabla_h \times v| \, dx \\ &\leq \mu_0^{-1} \|\nabla_h \times E\|_{0,K} \|\nabla_h \times v\|_{0,K}. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_f \alpha \llbracket E \rrbracket \cdot \llbracket v \rrbracket \, dS &\leq \|\alpha^{1/2} \llbracket E \rrbracket\|_{0,f} \|\alpha^{1/2} \llbracket v \rrbracket\|_{0,f}, \\ \int_{\Omega} \epsilon E \cdot v \, dx &\leq \epsilon_1 \|E\|_{0,\Omega} \|v\|_{0,\Omega}. \end{aligned}$$

Now let us bound the third term of the bilinear form $\tilde{a}_h(\cdot, \cdot)$, which is a modification of the proof in [37, Lemma 4].

$$\begin{aligned} &\sum_{f \in \mathcal{F}_h} \int_f \llbracket E \rrbracket \cdot \{\mu^{-1} \Pi_h(\nabla_h \times v)\} \, dS \\ &\leq \mu_0^{-1} \sum_{f \in \mathcal{F}_h} \int_f \left| \alpha^{1/2} \llbracket E \rrbracket \right| \cdot \left| \alpha^{-1/2} \{\Pi_h(\nabla_h \times v)\} \right| \, dS \\ &\leq \mu_0^{-1} \sum_{f \in \mathcal{F}_h} \|\alpha^{1/2} \llbracket E \rrbracket\|_{0,f} \cdot \|\alpha^{-1/2} \{\Pi_h(\nabla_h \times v)\}\|_{0,f} \\ &\leq \mu_0^{-1} \left(\sum_{f \in \mathcal{F}_h} \int_f \alpha \llbracket E \rrbracket^2 \, dS \right)^{1/2} \cdot \left(\sum_{f \in \mathcal{F}_h} \int_f \alpha^{-1} |\{\Pi_h(\nabla_h \times v)\}|^2 \, dS \right)^{1/2} \\ &= \mu_0^{-1} \eta^{-1/2} \left(\sum_{f \in \mathcal{F}_h} \|\alpha^{1/2} \llbracket E \rrbracket\|_{0,f}^2 \right)^{1/2} \cdot \left(\sum_{f \in \mathcal{F}_h} \int_f h_f |\{\Pi_h(\nabla_h \times v)\}|^2 \, dS \right)^{1/2}. \end{aligned}$$

Using the definition of h_f , we obtain

$$\begin{aligned} \sum_{f \in \mathcal{F}_h} \int_f h_f |\{\Pi_h(\nabla_h \times \mathbf{v})\}|^2 dS &\leq \frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K |\Pi_h(\nabla_h \times \mathbf{v})|^2 dS \\ &\leq \frac{1}{2} \sum_{K \in \mathcal{T}_h} h_K \|\Pi_h(\nabla_h \times \mathbf{v})\|_{0,\partial K}^2. \end{aligned}$$

Recalling the discrete trace theorem ([44, Chapter 12.2, Lemma 12.8])

$$\|\mathbf{w}\|_{0,\partial K}^2 \leq \tilde{C}^2 h_K^{-1} \|\mathbf{w}\|_{0,K}^2 \quad \forall \mathbf{w} \in \mathbf{P}^l(K), \quad \forall K \in \mathcal{T}_h,$$

where the positive constant $\tilde{C}$ depends only on the mesh regularity, the polynomial degree l , and the spatial dimension. From the L^2 -projection property,

$$\|\Pi_h \mathbf{w}\|_{0,K} \leq \|\mathbf{w}\|_{0,K} \quad \forall \mathbf{w} \in L^2(K).$$

Combining the above results, we obtain

$$\frac{1}{2} \sum_{K \in \mathcal{T}_h} h_K \|\Pi_h(\nabla_h \times \mathbf{v})\|_{0,\partial K}^2 \leq \frac{1}{2} \tilde{C}^2 \sum_{K \in \mathcal{T}_h} \|\nabla_h \times \mathbf{v}\|_{0,K}^2.$$

Finally, we obtain the bound

$$\begin{aligned} \sum_{f \in \mathcal{F}_h} \int_f \llbracket \mathbf{E} \rrbracket \cdot \{\mu^{-1} \Pi_h(\nabla_h \times \mathbf{v})\} dS &\leq \mu_0^{-1} (2\eta)^{-1/2} \tilde{C} \left(\sum_{f \in \mathcal{F}_h} \|\alpha^{1/2} \llbracket \mathbf{E} \rrbracket\|_{0,f}^2 \right)^{1/2} \\ &\quad \cdot \left(\sum_{K \in \mathcal{T}_h} \|\nabla_h \times \mathbf{v}\|_{0,K}^2 \right)^{1/2}. \end{aligned} \quad (4.13)$$

Similarly, we can bound the fourth term of the bilinear form as

$$\begin{aligned} \sum_{f \in \mathcal{F}_h} \int_f \llbracket \mathbf{v} \rrbracket \cdot \{\mu^{-1} \Pi_h(\nabla_h \times \mathbf{E})\} dS &\leq \mu_0^{-1} (2\eta)^{-1/2} \tilde{C} \left(\sum_{f \in \mathcal{F}_h} \|\alpha^{1/2} \llbracket \mathbf{v} \rrbracket\|_{0,f}^2 \right)^{1/2} \\ &\quad \cdot \left(\sum_{K \in \mathcal{T}_h} \|\nabla_h \times \mathbf{E}\|_{0,K}^2 \right)^{1/2}. \end{aligned} \quad (4.14)$$

Combining these results, we obtain (4.12). $\square$

Lemma 2 (Stability). Assume $\eta > \max\{1, \mu_1^2\} \tilde{C}^2 / (2\mu_0^2)$. There is a constant $C_s > 0$ such that

$$\tilde{a}_h(\mathbf{E}, \mathbf{E}) \geq C_s \|\mathbf{E}\|_h^2 \quad \forall \mathbf{E} \in \mathbf{V}(h). \quad (4.15)$$

Proof.

$$\begin{aligned} \int_{\Omega} \epsilon \mathbf{E} \cdot \mathbf{E} d\mathbf{x} &\geq \epsilon_0 \|\mathbf{E}\|_{0,\Omega}^2, \\ \sum_{K \in \mathcal{T}_h} \int_K \mu^{-1} (\nabla_h \times \mathbf{E}) \cdot (\nabla_h \times \mathbf{E}) d\mathbf{x} &\geq \mu_1^{-1} \sum_{K \in \mathcal{T}_h} \|\nabla_h \times \mathbf{E}\|_{0,K}^2, \\ \sum_{f \in \mathcal{F}_h} \int_f \alpha \llbracket \mathbf{E} \rrbracket \cdot \llbracket \mathbf{E} \rrbracket dS &= \sum_{f \in \mathcal{F}_h} \|\alpha^{1/2} \llbracket \mathbf{E} \rrbracket\|_{0,f}^2. \end{aligned}$$

Similar to the proof of Lemma 1, we have

$$\begin{aligned} &-2 \sum_{f \in \mathcal{F}_h} \int_f \llbracket \mathbf{E} \rrbracket \cdot \{\mu^{-1} \Pi_h(\nabla_h \times \mathbf{E})\} dS \\ &\geq -2\mu_0^{-1} (2\eta)^{-1/2} \tilde{C} \left(\sum_{f \in \mathcal{F}_h} \|\alpha^{1/2} \llbracket \mathbf{E} \rrbracket\|_{0,f}^2 \right)^{1/2} \cdot \left(\sum_{K \in \mathcal{T}_h} \|\nabla_h \times \mathbf{E}\|_{0,K}^2 \right)^{1/2} \\ &\geq -\mu_0^{-1} (2\eta)^{-1/2} \tilde{C} \left(\sum_{f \in \mathcal{F}_h} \|\alpha^{1/2} \llbracket \mathbf{E} \rrbracket\|_{0,f}^2 + \sum_{K \in \mathcal{T}_h} \|\nabla_h \times \mathbf{E}\|_{0,K}^2 \right). \end{aligned}$$

Therefore, when $\eta > \max\{1, \mu_1^2\} \tilde{C}^2/(2\mu_0^2)$, $\tilde{a}_h(E, E)$ is bounded from below by

$$\begin{aligned} \epsilon_0 \|E\|_{0,\Omega}^2 + (1 - \mu_0^{-1}(2\eta)^{-1/2}\tilde{C}) \sum_{f \in F_h} \|\alpha^{1/2} \llbracket E \rrbracket\|_{0,f}^2 \\ + (\mu_1^{-1} - \mu_0^{-1}(2\eta)^{-1/2}\tilde{C}) \sum_{K \in \mathcal{T}_h} \|\nabla_h \times E\|_{0,K}^2. \end{aligned} \quad (4.16)$$

Denote $C_0 = 1 - \mu_0^{-1}(2\eta)^{-1/2}\tilde{C}$, $C_1 = \mu_1^{-1} - \mu_0^{-1}(2\eta)^{-1/2}\tilde{C}$. Then,

$$\tilde{a}_h(E, E) \geq C_s \left(\sum_{f \in F_h} \|\alpha^{1/2} \llbracket E \rrbracket\|_{0,f}^2 + \sum_{K \in \mathcal{T}_h} \|\nabla_h \times E\|_{0,K}^2 + \|E\|_{0,\Omega}^2 \right)$$

holds, where $C_s = \min\{\epsilon_0, C_0, C_1\}$. $\square$

The next result establishes the uniform boundedness of the numerical solution defined by the IPDG scheme for the $H(\text{curl})$ -elliptic hemivariational inequality. We first record two consequences of the assumptions on ψ . From (2.1) and (2.8) (c), we have

$$|\psi^0(\xi_1; \xi_2)| \leq (c_0 + c_1 |\xi_1|) |\xi_2| \quad \forall \xi_1, \xi_2 \in \mathbb{R}^3. \quad (4.17)$$

Moreover, (2.8) (d) is equivalent to ([45, p. 124] or [46, Proposition 2.42])

$$(\eta_1 - \eta_2) \cdot (\xi_1 - \xi_2) \geq -m |\xi_1 - \xi_2|^2 \quad \text{a.e. } x \in \Omega, \quad \forall \xi_i \in \mathbb{R}^3, \quad \eta_i \in \partial\psi(x, \xi_i), \quad i = 1, 2. \quad (4.18)$$

Lemma 3. Assume (2.8), $m < \epsilon_0$ and $\eta > \max\{1, \mu_1^2\} \tilde{C}^2/(2\mu_0^2)$. If $E_h \in V_h$ is a solution to the problem (3.5), then $\|E_h\|_h$ is uniformly bounded with respect to the mesh size h .

Proof. By setting $v_h = -E_h$ in (3.5), we obtain

$$a_h(E_h, E_h) \leq \int_{\Omega} \psi^0(E_h; -E_h) dx + \langle f, E_h \rangle. \quad (4.19)$$

From assumption (2.8) (d), we have

$$\psi^0(E_h; 0 - E_h) + \psi^0(0; E_h - 0) \leq m |E_h|^2,$$

Using (4.17), we obtain

$$-\psi^0(0; E_h) \leq c_0 |E_h|;$$

hence,

$$\int_{\Omega} \psi^0(E_h; -E_h) dx \leq m \|E_h\|_{0,\Omega}^2 + \int_{\Omega} c_0 |E_h| dx.$$

Moreover,

$$\langle f, E_h \rangle \leq \|f\|_{V^*} \|E_h\|_{\text{curl},\Omega} \leq \|f\|_{V^*} \|E_h\|_h.$$

By the Cauchy-Schwarz inequality,

$$\int_{\Omega} c_0 |E_h| dx \leq c_0 |\Omega|^{1/2} \|E_h\|_{0,\Omega},$$

where $|\Omega|$ means the Lebesgue measure of the bounded domain Ω . Combining these inequalities with the lower bound (4.16) of $\tilde{a}_h(E_h, E_h)$, we obtain

$$\begin{aligned} (\epsilon_0 - m) \|E_h\|_{0,\Omega}^2 + C_0 \sum_{f \in F_h} \|\alpha^{1/2} \llbracket E_h \rrbracket\|_{0,f}^2 + C_1 \sum_{K \in \mathcal{T}_h} \|\nabla_h \times E_h\|_{0,K}^2 \\ \leq (c_0 |\Omega|^{1/2} + \|f\|_{V^*}) \|E_h\|_h. \end{aligned}$$

Since $\epsilon_0 - m > 0$ and $\eta > \max\{1, \mu_1^2\} \tilde{C}^2/(2\mu_0^2)$, then

$$\|E_h\|_h \leq \frac{c_0 |\Omega|^{1/2} + \|f\|_{V^*}}{\min\{\epsilon_0 - m, C_0, C_1\}}. \quad (4.20)$$

Therefore, $\|E_h\|_h$ is bounded by a constant independent of h . $\square$

Remark 1. We note that since $\epsilon_0 = k^{-1}\tilde{\epsilon}_0$, the condition $m < \epsilon_0$ can always be satisfied as long as the time step-size k is sufficiently small.

4.2. Existence and uniqueness

In this subsection, we establish the existence and uniqueness of the solution to (3.5) by following the approach developed in [19,47]. To analyze the existence and uniqueness of the discrete solution later, we first recall the following results.

Lemma 4 ([48, Theorem 3.4]). *Let V be a real Banach space, and let $g : V \rightarrow \mathbb{R}$ be locally Lipschitz continuous. Then g is strongly convex on V with a constant $\alpha > 0$ if and only if ∂g is strongly monotone on V with a constant 2α , i.e.,*

$$\langle \xi - \eta, u - v \rangle \geq 2\alpha \|u - v\|_V^2 \quad \forall u, v \in V, \xi \in \partial g(u), \eta \in \partial g(v).$$

Proposition 1 ([47, Proposition 2.5]). *Let V be a real Hilbert space, and let $g : V \rightarrow \mathbb{R}$ be a locally Lipschitz continuous, strongly convex functional on V with constant $\alpha > 0$. Then there exist two constants $\bar{c}_0$ and $\bar{c}_1$ such that*

$$g(v) \geq \alpha \|v\|_V^2 + \bar{c}_0 + \bar{c}_1 \|v\|_V \quad \forall v \in V. \quad (4.21)$$

Consequently, $g(\cdot)$ is coercive on V .

Define an energy functional

$$\mathcal{E}(\mathbf{v}_h) = \frac{1}{2} a_h(\mathbf{v}_h, \mathbf{v}_h) + \int_{\Omega} \psi(\mathbf{x}, \mathbf{v}_h(\mathbf{x})) d\mathbf{x} - \langle \mathbf{f}, \mathbf{v}_h \rangle, \quad \mathbf{v}_h \in \mathbf{V}_h. \quad (4.22)$$

We consider the minimization problem

$$\mathbf{E}_h \in \mathbf{V}_h, \quad \mathcal{E}(\mathbf{E}_h) = \inf \{ \mathcal{E}(\mathbf{v}_h) \mid \mathbf{v}_h \in \mathbf{V}_h \}. \quad (4.23)$$

Lemma 5. Assume (2.8), $m < \epsilon_0$ and $\eta > \max\{1, \mu_1^2\} \bar{C}^2 / (2\mu_0^2)$. Then the functional $\mathcal{E}(\cdot)$ is locally Lipschitz continuous, strongly convex and coercive on $\mathbf{V}_h$.

Proof. The local Lipschitz continuity of $\mathcal{E}(\cdot)$ follows immediately from the assumptions on ψ . Let us prove the strong convexity. For this purpose, define a linear operator $A_h : \mathbf{V}_h \rightarrow \mathbf{V}_h^*$ by

$$\langle A_h \mathbf{u}_h, \mathbf{v}_h \rangle = a_h(\mathbf{u}_h, \mathbf{v}_h) \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h. \quad (4.24)$$

Applying Lemma 1, we obtain

$$\|A_h \mathbf{u}_h\|_{\mathbf{V}_h^*} = \sup_{\|\mathbf{v}_h\|_h \neq 0} \frac{|\langle A_h \mathbf{u}_h, \mathbf{v}_h \rangle|}{\|\mathbf{v}_h\|_h} = \sup_{\|\mathbf{v}_h\|_h \neq 0} \frac{|a_h(\mathbf{u}_h, \mathbf{v}_h)|}{\|\mathbf{v}_h\|_h} \leq C_b \|\mathbf{u}_h\|_h \quad \forall \mathbf{u}_h \in \mathbf{V}_h.$$

By Lemma 2,

$$\langle A_h \mathbf{u}_h - A_h \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h \rangle = a_h(\mathbf{u}_h - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) \geq C_s \|\mathbf{u}_h - \mathbf{v}_h\|_h^2 \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h.$$

Hence $A_h \in \mathcal{L}(\mathbf{V}_h, \mathbf{V}_h^*)$ and it is strongly monotone. Define a functional $\Psi : L^2(\Omega) \rightarrow \mathbb{R}$ by

$$\Psi(\mathbf{v}) = \int_{\Omega} \psi(\mathbf{x}, \mathbf{v}(\mathbf{x})) d\mathbf{x} \quad \forall \mathbf{v} \in L^2(\Omega).$$

Then, by Thm. 4.37 in [49], under assumption (2.8), Ψ is well defined, locally Lipschitz continuous on $L^2(\Omega)$, and

$$\partial \Psi(\mathbf{v}) \subset \int_{\Omega} \partial \psi(\mathbf{v}) d\mathbf{x} \quad (4.25)$$

in the sense that for any $\xi \in \partial \Psi(\mathbf{v})$, there exists a function $\zeta \in L^2(\Omega)$ such that $\zeta(\mathbf{x}) \in \partial \psi(\mathbf{x}, \mathbf{v}(\mathbf{x}))$ for a.e. $\mathbf{x} \in \Omega$ and

$$\langle \xi, \mathbf{w} \rangle_{L^2(\Omega) \times L^2(\Omega)} = \int_{\Omega} \zeta(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) d\mathbf{x} \quad \forall \mathbf{w} \in L^2(\Omega).$$

For $\mathbf{v}_h \in \mathbf{V}_h$ and $\eta \in \partial \mathcal{E}(\mathbf{v}_h)$, by (2.4) we can write

$$\eta = A_h \mathbf{v}_h + \xi - \mathbf{f}, \quad \xi \in \partial \Psi(\mathbf{v}_h). \quad (4.26)$$

Thus, for $i = 1, 2$, with $\mathbf{v}_{h,i} \in \mathbf{V}_h$ and $\eta_i \in \partial \mathcal{E}(\mathbf{v}_{h,i})$, by (4.25) we have $\zeta_i \in L^2(\Omega)$ such that $\zeta_i(\mathbf{x}) \in \partial \psi(\mathbf{x}, \mathbf{v}_{h,i}(\mathbf{x}))$ for a.e. $\mathbf{x} \in \Omega$ and

$$\langle \eta_i, \mathbf{w} \rangle = \langle A_h \mathbf{v}_{h,i}, \mathbf{w} \rangle + \int_{\Omega} \zeta_i(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) d\mathbf{x} - \langle \mathbf{f}, \mathbf{w} \rangle \quad \forall \mathbf{w} \in L^2(\Omega).$$

Thus, from (4.18) and Lemma 2,

$$\begin{aligned} \langle \eta_1 - \eta_2, \mathbf{v}_{h,1} - \mathbf{v}_{h,2} \rangle &= \langle A_h \mathbf{v}_{h,1} - A_h \mathbf{v}_{h,2}, \mathbf{v}_{h,1} - \mathbf{v}_{h,2} \rangle + \int_{\Omega} (\zeta_1 - \zeta_2) \cdot (\mathbf{v}_{h,1} - \mathbf{v}_{h,2}) d\mathbf{x} \\ &\geq (\epsilon_0 - m) \|\mathbf{v}_{h,1} - \mathbf{v}_{h,2}\|_{0,\Omega}^2 + C_0 \sum_{f \in F_h} \|\alpha^{1/2} [\![\mathbf{v}_{h,1} - \mathbf{v}_{h,2}]\!]\|_{0,f}^2 \\ &\quad + C_1 \sum_{K \in \mathcal{T}_h} \|\nabla_h \times (\mathbf{v}_{h,1} - \mathbf{v}_{h,2})\|_{0,K}^2 \\ &\geq \min\{\epsilon_0 - m, C_0, C_1\} \|\mathbf{v}_{h,1} - \mathbf{v}_{h,2}\|_h^2. \end{aligned}$$

Thus, by Lemma 4, $\mathcal{E}(\cdot)$ is strongly convex. Moreover, by Proposition 1, $\mathcal{E}(\cdot)$ is coercive on $\mathbf{V}_h$. $\square$

Proposition 2. Under the assumptions (2.8), $m < \epsilon_0$ and $\eta > \max\{1, \mu_1^2\} \tilde{C}^2/(2\mu_0^2)$, the minimization problem (4.23) has a unique solution $E_h \in V_h$.

Proof. Since $\mathcal{E}(\cdot)$ is continuous, strictly convex and coercive on V_h , from [50, §3.3.2] the minimization problem (4.23) has a unique solution. $\square$

Theorem 2. Assume (2.8), $m < \epsilon_0$ and $\eta > \max\{1, \mu_1^2\} \tilde{C}^2/(2\mu_0^2)$. Then for any $f \in V^*$, the problem (3.5) has a unique solution $E_h \in V_h$, which is also the unique solution of the minimization problem (4.23).

Proof. For the solution $E_h \in V_h$ of the minimization problem (4.23), we apply (4.26) to get

$$\langle A_h E_h, v_h \rangle + \int_{\Omega} \zeta(x) \cdot v_h(x) dx - \langle f, v_h \rangle \geq 0,$$

where $\zeta \in L^2(\Omega)$ such that $\zeta(x) \in \partial\psi(x, E_h(x))$ for a.e. $x \in \Omega$. Using the property of the Clarke subdifferential (2.1) we have

$$\psi^0(E_h(x); v_h(x)) \geq \zeta(x) \cdot v_h(x) \quad \text{a.e. } x \in \Omega.$$

Combining the above two inequalities, we can see that E_h is a solution of the problem (3.5).

Now, let us prove uniqueness of the solution. Assume that $E_h, \tilde{E}_h \in V_h$ are two solutions of the problem (3.5). Then we have

$$a_h(\tilde{E}_h, v_h) + \int_{\Omega} \psi^0(\tilde{E}_h; v_h) dx \geq \langle f, v_h \rangle \quad \forall v_h \in V_h. \quad (4.27)$$

Take $v_h = \tilde{E}_h - E_h$ in (3.5) and $v_h = E_h - \tilde{E}_h$ in (4.27). Adding the two resulting inequalities and using Lemma 2, we obtain

$$\begin{aligned} & \epsilon_0 \|\tilde{E}_h - E_h\|_{0,\Omega}^2 + C_0 \sum_{f \in F_h} \|\alpha^{1/2} \|\tilde{E}_h - E_h\|_{0,f}^2 + C_1 \sum_{K \in \mathcal{T}_h} \|\nabla_h \times (\tilde{E}_h - E_h)\|_{0,K}^2 \\ & \leq a_h(\tilde{E}_h - E_h, \tilde{E}_h - E_h) \leq \int_{\Omega} (\psi^0(E_h; \tilde{E}_h - E_h) + \psi^0(\tilde{E}_h; E_h - \tilde{E}_h)) dx \\ & \leq m \|\tilde{E}_h - E_h\|_{0,\Omega}^2. \end{aligned}$$

By the smallness condition $m < \epsilon_0$, we deduce that $\tilde{E}_h = E_h$. $\square$

4.3. A priori error estimate

In this subsection, we derive a priori error estimates for the IPDG method. To this end, we first present the following lemma, which provides approximation estimates for the second-family Nédélec interpolation operator Π_N [51–53].

Lemma 6. Assume that $\{\mathcal{T}_h\}$ is a shape-regular family of tetrahedral or affine hexahedral (including parallelepiped) partitions of $\bar{\Omega}$, and assume $E \in H_0(\text{curl}; \Omega) \cap H^s(\Omega)$ with $\nabla \times E \in H^s(\Omega)$, where $s > 1/2$. Then the following error estimates hold:

$$\|E - \Pi_N E\|_{\text{curl}, \Omega} \leq C_N h^{\min\{s, l\}} (\|E\|_{s, \Omega} + \|\nabla \times E\|_{s, \Omega}), \quad (4.28)$$

$$\|E - \Pi_N E\|_h \leq C_N h^{\min\{s, l\}} (\|E\|_{s, \Omega} + \|\nabla \times E\|_{s, \Omega}), \quad (4.29)$$

where $C_N > 0$ is a constant depending on the mesh regularity and the polynomial degree l but independent of h , and for (4.29), C_N also depends on the upper and lower bounds of the coefficients μ and ϵ .

Moreover, if $E \in H_0(\text{curl}; \Omega) \cap H^{s+1}(\Omega)$ for some number $s > 0$, then

$$\|E - \Pi_N E\|_{0, \Omega} \leq C_N h^{\min\{s, l\}+1} \|E\|_{s+1, \Omega}. \quad (4.30)$$

The estimate (4.28) can be found in [51], and (4.30) appears in [36, Lemma 4.1]. The error bound (4.29) follows from (4.28) because the coefficients ϵ and μ are bounded and, for $E \in H_0(\text{curl}; \Omega)$, the tangential jump $\|E - \Pi_N E\|$ vanishes.

For $v \in V(h)$, we define

$$r_h(E; v) = \sum_{f \in F_h} \int_f \|v\| \cdot \{\mu^{-1} \nabla \times E - \mu^{-1} \Pi_h(\nabla \times E)\} dS. \quad (4.31)$$

For $r_h(E; v)$ to be well-defined, the condition $\nabla \times E \in H^s(\Omega)$, where $s > 1/2$, is assumed. Under this condition, we have the following result [37].

Lemma 7. Assume $\nabla \times E \in H^s(\Omega)$, $s > 1/2$. Then,

$$|r_h(E; v)| \leq C_R h^{\min\{s, l+1\}} \|v\|_h \|\nabla \times E\|_{s, \Omega} \quad \forall v \in V(h),$$

where the constant C_R is independent of the mesh size but depends on η , the upper and lower bounds of the coefficient μ , the mesh regularity, and the polynomial order l .

Theorem 3. Assume that $\mathcal{T}_h$ is a shape-regular family of tetrahedral or affine hexahedral (including parallelepiped) partitions of $\bar{\Omega}$. Let E and E_h be the solutions of (2.11) and (3.5), respectively. Suppose that

$$\mu^{-1} \nabla \times E \in H(\text{curl}, \Omega), \quad (\mu^{-1} \nabla \times E)|_K \in H^1(K) \quad \text{for all } K \in \mathcal{T}_h.$$

Assume that the penalty parameter η satisfies

$$\eta > \frac{\max\{1, \mu_1^2\} \bar{C}^2}{2\mu_0^2},$$

and that $m < \epsilon_0$.

If $E \in H^s(\Omega)$ and $\nabla \times E \in H^s(\Omega)$ with $s > 1/2$, then there exists a positive constant C , independent of h , such that

$$\|E - E_h\|_h \leq Ch^{\min\{s, l\}/2}. \quad (4.32)$$

Moreover, if $E \in H^{s+1}(\Omega)$ with $s > 1/2$, then

$$\|E - E_h\|_h \leq Ch^{(\min\{s, l\}+1)/2}. \quad (4.33)$$

Proof. Denote $E_I = \Pi_N E$ and write

$$\tilde{a}_h(E_I - E_h, E_I - E_h) = T_1 + T_2, \quad (4.34)$$

where $T_1 = \tilde{a}_h(E_I - E, E_I - E_h)$ and $T_2 = \tilde{a}_h(E - E_h, E_I - E_h)$. Using Young's inequality (4.1) and the boundedness estimate for $\tilde{a}_h$, we have, for any small $\varepsilon > 0$,

$$T_1 \leq C_b \|E_I - E\|_h \|E_I - E_h\|_h \leq \frac{\varepsilon}{4} \|E_I - E_h\|_h^2 + \frac{C_b^2}{\varepsilon} \|E_I - E\|_h^2. \quad (4.35)$$

Note that on $\mathcal{F}_h$, $\llbracket E \rrbracket = 0$, $\{E\} = E$, and $\llbracket \nabla \times E \rrbracket = 0$. Thus,

$$\begin{aligned} \tilde{a}_h(E, E_I - E_h) &= \int_{\Omega} \epsilon E \cdot (E_I - E_h) \, dx + \sum_{K \in \mathcal{T}_h} \int_K \mu^{-1} (\nabla \times E) \cdot (\nabla_h \times (E_I - E_h)) \, dx \\ &\quad - \sum_{f \in \mathcal{F}_h} \int_f \llbracket E_I - E_h \rrbracket \cdot \{\mu^{-1} \Pi_h(\nabla \times E)\} \, dS \\ &= a_h(E, E_I - E_h) + \sum_{f \in \mathcal{F}_h} \int_f \llbracket E_I - E_h \rrbracket \cdot \{\mu^{-1} \nabla \times E\} \, dS \\ &\quad - \sum_{f \in \mathcal{F}_h} \int_f \llbracket E_I - E_h \rrbracket \cdot \{\mu^{-1} \Pi_h(\nabla \times E)\} \, dS. \end{aligned} \quad (4.36)$$

Since $E_h - E_I \in V_h$, it follows from Theorem 1 that

$$a_h(E, E_h - E_I) + \int_{\Omega} \psi^0(E; E_h - E_I) \, dx \geq \langle f, E_h - E_I \rangle,$$

that is,

$$\langle f, E_I - E_h \rangle + \int_{\Omega} \psi^0(E; E_h - E_I) \, dx \geq a_h(E, E_I - E_h). \quad (4.37)$$

Combining (4.36) with (4.37), we obtain

$$\begin{aligned} \tilde{a}_h(E, E_I - E_h) &\leq \langle f, E_I - E_h \rangle + \int_{\Omega} \psi^0(E; E_h - E_I) \, dx \\ &\quad + \sum_{f \in \mathcal{F}_h} \int_f \llbracket E_I - E_h \rrbracket \cdot \{\mu^{-1} \nabla \times E - \mu^{-1} \Pi_h(\nabla \times E)\} \, dS \\ &= \langle f, E_I - E_h \rangle + \int_{\Omega} \psi^0(E; E_h - E_I) \, dx + r_h(E; E_I - E_h). \end{aligned} \quad (4.38)$$

Letting $v_h = E_I - E_h$ in (3.5), we get

$$-\tilde{a}_h(E_h, E_I - E_h) = -a_h(E_h, E_I - E_h) \leq \int_{\Omega} \psi^0(E_h; E_I - E_h) \, dx - \langle f, E_I - E_h \rangle. \quad (4.39)$$

Combining (4.38) and (4.39), and using the subadditivity of the generalized directional derivative, we obtain

$$\begin{aligned} T_2 &\leq \int_{\Omega} \psi^0(E; E_h - E_I) \, dx + \int_{\Omega} \psi^0(E_h; E_I - E_h) \, dx + r_h(E; E_I - E_h) \\ &\leq \int_{\Omega} \psi^0(E; E_h - E) \, dx + \int_{\Omega} \psi^0(E; E - E_I) \, dx \\ &\quad + \int_{\Omega} \psi^0(E_h; E_I - E) \, dx + \int_{\Omega} \psi^0(E_h; E - E_h) \, dx + r_h(E; E_I - E_h). \end{aligned} \quad (4.40)$$

Using (2.8) (d), we obtain

$$\int_{\Omega} \psi^0(E; E_h - E) \, dx + \int_{\Omega} \psi^0(E_h; E - E_h) \, dx \leq m \|E - E_h\|_{0, \Omega}^2.$$

By the triangle inequality $\|E - E_h\|_{0,\Omega} \leq \|E - E_I\|_{0,\Omega} + \|E_I - E_h\|_{0,\Omega}$ and Young's inequality (4.1),

$$\|E - E_h\|_{0,\Omega}^2 \leq (1 + \varepsilon)\|E_I - E_h\|_{0,\Omega}^2 + (1 + 1/\varepsilon)\|E - E_I\|_{0,\Omega}^2.$$

Therefore,

$$\begin{aligned} & \int_{\Omega} \psi^0(E; E_h - E) dx + \int_{\Omega} \psi^0(E_h; E - E_h) dx \\ & \leq (1 + \varepsilon)m\|E_I - E_h\|_{0,\Omega}^2 + (1 + 1/\varepsilon)m\|E - E_I\|_{0,\Omega}^2. \end{aligned}$$

Using (4.17), we have

$$\int_{\Omega} \psi^0(E; E - E_I) dx \leq \int_{\Omega} (c_0 + c_1|E|)|E - E_I| dx, \quad (4.41)$$

$$\int_{\Omega} \psi^0(E_h; E_I - E) dx \leq \int_{\Omega} (c_0 + c_1|E_h|)|E - E_I| dx. \quad (4.42)$$

Since Ω is bounded, and since E_h is uniformly bounded, the Cauchy-Schwarz inequality yields constants $C_{Eb}, C_{Ehb} > 0$ such that

$$\begin{aligned} \int_{\Omega} (c_0 + c_1|E|)|E - E_I| dx & \leq \left(\int_{\Omega} (c_0 + c_1|E|)^2 dx \right)^{1/2} \|E - E_I\|_{0,\Omega} = C_{Eb}\|E - E_I\|_{0,\Omega}, \\ \int_{\Omega} (c_0 + c_1|E_h|)|E - E_I| dx & \leq \left(\int_{\Omega} (c_0 + c_1|E_h|)^2 dx \right)^{1/2} \|E - E_I\|_{0,\Omega} \leq C_{Ehb}\|E - E_I\|_{0,\Omega}. \end{aligned}$$

Therefore,

$$\begin{aligned} T_2 & \leq (C_{Eb} + C_{Ehb})\|E - E_I\|_{0,\Omega} + (1 + \varepsilon)m\|E_I - E_h\|_{0,\Omega}^2 + (1 + 1/\varepsilon)m\|E - E_I\|_{0,\Omega}^2 \\ & \quad + r_h(E; E_I - E_h). \end{aligned} \quad (4.43)$$

Since $E \in H^{s+1}(\Omega)$ implies $\nabla \times E \in H^s(\Omega)$, the regularity assumption imposed on E in the theorem already ensures that $\nabla \times E \in H^s(\Omega)$ for $s > 1/2$. Combining (4.34), (4.35), (4.43), Lemmas 2 and 7, we obtain

$$\begin{aligned} & \epsilon_0\|E_I - E_h\|_{0,\Omega}^2 + C_0 \sum_{f \in \mathcal{F}_h} \|\alpha^{1/2} [E_I - E_h]\|_{0,f}^2 + C_1 \sum_{K \in \mathcal{T}_h} \|\nabla_h \times (E_I - E_h)\|_{0,K}^2 \\ & \leq (C_{Eb} + C_{Ehb})\|E - E_I\|_{0,\Omega} + (1 + \varepsilon)m\|E_I - E_h\|_{0,\Omega}^2 + (1 + 1/\varepsilon)m\|E - E_I\|_{0,\Omega}^2 \\ & \quad + \frac{\varepsilon}{4}\|E_I - E_h\|_h^2 + \frac{C_b^2}{\varepsilon}\|E_I - E\|_h^2 + C_R h^{\min\{s, l+1\}} |E_I - E_h|_h \|\nabla \times E\|_{s,\Omega}. \end{aligned} \quad (4.44)$$

Using Young's inequality (4.1), we have, for any $\varepsilon > 0$,

$$C_R h^{\min\{s, l+1\}} |E_I - E_h|_h \|\nabla \times E\|_{s,\Omega} \leq \frac{\varepsilon}{4} |E_I - E_h|_h^2 + \frac{C_R^2 h^{2\min\{s, l+1\}}}{\varepsilon} \|\nabla \times E\|_{s,\Omega}^2. \quad (4.45)$$

Since $m < \epsilon_0$, we can choose a sufficiently small $\varepsilon > 0$ such that $\epsilon_0 - m - (m + 1/4)\varepsilon > 0$ and $\min\{C_0, C_1\} > \varepsilon/4$. When $E \in H^s(\Omega)$ and $\nabla \times E \in H^s(\Omega)$ with $s > 1/2$, the term $\|E - E_I\|_{0,\Omega}$ on the right-hand side of (4.44) does not yield the optimal convergence order. In this case, it can only be controlled by (4.29). Invoking the estimate (4.29), we deduce from (4.44) that

$$\|E_I - E_h\|_h \leq C h^{\min\{s, l\}/2}.$$

Moreover, when $E \in H^{s+1}(\Omega)$, the term $\|E - E_I\|_{0,\Omega}$ attains the optimal convergence order given by (4.30). Therefore, we obtain

$$\|E_I - E_h\|_h \leq C h^{(\min\{s, l\}+1)/2}.$$

Finally, we use the triangle inequality $\|E - E_h\|_h \leq \|E - E_I\|_h + \|E_I - E_h\|_h$ to conclude the error bound (4.32) and (4.33). $\square$

In particular, when linear elements are used and the solution E of the hemivariational inequality (2.11) satisfies

$$E \in H^2(\Omega),$$

we obtain the optimal first-order convergence estimate:

$$\|E - E_h\|_h \leq C h.$$

5. Numerical example

This paper primarily focuses on the three-dimensional case. However, a two-dimensional vector field $E(x, y) = (E_1(x, y), E_2(x, y))$ can be embedded into three dimensions by setting

$$E(x, y, z) = (E_1(x, y), E_2(x, y), 0).$$

Similarly, a two-dimensional normal vector $\mathbf{n} = (n_1, n_2)$ can be identified with $\mathbf{n} = (n_1, n_2, 0)$ in $\mathbb{R}^3$. Under this identification, we have

$$\nabla \times \mathbf{E} = \left(0, 0, \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} \right), \quad \mathbf{n} \times \mathbf{E} = (0, 0, n_1 E_2 - n_2 E_1).$$

Therefore, the three-dimensional analysis presented in this paper applies directly to the two-dimensional case. In this section, we report numerical results for a two-dimensional example.

Let $\Omega = (0, 1)^2$, $\epsilon = 1$, $\mu = 1$, and the source term

$$\mathbf{f} = \begin{pmatrix} (1 + 2\pi^2) \cos(\pi x) \sin(\pi y) \\ -(1 + 2\pi^2) \sin(\pi x) \cos(\pi y) \end{pmatrix}.$$

The function ψ is chosen as follows:

$$\omega(t) = (a - b)e^{-\beta t} + b, \quad \psi(\mathbf{E}) = \int_0^{|\mathbf{E}|} \omega(t) dt,$$

where $a > b > 0$ and $\beta > 0$. It can be verified that this function ψ satisfies the conditions in (2.8), where the parameter m in (2.8) (d) is given by $m = \beta(a - b)$. We choose the parameters $a = 0.004$, $b = 0.002$, and $\beta = 100$ in the function $\omega(t)$.

To solve the problem (3.5), we employ the Uzawa iterative algorithm (cf. [54]), see Algorithm 1. We choose the penalty parameter $\eta = 10^3$. Since the analytic solution of the inequality problem is unknown, we will use the numerical solution with a grid size of $h = 2^{-6}$ as a reference solution to compute the error of the numerical solution with coarser grid sizes.

Algorithm 1: Uzawa iteration for the $\mathbf{H}(\text{curl})$ -elliptic HVI.

Input: Maximal number of iteration steps $l_{\max}$, error tolerance $\text{tol} > 0$.

Output: The numerical solution of the inequality problem $\mathbf{E}^*$.

```

1 Find  $\mathbf{E}_h^0 \in \mathbf{V}_h$  such that  $a_h(\mathbf{E}_h^0, \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle \quad \forall \mathbf{v}_h \in \mathbf{V}_h$ ;
2 for  $l = 1$  to  $l_{\max}$  do
3   Find  $\mathbf{E}_h^l \in \mathbf{V}_h$  such that  $a_h(\mathbf{E}_h^l, \mathbf{v}_h) + \int_{\Omega} \lambda_h^l \cdot \mathbf{v}_h = \langle \mathbf{f}, \mathbf{v}_h \rangle \quad \forall \mathbf{v}_h \in \mathbf{V}_h$ , where  $\lambda_h^l \in \omega(|\mathbf{E}_h^{l-1}|) \partial |\mathbf{E}_h^l|$ ;
4   if  $\|\mathbf{E}_h^l - \mathbf{E}_h^{l-1}\|_{0,\Omega} \leq \text{tol} \|\mathbf{E}_h^{l-1}\|_{0,\Omega}$  and  $\|\lambda_h^l - \lambda_h^{l-1}\|_{0,\Omega} \leq \text{tol} \|\lambda_h^{l-1}\|_{0,\Omega}$  then
5     break
6  $\mathbf{E}^* = \mathbf{E}_h^l$ ;
7 return  $\mathbf{E}^*$ ;
```

Table 1

Errors and convergence orders of the numerical solutions.

h	$\ \mathbf{E} - \mathbf{E}_h\ _{0,\Omega}$	Order	$\ \mathbf{E} - \mathbf{E}_h\ _h$	Order
2^{-1}	2.1929e-1	-	1.5957	-
2^{-2}	6.0856e-2	1.8494	8.3933e-1	0.9269
2^{-3}	1.5644e-2	1.9598	4.2970e-1	0.9659
2^{-4}	3.8917e-3	2.0071	2.1416e-1	1.0047

Numerical results are reported in Table 1. It is observed that for this example, the numerical solutions exhibit second-order convergence in the L^2 -norm and first-order convergence in the energy norm $\|\cdot\|_h$ with respect to the grid size h . The observed convergence rate in the energy norm $\|\cdot\|_h$ agrees with the theoretical result established in Theorem 3. Fig. 2 illustrates these convergence rates. Fig. 3 displays the streamline plots of the numerical solutions for the inequality problem with grid sizes $h = 2^{-4}$ and $h = 2^{-5}$.

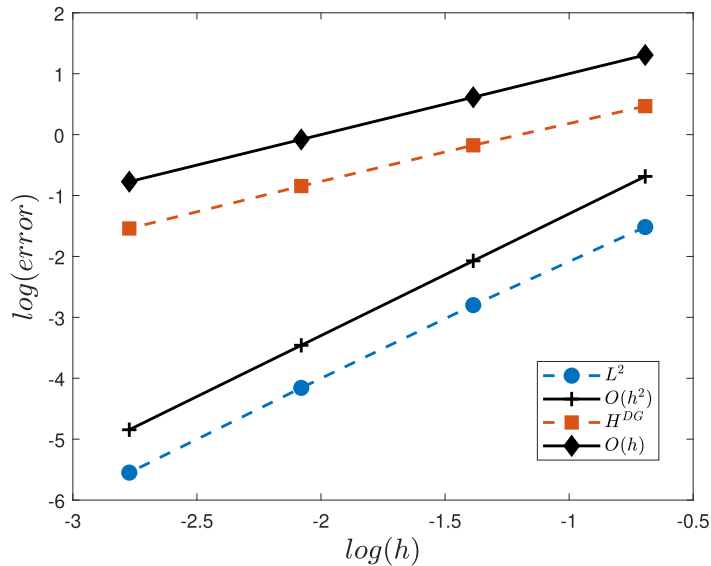


Fig. 2. Convergence of the numerical solutions.

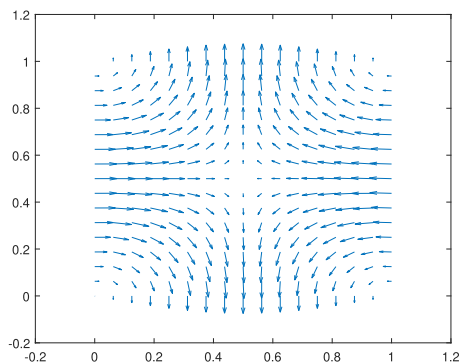
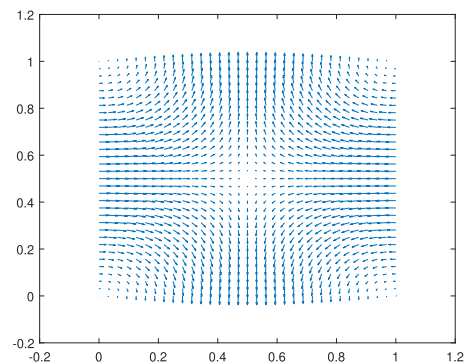
(a) E_h for $h = 2^{-4}$ (b) E_h for $h = 2^{-5}$

Fig. 3. Streamlines of numerical solutions.

CRediT authorship contribution statement

Xiajie Huang: Writing – original draft, Visualization, Software, Investigation, Formal analysis; **Fei Wang:** Writing – review & editing, Supervision, Resources, Project administration, Methodology, Funding acquisition, Conceptualization; **Weimin Han:** Writing – review & editing, Supervision, Resources, Methodology, Funding acquisition; **Min Ling:** Validation, Software, Formal analysis.

Data availability statement

No data was used for the research described in the article.

Declaration of competing interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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