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Analysis and Control of a General Elliptic Quasivariational-Hemivariational Inequality

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We consider a general elliptic quasivariational-hemivariational inequality in real Hilbert spaces for which we provide a solution existence and uniqueness result, a convergent iterative procedure, and a Lipschitz continuous dependence result that we use in order to deduce the existence of a solution to an associated optimal control problem. As an example for applications of the abstract results, we consider a new model of static contact problem which describes the equilibrium of an elastic body with a reactive foundation. The weak formulation of the model is a quasivariational-hemivariational inequality for the displacement field. We present theoretical results on the analysis and control of the contact problem.

Keywords: Quasivariational-hemivariational inequality, existence, uniqueness, optimal control, locking material, frictional contact problem, unilateral constraint, weak solution.

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1. Introduction

Introduced by Panagiotopoulous in the early 1980s ([21]), hemivariational inequalities and the more general variational-hemivariational inequalities represent an area of intensive research in the recent years, due to their applications in physical and engineering problems where non-smooth, non-monotone and/or set-valued relations are used among different physical quantities. While variational inequalities are featured by the presence of non-smooth convex functions in their formulations, variational-hemivariational inequalities generally include both nonsmooth convex functions and locally Lipschitz functions that are allowed to be nonconvex. Early references in the area include the books [19, 23]. The mathematical literature on variational-hemivariational inequalities concerns existence, uniqueness, regularity and convergence results, among others. The reader is referred to [17, 29] for recent advances on the analysis of variational-hemivariational inequalities, and is referred to [10] for recent advances on their numerical analysis.

An abstract form of an elliptic quasivariational-hemivariational inequality is as follows.

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Problem 1.1. *Find an element $u \in K_V$ such that*

$$\langle Au, v-u\rangle + \Phi(u,v) - \Phi(u,u) + \Psi^0(u; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K_V. \qquad (1)$$

Here K_V is a subset of a normed space V, A is an operator mapping V to its dual V^*, Φ is a bi-variate functional defined on V, Ψ is a locally Lipschitz function on V and Ψ^0 denotes the generalized directional derivative of Ψ. For a precise description of the problem setting, cf. Section 2. We use the term "quasivariational-hemivariational" inequality since the functional Φ depends on the solution u.

Existence and uniqueness of a solution to Problem 1.1 was first investigated in [18] through an application of abstract surjectivity results of pseudomonotone operators combined with the Banach fixed-point theorem. The problem was re-examined in [8] through an elementary approach of the Banach fixed-point argument, starting with a minimization principle for a special case of Problem 1.1 where $\Phi(u,v) \equiv \Phi(v)$ and A is a potential operator ([7]). See Section 2 for a typical statement on the solution existence and uniqueness for Problem 1.1.

An extension of Problem 1.1 is to allow the functional Ψ to depend on two arguments. In the following form of such an extension, Ψ^0 stands for the generalized directional derivative of Ψ with respect to its second argument.

Problem 1.2. *Find an element $u \in K_V$ such that*

$$\langle Au, v-u\rangle + \Phi(u,v) - \Phi(u,u) + \Psi^0(u, u; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K_V. \qquad (2)$$

Assumptions on the data of Problem 1.2 will be specified in Section 3. A problem from contact mechanics, similar to Problem 1.2 (or more precisely, of the form of Problem 4.2 in Section 4), is considered in [17, Section 7.1]. There, the existence of a solution to the problem is given, but not the uniqueness of the solution. One purpose of the paper is to fill this gap by providing a complete result on solution existence and uniqueness for Problem 1.2. Furthermore, we investigate an optimal control problem associated with Problem 1.2. We will illustrate the theoretical results in the study of a contact problem.

Processes of contact between deformable bodies abound in industry and everyday life. Their modeling, analysis and numerical simulation are the topics of a large number of references that continues to grow steadily. Comprehensive references in the area include books [2, 4, 5, 9, 11, 15, 17, 22, 29], for instance. In these references, various contact problems are studied, for different types of materials such as elastic, viscoelastic and viscoplastic materials, associated with different contact and friction boundary conditions. The contact problems are formulated as variational, hemivariational and variational-hemivariational inequalities, that allows well-posedness analysis with techniques from functional analysis and nonsmooth analysis. We consider in this paper a contact model for elastic materials with locking property which leads to unilateral constraint in the variational formulations of the model. This contact problem is similar to the one considered in [27] except that the contact is frictional.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries, including an existence and uniqueness result for Problem 1.1. In Section 3, we deal with analysis and control of Problem 1.2, which represents a general elliptic quasivariational-hemivariational inequality. We provide results on existence and uniqueness of a solution, a convergent iterative procedure, and Lipschitz continuous dependence for the solution, as well as the existence of an optimal control to an associated control problem. In Section 4 we consider a new model of contact for elastic materials, list assumptions on the data and introduce its variational formulation. Then, in Section 5 we apply the abstract results from Section 3 in the analysis and control of this contact model.

2. Preliminaries

In this section we present some notation and preliminary material which are needed in the rest of the paper. Let V be a real Hilbert space. Denote by $\|\cdot\|_V$ the norm on V, by V^* the dual space of V, and by $\|\cdot\|_{V^*}$, $\langle\cdot,\cdot\rangle$ the norm on V^* and the duality pairing over $V^*\times V$, respectively. Given a set $K_V\subset V$, an operator $A\colon V\to V^*$, functions $\Phi\colon V\times V\to\mathbb{R}$, $\Psi\colon V\to\mathbb{R}$ and $f\in V^*$, we consider Problem 1.1.

The function Φ is assumed to be convex with respect to its second argument whereas the function Ψ is assumed to be locally Lipschitz. In (1), $\Psi^0(u;v-u)$ denotes the generalized directional derivative of Ψ at the point u along the direction $v-u$. Recall that for a locally Lipschitz continuous function $\Psi\colon V\to\mathbb{R}$, the *Clarke* (or *generalized*) *directional derivative* of Ψ at $u\in V$ in the direction $v\in V$ is defined by

$$\Psi^0(u;v) := \limsup_{w\to u,\,\lambda\downarrow 0}\frac{\Psi(w+\lambda v)-\Psi(w)}{\lambda},$$

and the *Clarke subdifferential* (or *generalized gradient*) of Ψ at $u\in V$ is defined by

$$\partial\Psi(u) := \big\{\eta\in V^* \mid \Psi^0(u;v)\ge\langle\eta,v\rangle\ \forall\, v\in V\big\}. \tag{3}$$

Discussions and properties of the generalized directional derivative and the generalized gradient can be found in various references, e.g., [3, 17].

For a set K in a normed space X, the function $I_K: V\to\mathbb{R}$ defined by

$$I_K(u)=\begin{cases}0, & \text{if } u\in K,\\ +\infty, & \text{if } u\notin K,\end{cases}$$

is called the *indicator function* of K. Its *subdifferential* (in the sense of convex analysis) is

$$\partial^c I_K(u)=\begin{cases}\{\,\eta^*\in X^*\mid\langle\eta^*,u-v\rangle\ge 0\ \forall\, v\in K\,\} & \text{if } u\in K,\\ \emptyset & \text{if } u\notin K,\end{cases} \tag{4}$$

when K is convex. An element $\eta^*\in\partial^c I_K(u)$ (if any) is called a *subgradient* of I_K at u. We shall use these notions in Section 4 with a particular choice of the space V.

We use the notation " $\to$" and " $\rightharpoonup$" for the strong and weak convergence in various spaces that will be specified. Moreover, all the limits are considered as $n\to\infty$, even if we do not mention it explicitly.

Finally, we recall that a function $\varphi\colon K_V \to \mathbb{R}$ is said to be *weakly lower semicontinuous* (weakly l.s.c.) at $u \in K_V$ if

$$\liminf_{n\to\infty} \varphi(u_n) \geq \varphi(u) \tag{5}$$

for each sequence $\{u_n\} \subset K_V$ converging weakly to u in V. The function φ is weakly l.s.c. on K_V if it is weakly l.s.c. at every point $u \in K_V$.

For the analysis of Problem 1.1, we consider the following hypotheses on the data.

$H(K_V)$ V is a real Hilbert space, K_V is a non-empty, closed and convex set in V.

$H(A)$ $A\colon V \to V^*$ is Lipschitz continuous and strongly monotone.

$H(\Phi)_2$ $\Phi\colon V \times V \to \mathbb{R}$; for any $u \in V$, $\Phi(u,\cdot)\colon V \to \mathbb{R}$ is convex and bounded above on a non-empty open set; and there exists a constant $\alpha_\Phi \geq 0$ such that for all $u_1, u_2, v_1, v_2 \in V$ we have

$$\Phi(u_1, v_2) - \Phi(u_1, v_1) + \Phi(u_2, v_1) - \Phi(u_2, v_2) \leq \alpha_\Phi \|u_1 - u_2\|_V \|v_1 - v_2\|_V. \tag{6}$$

$H(\Psi)$ $\Psi\colon V \to \mathbb{R}$ is locally Lipschitz continuous, and for a constant $\alpha_\Psi \geq 0$,

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V. \tag{7}$$

$H(f)$ $f \in V^*$.

We denote by $m_A > 0$ the constant in the strong monotonicity inequality of A.

Then, $$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V.$$

The subscript 2 in $H(\Phi)_2$ reminds the reader that this is a condition for the case where Φ has two independent variables. The condition $\Phi(u,\cdot)\colon V \to \mathbb{R}$ being convex and bounded above on a non-empty open set guarantees that $\Phi(u,\cdot)$ is locally Lipschitz continuous on V (cf. [6, Corollary 2.4, p. 12]).

The unique solvability of Problem 1.1 was shown in [18] and [29]. The statement of the next result follows [8]; see also [28].

Theorem 2.1. *Assume $H(K_V)$, $H(A)$, $H(\Phi)_2$, $H(\Psi)$, $H(f)$, and the smallness condition*

$$\alpha_\Phi + \alpha_\Psi < m_A.$$

Then, Problem 1.1 *has a unique solution $u \in K_V$.*

We end this section by recalling a well-known general result on existence of a minimizer ([1, Section 3.3.2], [14]).

Theorem 2.2. *Let $(W, \|\cdot\|_W)$ be a reflexive Banach space, K_W a nonempty weakly closed subset of W and $J : K_W \to \mathbb{R}$ a weakly lower semicontinuous function. In addition, assume that either K_W is bounded or J is coercive, i.e., $J(w_n) \to \infty$ as $\|w_n\|_W \to \infty$. Then, there exists at least one element $\overline{w}$ such that*

$$\overline{w} \in K_W, \qquad J(\overline{w}) \leq J(w) \qquad \forall\, w \in K_W. \tag{8}$$

3. Analysis and control of a general quasivariational-hemivariational inequality

In this section we deal with the analysis and control of Problem 1.2. Note that the difference between the Problems 1.1 and 1.2 is that in Problem 1.2, $\Psi\colon V\times V\to\mathbb{R}$ is allowed to be a functional of two arguments. For this reason we refer to Problem 1.2 as a general elliptic quasivariational-hemivariational inequality and view Problem 1.1 a standard elliptic quasivariational-hemivariational inequality. We use Ψ^0 to mean the Clarke generalized directional derivative with respect to its second argument.

In the study of Problem 1.2, condition $H(\Psi)$ is replaced by the following.

$H(\Psi)_2$ $\Psi\colon V\times V\to\mathbb{R}$ is locally Lipschitz continuous with respect to its second argument, and for two constants $\alpha_{\Psi,1},\alpha_{\Psi,2}\ge 0$ and for all $w_1,w_2,v_1,v_2\in V$,

$$\Psi^0(w_1,v_1;v_2-v_1)+\Psi^0(w_2,v_2;v_1-v_2)\\ \le \alpha_{\Psi,1}\|v_1-v_2\|_V^2+\alpha_{\Psi,2}\|w_1-w_2\|_V\|v_1-v_2\|_V. \tag{9}$$

Similar to $H(\Phi)_2$, the subscript 2 in $H(\Psi)_2$ reminds the reader that this is a condition for the case where Ψ has two independent variables.

Our first result in this section is the following.

Theorem 3.1. *Assume $H(K_V)$, $H(A)$, $H(\Phi)_2$, $H(\Psi)_2$, $H(f)$, and the smallness condition*

$$\alpha_\Phi+\alpha_{\Psi,1}+\alpha_{\Psi,2}<m_A. \tag{10}$$

Then, Problem 1.2 has a unique solution $u\in K_V$. Moreover, the operator $f\mapsto u=u(f)$ which maps the element $f\in V^*$ to the solution $u\in K_V$ of Problem 1.2 is Lipschitz continuous.

Proof. By Theorem 2.1, for any $w\in K_V$, the auxiliary problem of finding $u\in K_V$ such that

$$\langle Au,v-u\rangle+\Phi(u,v)-\Phi(u,u)+\Psi^0(w,u;v-u)\ge\langle f,v-u\rangle\quad\forall\,v\in K_V \tag{11}$$

has a unique solution. This defines a mapping $P\colon K_V\to K_V$ by $u=P(w)$. Let us show that the mapping P is a contraction on K_V. For this purpose, let $w_1,w_2\in K_V$, and denote $u_1=P(w_1)$, $u_2=P(w_2)$. Then we have the inequalities

$$\langle Au_1,u_2-u_1\rangle+\Phi(u_1,u_2)-\Phi(u_1,u_1)+\Psi^0(w_1,u_1;u_2-u_1)\ge\langle f,u_2-u_1\rangle,$$
$$\langle Au_2,u_1-u_2\rangle+\Phi(u_2,u_1)-\Phi(u_2,u_2)+\Psi^0(w_2,u_2;u_1-u_2)\ge\langle f,u_1-u_2\rangle.$$

We add the two inequalities to get

$$\langle Au_1-Au_2,u_1-u_2\rangle\le\Phi(u_1,u_2)+\Phi(u_2,u_1)-\Phi(u_1,u_1)-\Phi(u_2,u_2)\\ +\Psi^0(w_1,u_1;u_2-u_1)+\Psi^0(w_2,u_2;u_1-u_2).$$

Now we apply the assumptions on the operator A and the functions Φ, Ψ and find

$$m_A\|u_1-u_2\|_V^2\le\alpha_\Phi\|u_1-u_2\|_V^2+\alpha_{\Psi,1}\|u_1-u_2\|_V^2+\alpha_{\Psi,2}\|w_1-w_2\|_V\|u_1-u_2\|_V. \tag{12}$$

Thus,

$$\|u_1-u_2\|_V\le\frac{\alpha_{\Psi,2}}{m_A-(\alpha_\Phi+\alpha_{\Psi,1})}\,\|w_1-w_2\|_V.$$

From the smallness condition (10),

$$\frac{\alpha_{\Psi,2}}{m_A - (\alpha_\Phi + \alpha_{\Psi,1})} < 1.$$

Hence, $P\colon K_V \to K_V$ is contractive and, by the Banach fixed-point theorem, P has a unique fixed-point $u \in K_V$. It is easy to see that the fixed-point u is the unique solution of Problem 1.2. This proves the existence and uniqueness part of the theorem.

In order to prove the Lipschitz continuity of the solution with respect to $f \in V^*$, we consider two elements $f_1, f_2 \in V^*$ and denote by u_1, u_2 the solutions of Problem 1.2 for $f = f_1$ and $f = f_2$, respectively. We use arguments similar to those used in the proof of (12) to see that

$$m_A\|u_1 - u_2\|_V^2 \le (\alpha_\Phi + \alpha_{\Psi,1} + \alpha_{\Psi,2})\|u_1 - u_2\|_V^2 + \|f_1 - f_2\|_{V^*}\|u_1 - u_2\|_V. \quad (13)$$

Then, using the smallness condition (10) we deduce that

$$\|u_1 - u_2\|_V \le \frac{1}{m_A - (\alpha_\Phi + \alpha_{\Psi,1} + \alpha_{\Psi,2})}\,\|f_1 - f_2\|_{V^*},$$

which conclude the proof. □

A natural iterative procedure can be introduced and studied for solving Problem 1.2 (to be implemented at the discrete level). Let

$$u_0 \in K_V, \quad (14)$$

and for $n \ge 1$, find $u_n \in K_V$ such that, for all $v \in K_V$,

$$\langle Au_n, v - u_n\rangle + \Phi(u_n, v) - \Phi(u_n, u_n) + \Psi^0(u_{n-1}, u_n; v - u_n) \ge \langle f, v - u_n\rangle. \quad (15)$$

Observe that each iteration (15) requires the solution of a quasivariational-hemivariational inequality of the form stated in Problem 1.1.

Theorem 3.2. *Under the assumptions of Theorem* 3.1*, the iteration procedure* (14)–(15) *is well-defined and the sequence* $\{u_n\}$ *converges to the solution* $u \in K_V$ *of Problem* 1.2*.*

Proof. Under the assumptions of Theorem 3.1, we have the existence of a unique solution $u_n \in K_V$ to (15) (cf. e.g. [8, Theorem 3.7]).

To prove the convergence, we choose $v = u_n$ in (2) and $v = u$ in (15), and add the two inequalities to obtain

$$\begin{aligned}\langle Au - Au_n, u - u_n\rangle \le \Phi(u, u_n) - \Phi(u, u) + \Phi(u_n, u) - \Phi(u_n, u_n)\\ + \Psi^0(u, u; u_n - u) + \Psi^0(u_{n-1}, u_n; u - u_n).\end{aligned}$$

Then we use the conditions $H(A)$, $H(\Phi)_2$, $H(\Psi)_2$ to get

$$m_A\|u - u_n\|_V^2 \le \alpha_\Phi\|u - u_n\|_V^2 + \alpha_{\Psi,1}\|u - u_n\|_V^2 + \alpha_{\Psi,2}\|u - u_{n-1}\|_V\|u - u_n\|_V,$$

i.e.,
$$\|u - u_n\|_V \le \kappa\,\|u - u_{n-1}\|_V, \quad \kappa := \frac{\alpha_{\Psi,2}}{m_A - (\alpha_\Phi + \alpha_{\Psi,1})}.$$

Therefore,
$$\|u - u_n\|_V \le \kappa^n\|u - u_0\|_V.$$

By the smallness condition (10), we know $\kappa < 1$ and hence, $\|u - u_n\|_V \to 0$ as $n \to \infty$. $\square$

We now proceed with the study of an optimal control problem associated with the quasivariational-hemivariational inequality (2). To this end, we keep the assumption of Theorem 3.1 and for any $f \in V^*$ we denote by $u(f) \in K_V$ the solution of Problem 1.2. We assume that W is a normed space endowed with the norm $\|\cdot\|_W$, $K_W \subset W$ and $B : K_W \to V^*$ is a given operator. In addition, we consider a cost function $J : K_V \times K_W \to \mathbb{R}$. In this framework we state the following optimal control problem.

Problem 3.3. *Find $\overline{w} \in K_W$ such that*

$$J(u(B\overline{w}), \overline{w}) = \min_{w \in K_W} J(u(Bw), w). \tag{16}$$

In the study of this problem we consider the following assumptions.

$H(K_W)$ W is a reflexive Banach space, K_W is a nonempty weakly closed set in W.

$H(B)$ $B\colon K_W \to V^*$ is completely continuous, that is, if $w_n \rightharpoonup w$ in W, then $Bw_n \to Bw$ in V^*.

$H(J)$ $J\colon V \times W \to \mathbb{R}$ and

$$\left\{ \begin{array}{l} \text{for all sequences } \{u_n\} \subset V \text{ and } \{w_n\} \subset W \text{ such that } u_n \to u \\ \text{in } V,\ w_n \rightharpoonup w \ \text{ in } W,\ \text{we have } \liminf\limits_{n\to\infty} J(u_n, w_n) \ge J(u, w). \end{array} \right. \tag{17}$$

$H(K_W)'$ K_W is a bounded set in W.

$H(J)'$ K_W is unbounded and there exists $m_J : K_W \to \mathbb{R}$ such that

$$\left\{ \begin{array}{l} \text{(a)} \ \ J(u, w) \ge m_J(w) \quad \forall\, u \in V,\ w \in K_W. \\ \text{(b)} \ \ \|w_n\|_W \to +\infty \implies m_J(w_n) \to +\infty. \end{array} \right. \tag{18}$$

Theorem 3.4. *Assume $H(K_V)$, $H(A)$, $H(\Phi)_2$, $H(\Psi)_2$, (10), $H(K_W)$, $H(B)$, $H(J)$, and either $H(K_W)'$ or $H(J)'$. Then, there exists at least one solution $\overline{w} \in K_W$ of Problem* 3.3.

Proof. Let $J_1 : K_W \to \mathbb{R}$ be the function defined by

$$J_1(w) = J(u(Bw), w) \quad \forall\, w \in K_W \tag{19}$$

and consider the following auxiliary problem:

$$\text{find } \overline{w} \in K_W \ \text{ such that } J_1(\overline{w}) = \min_{w \in K_W} J_1(w). \tag{20}$$

We claim that this problem has at least one solution $\overline{w}$. For this purpose, we will apply Theorem 2.2.

Let $\{w_n\} \subset W$ be a sequence such that $w_n \rightharpoonup w$ in W. Then, assumption $H(B)$ shows that $Bw_n \to Bw$ in V^* and, therefore, Theorem 3.1 implies the convergence $u(Bw_n) \to u(Bw)$ in V. We now use definition (19) and assumption (17) to see that

$$\liminf_{k\to\infty} J_1(w_n) \ge J_1(w).$$

We conclude from here that the function $J_1 : K_W \to \mathbb{R}$ is weakly lower semicontinuous.

We now distinguish two cases. First, we assume $H(K_W)'$. Then, existence of a solution to problem (20) follows directly from Theorem 2.2. In the second case, assume $H(J)'$. Then, using (18)(a), for any sequence $\{w_n\} \subset K_W$ we have

$$J_1(w_n) = J(u(Bw_n), w_n) \geq m_J(w_n) \qquad \forall\, n \in \mathbb{N}.$$

Therefore, if $\|w_n\|_W \to \infty$, by assumption (18)(b) we deduce that $J_1(w_n) \to \infty$ which shows that the function J_1 is coercive. We again apply Theorem 2.2 to conclude the solvability of the minimization problem (20).

Since the problem (20) is a reformulation of Problem 1.2, we conclude that under the stated assumptions, Problem 1.2 has a solution. □

4. A contact model

The physical setting, available in many references on contact problems (e.g., those cited in Section 1), can be summarized as follows. A deformable body is fixed on a part of its boundary, is acted upon by body forces and surface tractions and can arrive in contact with an obstacle, the so called foundation. The equilibrium of the body in this physical setting can be described by various mathematical models, obtained by using different mechanical assumptions.

Let the reference configuration of an elastic body be a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$. The boundary $\partial\Omega$ of Ω is assumed to be Lipschitz continuous and

$$\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_C}$$

where Γ_D, Γ_N and Γ_C are mutually disjoint relatively open subsets. We assume $\text{meas}\,(\Gamma_D) > 0$ and $\text{meas}\,(\Gamma_C) > 0$, yet Γ_N is allowed to be empty. We use $\boldsymbol{\nu}$ for the unit outward normal vector on $\partial\Omega$. The displacement variable is an $\mathbb{R}^d$-valued function $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ with the components u_i, $1 \leq i \leq d$. We adopt the summation convention over a repeated index. Over $\mathbb{R}^d$, we use the canonical inner product

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d.$$

Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. The canonical inner product over $\mathbb{S}^d$ is

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d$$

and $\mathbf{0}$ will represent the zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$. For a differentiable (in the classical sense or the weak sense) displacement field $\boldsymbol{u}$, the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)/2$ is an $\mathbb{S}^d$-valued function. The stress tensor $\boldsymbol{\sigma}$ is also an $\mathbb{S}^d$-valued function in Ω. For a vector field $\boldsymbol{v}$, its normal and tangential components on the boundary $\partial\Omega$ are defined as $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$.

For a tensor field $\boldsymbol{\sigma}$, its normal and tangential components on $\partial\Omega$ are defined by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$. Note that, here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \partial\Omega$.

The pointwise formulation of the contact problem we consider is as follows.

Problem 4.1. *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma} \in \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \partial^c I_{K_0}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \Omega, \tag{21}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{22}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D, \tag{23}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad \text{on} \quad \Gamma_N, \tag{24}$$

$$-\sigma_\nu = p(u_\nu - g_0) \quad \text{on} \quad \Gamma_C, \tag{25}$$

$$-\boldsymbol{\sigma}_\tau \in h_\tau(u_\nu - g_0)\partial\psi_\tau(\boldsymbol{u}_\tau) \quad \text{on} \quad \Gamma_C. \tag{26}$$

We now provide a short description of the equations and boundary conditions in Problem 4.1. First, equation (21) represents the constitutive law of a locking material in which $\mathcal{F}$ is the elasticity operator, assumed to be nonlinear, I_{K_0} is the indicator function of the set $K_0 \subset \mathbb{S}^d$ and $\partial^c I_{K_0}$ represents its subdifferential in the sense of convex analysis, see (4). The study of locking materials started with the pioneering works of Prager [24, 25, 26]. There, the constitutive law of such materials was introduced and various mechanical interpretations have been provided. References in the field include [4, 22, 23], for instance. Examples of operators $\mathcal{F}$ which satisfy the condition (30) below can be found in [17, 29]. For the set K_0, which describes the locking constraints of the material, various examples can be found in the literature, as explained in [4]. A standard example is

$$K_0 = \left\{\boldsymbol{\tau} \in \mathbb{S}^d \mid \mathcal{F}(\boldsymbol{\tau}) \le k\right\}, \tag{27}$$

where $\mathcal{F} : \mathbb{S}^d \to \mathbb{R}$ is a convex continuous function such that $\mathcal{F}(\boldsymbol{0}) = 0$ and k is a positive constant. It is easy to see that in this case the set K_0 is a nonempty convex closed subset of $\mathbb{S}^d$. Using (27) with the choice

$$\mathcal{F}(\boldsymbol{\tau}) = \operatorname{tr}(\boldsymbol{\tau}), \tag{28}$$

where $\operatorname{tr}(\boldsymbol{\tau})$ denotes the trace of the tensor $\boldsymbol{\tau} \in \mathbb{S}^d$, leads to the class of materials with limited compressibility ([25]). The choice

$$\mathcal{F}(\boldsymbol{\tau}) = \frac{1}{2}\,\|\boldsymbol{\tau}^D\|, \tag{29}$$

where $\boldsymbol{\tau}^D := \boldsymbol{\tau} - (\operatorname{tr}(\boldsymbol{\tau})/d)\,\boldsymbol{I}$ denotes the deviator of the tensor $\boldsymbol{\tau} \in \mathbb{S}^d$, leads to the von Mises convex. This convex set was considered in [24, 25] to model the ideal-locking effect.

Equation (22) is the equation of equilibrium in which $\boldsymbol{f}_0$ represents a given density of the external body force; we use this equation since the process is assumed to be static and, therefore, we neglect the inertial term in the equation of motion. Conditions (23), (24) represent the displacement and the traction boundary conditions, respectively. Condition (23) reflects the fact that the body is clamped on Γ_D and condition (24) describes the force boundary condition on Γ_N, $\boldsymbol{f}_N$ being a given density of the surface traction.

On the contact boundary Γ_C, along the normal direction, the contact condition is (25), p_ν being a given normal compliance function and g_0 being the initial gap between the body and a foundation. Such normal compliance contact conditions have been introduced in [20] and then used in a large number of papers, including [12, 13, 16]. Along the tangential direction, the friction law is (26), in which h_τ and ψ_τ are given functions and, as usual $\partial\psi_\tau$ represents the generalized gradient (or the Clarke subdifferential) of the function ψ_τ. The conditions (25) and (26) are of general forms, and they contain many particular contact conditions and friction laws as special cases, as explained in [17, Section 6.3]. Note that the functions p_ν and h_τ in these conditions are supposed to vanish for a negative argument. This restriction is imposed from physical reasons, since it reflects the fact that when there is separation between the body and the foundation then the reaction of the foundation vanishes. Moreover, the function h_τ can be interpreted as the coefficient of friction.

We now introduce assumptions on the problem data. For the elasticity operator $\mathcal{F}\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d$, assume

$$\left\{\begin{array}{l}\text{(a) there exists a constant } L_{\mathcal{F}}>0 \text{ such that}\\ \qquad \|\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\le L_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\ \qquad \text{for all } \ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega;\\ \text{(b) there exists a constant } m_{\mathcal{F}}>0 \text{ such that}\\ \qquad (\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)):(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\ge m_{\mathcal{F}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\ \qquad \text{for all } \ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega;\\ \text{(c) } \mathcal{F}(\cdot,\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d;\\ \text{(d) } \mathcal{F}(\boldsymbol{x},\boldsymbol{0})=\boldsymbol{0} \ \text{ for a.e. } \boldsymbol{x}\in\Omega.\end{array}\right. \tag{30}$$

For the normal compliance function $p_\nu\colon \Gamma_C\times\mathbb{R}\to\mathbb{R}$, assume

$$\left\{\begin{array}{l}\text{(a) there exists a constant } L_{p_\nu}\ge 0 \text{ such that}\\ \qquad |p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2)|\le L_{p_\nu}|r_1-r_2|\\ \qquad\quad \text{for all } r_1,\ r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_C;\\ \text{(b) } p_\nu(\cdot,r) \text{ is measurable on } \Gamma_C \text{ for all } r\in\mathbb{R};\\ \text{(c) } p_\nu(\boldsymbol{x},r)=0 \ \text{ for a.e. } \boldsymbol{x}\in\Gamma_C,\ \text{all } r\le 0.\end{array}\right. \tag{31}$$

For the tangential potential function $\psi_\tau\colon \Gamma_C\times\mathbb{R}^d\to\mathbb{R}$, assume

$$\left\{\begin{array}{l}\text{(a) } \psi_\tau(\cdot,\boldsymbol{\xi}) \text{ is measurable on } \Gamma_C \text{ for all } \boldsymbol{\xi}\in\mathbb{R}^d \text{ and there}\\ \qquad \text{exists } \bar{\boldsymbol{e}}_\tau\in L^2(\Gamma_C)^d \text{ such that } \psi_\tau(\cdot,\bar{\boldsymbol{e}}_\tau(\cdot))\in L^1(\Gamma_C);\\ \text{(b) } \psi_\tau(\boldsymbol{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R}^d \text{ for a.e. } \boldsymbol{x}\in\Gamma_C;\\ \text{(c) } |\partial\psi_\tau(\boldsymbol{x},\boldsymbol{\xi})|\le\bar{c}_{0\tau} \text{ for a.e. } \boldsymbol{x}\in\Gamma_C,\ \text{for } \boldsymbol{\xi}\in\mathbb{R}^d \text{ with } \bar{c}_{0\tau}\ge 0;\\ \text{(d) } \psi_\tau^0(\boldsymbol{x},\boldsymbol{\xi}_1;\boldsymbol{\xi}_2-\boldsymbol{\xi}_1)+\psi_\tau^0(\boldsymbol{x},\boldsymbol{\xi}_2;\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\le\alpha_{\psi_\tau}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|^2\\ \qquad \text{for a.e. } \boldsymbol{x}\in\Gamma_C,\ \text{all } \boldsymbol{\xi}_1,\ \boldsymbol{\xi}_2\in\mathbb{R}^d \text{ with } \alpha_{\psi_\tau}\ge 0.\end{array}\right. \tag{32}$$

For the friction coefficient function $h_\tau\colon \Gamma_C \times \mathbb{R} \to \mathbb{R}$, assume

$$\left\{\begin{array}{l}\text{(a) there exists a constant } c_{h_\tau} \ge 0 \text{ such that} \\ \qquad |h_\tau(\boldsymbol{x}, r_1) - h_\tau(\boldsymbol{x}, r_2)| \le c_{h_\tau}\, |r_1 - r_2| \\ \qquad\quad \text{for all } r_1,\ r_2 \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Gamma_C; \\ \text{(b) } h_\tau(\cdot, r) \text{ is measurable on } \Gamma_C \text{ for all } r \in \mathbb{R}; \\ \text{(c) } 0 \le h_\tau(\boldsymbol{x}, r) \le \overline{h}_\tau \text{ for all } r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_C \text{ with } \overline{h}_\tau \ge 0; \\ \text{(d) } h_\tau(\boldsymbol{x}, r) = 0 \ \text{ for a.e. } \boldsymbol{x} \in \Gamma_C,\ \text{ all } r \le 0. \end{array}\right. \tag{33}$$

We note that (32) (b) and (c) are equivalent to the property that $\psi_\tau(\boldsymbol{x}, \cdot)$ is Lipschitz continuous on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Gamma_C$ with a Lipschitz constant $\bar{c}_{0\tau}$. Finally, we assume that the set of locking constraints, the density of body forces and surface traction and the gap function are such that

$$K_0 \ \text{ is a closed convex subset of } \mathbb{S}^d \text{ and } \mathbf{0} \in K_0, \tag{34}$$

$$\boldsymbol{f}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \boldsymbol{f}_N \in L^2(\Gamma_N; \mathbb{R}^d), \tag{35}$$

$$g_0 \in L^2(\Gamma_C), \quad g_0(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_C. \tag{36}$$

In the variational analysis of Problem 4.1 we use the following function spaces

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_D \right\}, \tag{37}$$

which is a Hilbert space with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx.$$

Moreover, we consider the set $K_V \subset V$, the operator $A\colon V \to V^*$, the function $\Phi : V \times V \to \mathbb{R}$ and the element $f \in V^*$ defined as follows.

$$K_V = \{ \boldsymbol{v} \in V \mid \boldsymbol{\varepsilon}(\boldsymbol{v}) \in K_0 \text{ a.e. in } \Omega \}, \tag{38}$$

$$\langle A\boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{39}$$

$$\Phi(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} p(u_\nu - g_0) v_\nu\, da \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{40}$$

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_N} \boldsymbol{f}_N \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V. \tag{41}$$

Assume $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (21)–(26) and let $\boldsymbol{v} \in K_V$. Then, using standard arguments based on integration by parts and the definitions (3), (4) of the generalized gradient and convex subdifferential, respectively, it follows that $\boldsymbol{u} \in K_V$ and, moreover,

$$\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}))\, dx + \int_{\Gamma_C} \left[p_\nu(u_\nu - g_0)(v_\nu - u_\nu) + h_\tau(u_\nu - g_0)\, \psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau - \boldsymbol{u}_\tau) \right] da$$
$$\ge \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_N} \boldsymbol{f}_N \cdot (\boldsymbol{v} - \boldsymbol{u})\, da.$$

Therefore, using the notation (38)–(41), we deduce the following variational formulation of the contact problem (21)–(26).

Problem 4.2. *Find a displacement field* $\boldsymbol{u} \in K_V$ *such that*

$$\langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle + \Phi(\boldsymbol{u}, \boldsymbol{v}) - \Phi(\boldsymbol{u}, \boldsymbol{u}) + \int_{\Gamma_C} h_\tau(u_\nu - g_0)\, \psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, da$$
$$\geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle \quad \forall\, \boldsymbol{v} \in K_V. \tag{42}$$

Note that the solution of Problem 4.2 depends on $\boldsymbol{f}$ which in turns, depends on the data $\boldsymbol{f}_0$ and $\boldsymbol{f}_N$, as it follows from definition (41). For this reason, in Section 5 below we sometimes use the notation $\boldsymbol{u}(\boldsymbol{f})$ or $\boldsymbol{u}(\boldsymbol{f}_0; \boldsymbol{f}_N)$.

We end this section by noting the inequalities

$$\|v_\nu\|^2_{L^2(\Gamma_C)} \leq \lambda_\nu^{-1} \|\boldsymbol{v}\|_V^2 \quad \forall\, \boldsymbol{v} \in V,$$
$$\|\boldsymbol{v}_\tau\|^2_{L^2(\Gamma_C;\mathbb{R}^d)} \leq \lambda_\tau^{-1} \|\boldsymbol{v}\|_V^2 \quad \forall\, \boldsymbol{v} \in V.$$

Here $\lambda_\nu > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_C} u_\nu v_\nu da \quad \forall\, \boldsymbol{v} \in V,$$

whereas $\lambda_\tau > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_C} \boldsymbol{u}_\tau \cdot \boldsymbol{v}_\tau da \quad \forall\, \boldsymbol{v} \in V.$$

These inequalities will be used in the variational analysis of Problem 4.2 we provide in the next section.

5. Analysis and control of the contact model

Our first result in this section is the following existence and uniqueness result.

Theorem 5.1. *Assume* (30)–(36) *and*

$$L_{p_\nu} \lambda_\nu^{-1} + \overline{h}_\tau m_\tau \lambda_\tau^{-1} + c_{h_\tau} c_{\psi_\tau} (\lambda_\nu \lambda_\tau)^{-1/2} < m_{\mathcal{F}}. \tag{43}$$

Then, Problem 4.2 *has a unique solution* $\boldsymbol{u} \in K_V$. *Moreover, the operator*

$$(\boldsymbol{f}_0, \boldsymbol{f}_N) \mapsto \boldsymbol{u} = \boldsymbol{u}(\boldsymbol{f}_0, \boldsymbol{f}_N)$$

which maps any pair $(\boldsymbol{f}_0, \boldsymbol{f}_N) \in L^2(\Omega, \mathbb{R}^d) \times L^2(\Gamma_N, \mathbb{R}^d)$ *to the solution* $\boldsymbol{u} \in K_V$ *of Problem* 4.2 *is Lipschitz continuous.*

Proof. We start by considering the function $\Psi : V \times V \to \mathbb{R}$ defined by

$$\Psi(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} h_\tau(u_\nu - g_0)\, \psi_\tau(\boldsymbol{v}_\tau)\, da, \quad \boldsymbol{u}, \boldsymbol{v} \in V. \tag{44}$$

Then, using [17, Theorem 3.47], we find that the following inequality holds:

$$\Psi^0(\boldsymbol{w}, \boldsymbol{u}; \boldsymbol{v}) \leq \int_{\Gamma_C} h_\tau(w_\nu - g_0)\, \psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau) da \quad \forall\, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{45}$$

Next, we consider the auxiliary problem of finding an element $\boldsymbol{u} \in K_V$ such that

$$\langle A\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}\rangle + \Phi(\boldsymbol{u},\boldsymbol{v}) - \Phi(\boldsymbol{u},\boldsymbol{u}) + \Psi^0(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v}-\boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle \quad \forall\, \boldsymbol{v} \in K_V \quad (46)$$

to which we shall apply Theorem 3.1 with the space V defined in (37). To this end we check the validity of the conditions of this theorem. First, it is easy to see that $H(K_V)$, $H(A)$ and $H(f)$ hold, and moreover, the strong monotonicity constant of the operator A is $m_A = m_{\mathcal{F}}$.

Let us examine the condition $H(\Phi)_2$. It is easy to see that $\Phi(\boldsymbol{u},\cdot) : V \to \mathbb{R}$ is a convex continuous function and, therefore, it is bounded above on a non-empty open set. Moreover, for $\boldsymbol{u}_1, \boldsymbol{u}_2,\ \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, we have

$$p_\nu(u_{1,\nu}-g_0)(v_{2,\nu}-v_{1,\nu}) + p_\nu(u_{2,\nu}-g_0)(v_{1,\nu}-v_{2,\nu}) \leq L_{p_\nu}|u_{1,\nu}-u_{2,\nu}||v_{1,\nu}-v_{2,\nu}|$$

a.e. on Γ_C, and then

$$\begin{aligned}&\int_{\Gamma_C} [p_\nu(u_{1,\nu}-g_0)(v_{2,\nu}-v_{1,\nu}) + p_\nu(u_{2,\nu}-g_0)(v_{1,\nu}-v_{2,\nu})]\, da\\ &\quad \leq L_{p_\nu}\|u_{1,\nu}-u_{2,\nu}\|_{L^2(\Gamma_C)}\|v_{1,\nu}-v_{2,\nu}\|_{L^2(\Gamma_C)} \leq L_{p_\nu}\lambda_\nu^{-1}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_V.\end{aligned}$$

It follows from here that the function Φ satisfies condition $H(\Phi)_2$ with

$$\alpha_\Phi = L_{p_\nu}\lambda_\nu^{-1}. \quad (47)$$

Let us examine the assumption $H(\Psi)_2$. It is easy to see that Ψ is locally Lipschitz continuous with respect to its second argument. Now we check the condition (9). For $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{u}_1, \boldsymbol{u}_2 \in V$, we write

$$\begin{aligned}&h_\tau(w_{1,\nu}-g_0)\,\psi_\tau^0(\boldsymbol{u}_{1,\tau};\boldsymbol{u}_{2,\tau}-\boldsymbol{u}_{1,\tau}) + h_\tau(w_{2,\nu}-g_0)\,\psi_\tau^0(\boldsymbol{u}_{2,\tau};\boldsymbol{u}_{1,\tau}-\boldsymbol{u}_{2,\tau})\\ &\quad = h_\tau(w_{1,\nu}-g_0)\left[\psi_\tau^0(\boldsymbol{u}_{1,\tau};\boldsymbol{u}_{2,\tau}-\boldsymbol{u}_{1,\tau}) + \psi_\tau^0(\boldsymbol{u}_{2,\tau};\boldsymbol{u}_{1,\tau}-\boldsymbol{u}_{2,\tau})\right]\\ &\qquad + [h_\tau(w_{2,\nu}-g_0) - h_\tau(w_{1,\nu}-g_0)]\,\psi_\tau^0(\boldsymbol{u}_{2,\tau};\boldsymbol{u}_{1,\tau}-\boldsymbol{u}_{2,\tau})\end{aligned}$$

a.e. on Γ_C. Then, by assumption (33) we find that

$$0 \leq h_\tau(\boldsymbol{x},r) \leq \overline{h}_\tau, |h_\tau(w_{1,\nu}-g_0) - h_\tau(w_{2,\nu}-g_0)| \leq c_{h_\tau}\,|w_{1,\nu}-w_{2,\nu}|$$

a.e. on Γ_C. Also, by the Lipschitz continuity of ψ_τ,

$$\left|\psi_\tau^0(\boldsymbol{u}_{2,\tau};\boldsymbol{u}_{1,\tau}-\boldsymbol{u}_{2,\tau})\right| \leq c_{\psi_\tau}\,|\boldsymbol{u}_{1,\tau}-\boldsymbol{u}_{2,\tau}|$$

a.e. on Γ_C. Thus,

$$\begin{aligned}&\int_{\Gamma_C}\left[h_\tau(w_{1,\nu}-g_0)\,\psi_\tau^0(\boldsymbol{u}_{1,\tau};\boldsymbol{u}_{2,\tau}-\boldsymbol{u}_{1,\tau}) + h_\tau(w_{2,\nu}-g_0)\,\psi_\tau^0(\boldsymbol{u}_{2,\tau};\boldsymbol{u}_{1,\tau}-\boldsymbol{u}_{2,\tau})\right] da\\ &\quad \leq \int_{\Gamma_C}\left[\overline{h}_\tau m_\tau\,|\boldsymbol{u}_{1,\tau}-\boldsymbol{u}_{2,\tau}|^2 + c_{h_\tau}c_{\psi_\tau}\,|w_{1,\nu}-w_{2,\nu}|\,|\boldsymbol{u}_{1,\tau}-\boldsymbol{u}_{2,\tau}|\right] da\\ &\quad \leq \overline{h}_\tau m_\tau\lambda_\tau^{-1}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2 + c_{h_\tau}c_{\psi_\tau}(\lambda_\nu\lambda_\tau)^{-1/2}\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_V\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V. \quad (48)\end{aligned}$$

Hence,

$$\begin{aligned}&\Psi^0(\boldsymbol{w}_1,\boldsymbol{u}_1;\boldsymbol{u}_2-\boldsymbol{u}_1)+\Psi^0(\boldsymbol{w}_2,\boldsymbol{u}_2;\boldsymbol{u}_1-\boldsymbol{u}_2)\\&\le \overline{h}_\tau m_\tau\lambda_\tau^{-1}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2+c_{h_\tau}c_{\psi_\tau}(\lambda_\nu\lambda_\tau)^{-1/2}\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_V\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V.\end{aligned}$$

Therefore, (9) holds with

$$\alpha_{\Psi,1}=\overline{h}_\tau m_\tau\lambda_\tau^{-1},\quad \alpha_{\Psi,2}=c_{h_\tau}c_{\psi_\tau}(\lambda_\nu\lambda_\tau)^{-1/2}. \tag{49}$$

We now combine (47), (49) and (43) to see that the smallness condition (10) is satisfied.

In conclusion, we are in a position to use Theorem 3.1. In this way we deduce that there exists a unique element $\boldsymbol{u}\in K_V$ such that (46) holds. Then, combining (46) and (45) we deduce that $\boldsymbol{u}$ is a solution of Problem 4.2. This proves the existence part in Theorem 5.1.

We now prove the uniqueness part. To this end, let $\boldsymbol{u}_1,\boldsymbol{u}_2\in V$ be solutions to Problem 4.2. Then, $\boldsymbol{u}_1,\boldsymbol{u}_2\in K_V$ satisfy the inequalities

$$\begin{aligned}&\langle A\boldsymbol{u}_1,\boldsymbol{v}-\boldsymbol{u}_1\rangle+\Phi(\boldsymbol{u}_1,\boldsymbol{v})-\Phi(\boldsymbol{u}_1,\boldsymbol{u}_1)+\int_{\Gamma_C}h_\tau(u_{1,\nu}-g_0)\,\psi_\tau^0(\boldsymbol{u}_{1,\tau};\boldsymbol{v}_\tau-\boldsymbol{u}_{1,\tau})\,da\\&\qquad\ge\langle\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}_1\rangle\quad\forall\,\boldsymbol{v}\in K_V,\end{aligned}\tag{50}$$

$$\begin{aligned}&\langle A\boldsymbol{u}_2,\boldsymbol{v}-\boldsymbol{u}_2\rangle+\Phi(\boldsymbol{u}_2,\boldsymbol{v})-\Phi(\boldsymbol{u}_2,\boldsymbol{u}_2)+\int_{\Gamma_C}h_\tau(u_{2,\nu}-g_0)\,\psi_\tau^0(\boldsymbol{u}_{2,\tau};\boldsymbol{v}_\tau-\boldsymbol{u}_{2,\tau})\,da\\&\qquad\ge\langle\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}_2\rangle\quad\forall\,\boldsymbol{v}\in K_V.\end{aligned}\tag{51}$$

We take $\boldsymbol{v}=\boldsymbol{u}_2$ in (50) and $\boldsymbol{v}=\boldsymbol{u}_1$ in (51), then add the resulting inequalities and use conditions $H(A)$, $H(\Phi)_2$ to find that

$$\begin{aligned}&m_{\mathcal{F}}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2\\&\le\int_{\Gamma_C}\big[h_\tau(u_{1,\nu}-g_0)\,\psi_\tau^0(\boldsymbol{u}_{1,\tau};\boldsymbol{u}_{2,\tau}-\boldsymbol{u}_{1,\tau})+h_\tau(u_{2,\nu}-g_0)\,\psi_\tau^0(\boldsymbol{u}_{2,\tau};\boldsymbol{u}_{1,\tau}-\boldsymbol{u}_{2,\tau})\big]\,da\\&\qquad+L_{p_\nu}\lambda_\nu^{-1}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2.\end{aligned}\tag{52}$$

On the other hand, arguments similar to those used in the proof of (48) yield

$$\begin{aligned}&\int_{\Gamma_C}\big[h_\tau(u_{1,\nu}-g_0)\,\psi_\tau^0(\boldsymbol{u}_{1,\tau};\boldsymbol{u}_{2,\tau}-\boldsymbol{u}_{1,\tau})+h_\tau(u_{2,\nu}-g_0)\,\psi_\tau^0(\boldsymbol{u}_{2,\tau};\boldsymbol{u}_{1,\tau}-\boldsymbol{u}_{2,\tau})\big]\,da\\&\qquad\le(\overline{h}_\tau m_\tau\lambda_\tau^{-1}+c_{h_\tau}c_{\psi_\tau}(\lambda_\nu\lambda_\tau)^{-1/2})\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2.\end{aligned}\tag{53}$$

We now combine inequalities (52) and (53) to deduce that

$$\big(m_{\mathcal{F}}-L_{p_\nu}\lambda_\nu^{-1}-\overline{h}_\tau m_\tau\lambda_\tau^{-1}-c_{h_\tau}c_{\psi_\tau}(\lambda_\nu\lambda_\tau)^{-1/2}\big)\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V^2\le 0.$$

By the smallness assumption (43), we derive from the above inequality that $\boldsymbol{u}_1=\boldsymbol{u}_2$. This proves the uniqueness part in Theorem 5.1.

Finally, we recall that Theorem 3.1 guarantees that the operator $\boldsymbol{f} \mapsto \boldsymbol{u}(\boldsymbol{f})$ which maps any element $\boldsymbol{f} \in V^*$ to the solution $\boldsymbol{u} \in K_V$ of the quasivariational-hemivariational inequality (46) is Lipshitz continuous. Therefore, since the operator

$$(\boldsymbol{f}_0, \boldsymbol{f}_N) \mapsto \boldsymbol{f} : L^2(\Omega;\mathbb{R}^d) \times L^2(\Gamma_N;\mathbb{R}^d) \to V^*$$

defined by (41) is linear and continuous, we deduce that the operator

$$(\boldsymbol{f}_0, \boldsymbol{f}_N) \mapsto \boldsymbol{u}(\boldsymbol{f}_0, \boldsymbol{f}_N) : L^2(\Omega;\mathbb{R}^d) \times L^2(\Gamma_N;\mathbb{R}^d) \to V^*$$

is a Lipschitz continuous operator. This concludes the proof of Theorem 5.1. □

We now follow the statement of Theorem 5.1 with some comments.

Remark 5.2. (a) The proof of Theorem 5.1 shows that under assumptions (30)–(36) and (43), the quasivariational-hemivariational inequality (42) has a unique solution which, in the meantime, is the unique solution of inequality (46). We conclude from here that inequalities (42) and (46) are equivalent.

(b) In addition to the mathematical interest in the Lipschitz continuous dependence result in Theorem 5.1, it is important from mechanical point of view, since it shows that small perturbation on the density of body forces and surface tractions give rise to small perturbation on the weak solution of the contact problem (21)–(26). □

We then consider an iterative procedure to approximate Problem 4.2. Define a sequence $\{\boldsymbol{u}_n\}_{n\geq 0} \subset K_V$ by choosing an initial guess $\boldsymbol{u}_0 \in K_V$, and for $n \geq 1$, by finding $\boldsymbol{u}_n \in K_V$ such that

$$\langle A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n\rangle + \Phi(\boldsymbol{u}_n, \boldsymbol{v}) - \Phi(\boldsymbol{u}_n, \boldsymbol{u}_n) + \int_{\Gamma_C} h_\tau(u_{n-1,\nu} - g_0)\, \psi_\tau^0(\boldsymbol{u}_{n,\tau}; \boldsymbol{v}_\tau - \boldsymbol{u}_{n,\tau})\, da \\ \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_n\rangle \quad \forall\, \boldsymbol{v} \in K_V.$$

By Theorem 3.2, under the assumptions stated in Theorem 5.1, we have the convergence

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in } V.$$

We now turn to the optimal control of Problem 4.2. Several situations can be considered but, for simplicity, we restrict here to use Theorem 3.4 in the particular case when the control is the density of applied tractions $\boldsymbol{f}_N$, $W = L^2(\Gamma_N, \mathbb{R}^d)$, K_W is a bounded set and the cost function J does not depend on the control function. To this end assume in what follows that (30)–(34), (36), (43) hold and, moreover,

$$\boldsymbol{f}_0 \in L^2(\Omega;\mathbb{R}^d). \tag{54}$$

We also consider the operator $B\colon W \to V^*$ defined by

$$\langle B\boldsymbol{f}_N, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_N} \boldsymbol{f}_N \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V,\ \boldsymbol{f}_N \in L^2(\Gamma_N;\mathbb{R}^d). \tag{55}$$

Let $\widetilde{K}_W \subset L^2(\Gamma_N, \mathbb{R}^d)$. Then, Theorem 3.4 guarantees that for each $\boldsymbol{f}_N \in \widetilde{K}_W$, there exists a unique solution $\boldsymbol{u} = \boldsymbol{u}(B\boldsymbol{f}_N)$ to Problem 4.2. Next, given $\widetilde{J} : K_V \to \mathbb{R}$, the control problem we consider below is as follows.

Problem 5.3. *Find* $\overline{\boldsymbol{f}}_N \in \widetilde{K}_W$ *such that*

$$\widetilde{J}(\boldsymbol{u}(B\overline{\boldsymbol{f}}_N)) = \min_{\boldsymbol{f}_2 \in \widetilde{K}_W} \widetilde{J}(\boldsymbol{u}(B\boldsymbol{f}_N)). \tag{56}$$

In the study of this problem we consider the following assumptions.

$$\widetilde{K}_W \text{ is a nonempty weakly closed bounded set in } L^2(\Gamma_N;\mathbb{R}^d). \tag{57}$$

$$\widetilde{J} : K_V \to \mathbb{R} \text{ is a continuous function.} \tag{58}$$

Our result in the study of Problem 5.3 is the following.

Theorem 5.4. *Assume* (30)–(34), (36), (43), (54), (57) *and* (58). *Then, there exists at least one solution* $\overline{\boldsymbol{f}}_N \in \widetilde{K}_W$ *of Problem* 5.3.

Theorem 5.4 is a direct consequence of Theorem 3.4 as well as on Remark 5.2 (a) which states that inequalities (42) and (46) are equivalent. Its proof is based on the compactness of the operator B defined by (55). Since the details in proof are obvious, we skip them. Nevertheless, we provide two relevant examples of sets $\widetilde{K}_W$ and function $\widetilde{J}$ together with the mechanical interpretations of the corresponding optimal control problems.

Example 5.5. A first example of can be obtained by taking

$$\widetilde{K}_W = \left\{\boldsymbol{f} \in L^2(\Gamma_N,\mathbb{R}^d) \mid \|\boldsymbol{f}_N\|_{L^2(\Gamma_N,\mathbb{R}^d)} \le M\right\},$$
$$J(\boldsymbol{u}) = \int_{\Gamma_3} (u_\nu - \phi)^2\, da$$

where $M > 0$ and $\phi \in L^2(\Gamma_3)$ are given. With this choice, the mechanical interpretation of Problem 5.3 is the following: given a contact process of the form (21)–(26) with the data $\mathcal{F}$, K_0, $\boldsymbol{f}_0$, p_ν, h_τ, ψ_τ, g_0 and given $M > 0$, we look for a density of tractions $\boldsymbol{f}_N$ which satisfies the inequality $\|\boldsymbol{f}_N\|_{L^2(\Gamma_N,\mathbb{R}^d)} \le M$ such that the normal component of the corresponding solution is as close as possible to the "desired normal displacement" ϕ on Γ_3. □

Example 5.6. A second example of Problem 5.3 can be obtained by taking $\widetilde{K}_W$ as above and

$$\widetilde{J}(\boldsymbol{u}) = \int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2\, dx.$$

With this choice, the mechanical interpretation of Problem 5.3 is the following: given a contact process of the form (21)–(26), with the data $\mathcal{F}$, K_0, $\boldsymbol{f}_0$, p_ν, h_τ, ψ_τ, g_0 and given $M > 0$, we look for a density of tractions $\boldsymbol{f}_N$ which satisfies the inequality $\|\boldsymbol{f}_N\|_{L^2(\Gamma_N,\mathbb{R}^d)} \le M$, such that the corresponding deformation in the body is as small as possible. □

Theorem 5.4 guarantees the existence of the solutions of all these optimal control problems.

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