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NUMERICAL ANALYSIS OF A STEADY OSEEN FLOW PROBLEM WITH FRICTIONAL TYPE BOUNDARY CONDITIONS

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Abstract. In this paper, a hemivariational inequality is studied for the steady Oseen fluid flow problem in the presence of a nonsmooth slip boundary condition of friction type. Well-posedness of the Oseen hemivariational inequality is discussed. Mixed finite element methods are introduced to solve the Oseen hemivariational inequality and error estimates are derived for the mixed finite element solutions. The error estimates are of optimal order for low-order mixed element pairs under suitable solution regularity assumptions. Numerical results are reported illustrating the theoretical prediction of convergence orders.

1. Introduction. The notion of hemivariational inequalities was first introduced by Panagiotopoulos in early 1980s ([36]). In contrast to variational inequalities in which the nonsmooth terms have a convex structure, the nonsmooth terms in hemivariational inequalities are not assumed to be convex. Early comprehensive references on the mathematical theory and engineering applications of hemivariational inequalities are [34, 37]. In the last twenty years, there have been steadily more research activities on modeling, mathematical analysis, numerical computations and applications of hemivariational in-

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equalities. This is reflected by the large number of recent and new publications in the area, including comprehensive references such as [4, 5, 33, 39] on mathematical theories of hemivariational inequalities. For applications, numerical methods are needed to solve hemivariational inequalities. One is referred to [22] for an early comprehensive reference on the finite element method and solution algorithms for solving hemivariational inequalities. An optimal order error estimate for the numerical solution of hemivariational inequalities was first presented in the paper [19]. Afterwards, a number of papers have been published on the numerical solution and error analysis of numerical solutions of hemivariational inequalities. The reader is referred to the survey paper [21] for a summary account of numerical analysis of hemivariational inequalities.

In early 1990s, Fujita ([11, 12]) started the research on viscous incompressible fluid flows, modeled by the Stokes or Navier-Stokes equations and subject to nonsmooth monotone slip or leak boundary conditions of friction type. The weak formulations of the problems are variational inequalities. Numerous publications can be found on the theory and numerical solution of variational inequalities arising in fluid mechanics, e.g., [8, 23, 25, 26, 27, 28, 29, 38, 41]. When the slip or leak boundary conditions involve nonmonotone relations, the weak formulations of the corresponding problems become hemivariational inequalities, cf. [9, 10, 20, 30, 32]. A new class of variational-hemivariational inequalities is studied in [31], where the existence of weak solutions is proved by applying a surjectivity result for an operator inclusion problem involving Clarke's subdifferential. Then the result is applied to show the solution existence of the steady Oseen model for a generalized Newtonian incompressible fluid flow problem subject to a variety of different boundary conditions. In this paper, we study an Oseen hemivariational inequality, present an existence and uniqueness result, introduce and analyze mixed finite element methods for solving the Oseen hemivariational inequality, and report numerical results.

We note that traditionally, the existence of a solution to a hemivariational inequality is proved through an application of an abstract surjectivity result for pseudomonotone operators. An alternative and more accessible approach is developed in [13, 14] to prove the existence of a solution to hemivariational inequalities. This approach starts with minimization principles for special hemivariational inequalities, and extends the well-posedness analysis to general hemivariational inequalities via fixed-point arguments. The idea is further extended in [17, 18] for well-posedness analysis of mixed hemivariational-variational inequalities. In [16], well-posedness analysis is provided on a new class of mixed hemivariational-variational inequalities well suited for the study of pressure stabilized mixed finite element methods for solving a Navier–Stokes hemivariational inequality. In this paper, we analyze the Oseen hemivariational inequality and consider a mixed finite element method for its solution. The solution existence and uniqueness for the Oseen hemivariational inequality and its finite element approximation are shown through an application of a general result proved in [16].

The rest of the paper is organized as follows. In Section 2, we recall basic notions and results needed in later sections. In Section 3, we describe the physical setting of the fluid flow, introduce the Oseen hemivariational inequality and provide a well-posedness result on the problem. In Section 4, we apply the mixed finite element method to solve the Oseen hemivariational inequality, and derive error estimates for the finite element solutions. The error estimates are of optimal order for low-order mixed element pairs under suitable solution regularity assumptions. In Section 5, we report numerical simulation results on the mixed finite element method for solving the Oseen hemivariational inequality.

2. Preliminaries. Studies of hemivariational inequalities rely on the notions of the generalized directional derivative and the generalized subdifferential in the sense of Clarke ([7]) that we recall next.

DEFINITION 2.1. Assume $\Psi\colon V \to \mathbb{R}$ is a locally Lipschitz continuous function defined on a real Banach space V. The generalized (Clarke) directional derivative of Ψ at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u;v) := \limsup_{w\to u,\, \lambda\downarrow 0} \frac{\Psi(w+\lambda v)-\Psi(w)}{\lambda},$$

and the generalized subdifferential of Ψ at $u \in V$ is defined by

$$\partial\Psi(u) := \left\{\eta \in V^* \mid \Psi^0(u;v) \geq \langle \eta, v\rangle \;\forall\, v \in V\right\}.$$

We summarize some basic properties of the generalized directional derivative and the generalized subdifferential in the next result. Detailed discussions can be found in [7].

PROPOSITION 2.2. *Let $\Psi\colon V \to \mathbb{R}$ be locally Lipschitz continuous.*

(i) *If additionally, $\Psi\colon V \to \mathbb{R}$ is convex, then the generalized subdifferential $\partial\Psi(u)$ at any $u \in V$ in the sense of Clarke coincides with the convex subdifferential $\partial\Psi(u)$.*

(ii) *For all $\lambda \in \mathbb{R}$ and all $u \in V$, $\partial(\lambda\,\Psi)(u) = \lambda\,\partial\Psi(u)$.*

(iii) Ψ^0 *is positively homogeneous and subadditive, i.e.,*

$$\Psi^0(u;\lambda\, v) = \lambda\,\Psi^0(u;v) \quad \forall\,\lambda \geq 0,\; u, v \in V,$$
$$\Psi^0(u;v_1+v_2) \leq \Psi^0(u;v_1) + \Psi^0(u;v_2) \quad \forall\, u, v_1, v_2 \in V.$$

(iv) *For locally Lipschitz continuous functions $\Psi_1, \Psi_2\colon V \to \mathbb{R}$, the inclusion*

$$\partial(\Psi_1+\Psi_2)(u) \subseteq \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall\, u \in V \tag{1}$$

holds, or equivalently,

$$(\Psi_1+\Psi_2)^0(u;v) \leq \Psi_1^0(u;v) + \Psi_2^0(u;v) \quad \forall\, u, v \in V. \tag{2}$$

Because of Proposition 2.2 (i), we use the same symbol ∂ to mean both the convex subdifferential and the Clarke subbdifferential.

Let V and Q be two real Hilbert spaces. Their dual spaces are denoted by V^* and Q^*. The symbol $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V^* and V, or between Q^* and Q; it should be clear from the context which duality pairing is meant by $\langle\cdot,\cdot\rangle$. Let there be given $a\colon V\times V \to \mathbb{R}$, $b\colon V \times Q \to \mathbb{R}$, and $f \in V^*$. Denote by Δ the spatial domain of the problem under consideration, or a sub-domain, or the boundary or part of the boundary of the domain. Let ψ be a real-valued function defined on $\Delta \times \mathbb{R}^m$ for some positive integer m, and let γ_ψ be a linear continuous operator from V to $L^2(\Delta;\mathbb{R}^m)$. For applications in mechanics, either the operator γ_ψ is the normal component trace operator and $m = 1$, or γ_ψ is the tangential component trace operator and m equals the dimension of the spatial domain.

Consider an abstract mixed hemivariational inequality.

PROBLEM 2.3. Find $(u,p) \in V \times Q$ such that

$$a(u,v) - b(v,p) + I_\Delta(\psi^0(\gamma_\psi u; \gamma_\psi v)) \geq \langle f, v\rangle \quad \forall\, v \in V, \tag{3}$$

$$b(u,q) = 0 \quad \forall\, q \in Q. \tag{4}$$

We assume the following properties on the problem data.

<u>$H(a)$</u> $a\colon V \times V \to \mathbb{R}$ is bilinear, bounded and V-elliptic.

We will use $M_a > 0$ for the boundedness constant and $m_a > 0$ for the V-ellipticity constant:

$$|a(u,v)| \leq M_a \|u\|_V \|v\|_V \quad \forall\, u, v \in V, \tag{5}$$

$$a(v,v) \geq m_a \|v\|_V^2 \quad \forall\, v \in V. \tag{6}$$

<u>$H(b)$</u> $b : V \times Q \to \mathbb{R}$ is bilinear, bounded, and there exists a constant $\alpha_b > 0$ such that

$$\sup_{v\in V} \frac{b(v,q)}{\|v\|_V} \geq \alpha_b \|q\|_Q \quad \forall\, q \in Q. \tag{7}$$

We will use $M_b > 0$ for the boundedness constant of $b(\cdot,\cdot)$:

$$|b(v,q)| \leq M_b \|v\|_V \|q\|_Q \quad \forall\, v \in V,\, q \in Q. \tag{8}$$

The property (7) is usually called the inf-sup condition.

<u>$H(\psi)$</u> $\gamma_\psi \in \mathcal{L}(V; L^2(\Delta;\mathbb{R}^m))$; $\psi\colon \Delta \times \mathbb{R}^m \to \mathbb{R}$; $\psi(\cdot, z)$ is measurable on Δ for all $z \in \mathbb{R}^m$; for a function $z_0 \in L^2(\Delta;\mathbb{R}^m)$, $\psi(\cdot, z_0(\cdot)) \in L^1(\Delta)$; $\psi(\boldsymbol{x}, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^m$ for a.e. $\boldsymbol{x} \in \Delta$; and there exist non-negative constants c_1, c_1 and α_ψ such that

$$|\partial\psi(z)| \leq c_0 + c_1 |z|_{\mathbb{R}^m} \quad \forall\, z \in \mathbb{R}^m, \text{ a.e. on } \Delta, \tag{9}$$

$$\psi^0(z_1; z_2 - z_1) + \psi^0(z_2; z_1 - z_2) \leq \alpha_\psi |z_1 - z_2|_{\mathbb{R}^m}^2 \quad \forall\, z_1, z_2 \in \mathbb{R}^m, \text{ a.e. on } \Delta. \tag{10}$$

Note that to simplify the notation, we usually write $\psi(z)$ by suppressing its first argument. The relation (9) is a short-hand notation for

$$|\eta| \leq c_0 + c_1 |z|_{\mathbb{R}^m} \quad \forall\, z \in \mathbb{R}^m,\ \eta \in \partial\psi(z), \text{ a.e. on } \Delta,$$

and it is equivalent to

$$\left|\psi^0(z_1; z_2)\right| \leq (c_0 + c_1 |z_1|_{\mathbb{R}^m})\, |z_2|_{\mathbb{R}^m} \quad \forall\, z_1, z_2 \in \mathbb{R}^m, \text{ a.e. on } \Delta.$$

The inequality (10) is equivalent to ([39, p. 124])

$$\langle \eta_1 - \eta_2, z_1 - z_2 \rangle \geq -\alpha_\psi |z_1 - z_2|_{\mathbb{R}^m}^2 \quad \forall\, z_i \in \mathbb{R}^m,\, \eta_i \in \partial\psi(z_i),\, i = 1, 2, \text{ a.e. on } \Delta.$$

Let I_Δ stand for the integration operator over Δ. Denote by $c_\Delta > 0$ the smallest constant in the inequality

$$I_\Delta(|\gamma_\psi v|_{\mathbb{R}^m}^2) \leq c_\Delta^2 \|v\|_V^2 \quad \forall\, v \in V. \tag{11}$$

From the assumptions (9), (10), we can deduce that

$$\psi^0(z; -z) \leq c_0 |z|_{\mathbb{R}^m} + \alpha_\psi |z|_{\mathbb{R}^m}^2 \quad \forall\, z \in \mathbb{R}^m, \text{ a.e. on } \Delta. \tag{12}$$

Then it is easy to show that

$$I_\Delta(\psi^0(\gamma_\psi v; -\gamma_\psi v)) \leq c_0 c_\Delta |\Delta|^{1/2} \|v\|_V + \alpha_\psi c_\Delta^2 \|v\|_V^2 \quad \forall\, v \in V, \tag{13}$$

where $|\Delta| := I_\Delta(1)$ is the measure of Δ.

$\underline{H(f)}$ $f \in V^*$.

The following result is deduced from [16, Theorem 2.6].

THEOREM 2.4. *Assume $H(a)$, $H(b)$, $H(\psi)$, $H(f)$, and*

$$\alpha_\psi c_\Delta^2 < m_a. \tag{14}$$

Then Problem 2.3 has a solution $(u, p) \in V \times Q$, u being unique and

$$\|u\|_V \leq \frac{c_0 c_\Delta |\Delta|^{1/2} + \|f\|_{V^*}}{m_a - \alpha_\psi c_\Delta^2}.$$

Moreover, u depends Lipschitz continuously on f.

We comment that the uniqueness of $p \in Q$ can usually be shown in concrete application problems.

3. The Oseen hemivariational inequality. We describe the Oseen hemivariational inequality problem in this section. Consider an incompressible fluid with a viscosity coefficient $\mu > 0$. Assume the fluid occupies a domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$. The boundary $\partial\Omega$ of the domain is assumed to be Lipschitz continuous. Consequently, the unit outward normal vector $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_d)^T$ is defined a.e. on the boundary. For a vector-valued field $\boldsymbol{u}$ defined on the boundary, its normal and tangential components are $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, respectively. With the velocity field $\boldsymbol{u}$ and the pressure p, we define the deformation rate tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ and the stress tensor $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})$, where $\boldsymbol{I}$ is the identity matrix. We call $\sigma_\nu = \boldsymbol{\nu} \cdot \boldsymbol{\sigma} \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ the normal and tangential components of $\boldsymbol{\sigma}$ on the boundary. The boundary $\partial\Omega$ is split into two non-trivial parts where different kind of boundary conditions will be imposed: $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$ with Γ_0 and Γ_1 relatively open, $|\Gamma_0| > 0$, $|\Gamma_1| > 0$, and $\Gamma_0 \cap \Gamma_1 = \emptyset$.

Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. We use the canonical inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^d u_i v_i, \quad |\boldsymbol{v}| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \forall\, \boldsymbol{u} = (u_i),\, \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{i,j=1}^d \sigma_{ij} \tau_{ij}, \quad |\boldsymbol{\sigma}| = (\boldsymbol{\sigma} : \boldsymbol{\sigma})^{1/2} \quad \forall\, \boldsymbol{\sigma} = (\sigma_{ij}),\, \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

Given a velocity field $\boldsymbol{b}$, the pointwise formulation of the Oseen problem is the following.

PROBLEM 3.1. Find a velocity field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ and a pressure $p\colon \Omega \to \mathbb{R}$ such that

$$-\operatorname{div}(2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})) + (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{15}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{16}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0, \tag{17}$$

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi_\tau(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_1. \tag{18}$$

The equations (15)–(16) are the Oseen equations. Generally, the Oseen equations also contain a reactive term of the form $b_0\boldsymbol{u}$ on the left side of (15). Since the analysis and numerical approximation of the general Oseen equations can be carried out similarly, to simplify the exposition, we consider the Oseen equations (15)–(16) without the reactive term in this paper. The Oseen equations arise as intermediate equations in many methods to solve the Navier–Stokes equations numerically (cf. [24, Chapter 5]). In (15), $\boldsymbol{f}$ is a given force density function. The equations (15)–(16) are supplemented by the boundary conditions (17)–(18). The super-potential $\psi_\tau : \mathbb{R}^d \to \mathbb{R}$ is assumed locally Lipschitz and $\partial\psi_\tau$ is the subdifferential of ψ_τ in the sense of Clarke. The relation (18) is a non-leak slip boundary condition. The first part of the boundary condition (18) means that the fluid can not pass through Γ_1 outside the domain, whereas the second part of (18) represents a friction-type condition for the friction $\boldsymbol{\sigma}_\tau$ in terms of the tangential velocity $\boldsymbol{u}_\tau$. As we will see later, in general, the weak formulation of the problem (15)–(18) is a hemivariational inequality. If the super-potential ψ_τ is convex, then the weak formulation is reduced to a variational inequality.

The function spaces for the velocity variable and the pressure variable are

$$V = \left\{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0,\, v_\nu = 0 \text{ on } \Gamma_1\right\}, \tag{19}$$

$$Q = \left\{q \in L^2(\Omega) \mid I_\Omega(q) = 0\right\}, \tag{20}$$

where $I_\Omega(q)$ is the integral of q over Ω. Since $|\Gamma_0| > 0$, Korn's inequality holds (cf. [35, p. 79]): for a constant $c > 0$ depending only on Ω and Γ_0,

$$\|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)} \le c\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2(\Omega;\mathbb{S}^d)} \quad \forall\,\boldsymbol{v} \in V. \tag{21}$$

Thus, over the space V, the quantity $\|\boldsymbol{\varepsilon}(\cdot)\|_{L^2(\Omega;\mathbb{S}^d)}$ defines a norm and it is equivalent to the standard $H^1(\Omega;\mathbb{R}^d)$-norm. We use the norm $\|\cdot\|_V = \|\boldsymbol{\varepsilon}(\cdot)\|_{L^2(\Omega;\mathbb{S}^d)}$ on V. We have the trace inequality

$$\|\boldsymbol{v}_\tau\|_{L^2(\Gamma_1;\mathbb{R}^d)} \le \lambda_0^{-1/2}\|\boldsymbol{v}\|_V \quad \forall\,\boldsymbol{v} \in V, \tag{22}$$

where $\lambda_0 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = \lambda \int_{\Gamma_1} \boldsymbol{u}_\tau\cdot\boldsymbol{v}_\tau\,ds \quad \forall\,\boldsymbol{v} \in V. \tag{23}$$

For the given data in Problem 3.1, we assume

$$\boldsymbol{b} \in L^\infty(\Omega;\mathbb{R}^d), \quad \operatorname{div}\boldsymbol{b} = 0 \text{ in } \Omega, \quad \boldsymbol{b}\cdot\boldsymbol{\nu} \ge 0 \text{ on } \Gamma_1, \tag{24}$$

$$\boldsymbol{f} \in V^*. \tag{25}$$

and

$\underline{H(\psi_\tau)}$. $\psi_\tau\colon \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz on $\mathbb{R}^d$ and there exist non-negative constants $\alpha_\psi, c_0, c_1 \ge 0$ such that

$$\psi_\tau^0(\boldsymbol{\xi}_1;\boldsymbol{\xi}_2-\boldsymbol{\xi}_1)+\psi_\tau^0(\boldsymbol{\xi}_2;\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\le \alpha_\psi|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2| \quad \forall\,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{R}^d, \tag{26}$$

$$|\boldsymbol{\eta}|\le c_0+c_1|\boldsymbol{\xi}| \quad \forall\,\boldsymbol{\xi}\in\mathbb{R}^d, \boldsymbol{\eta}\in\partial\psi_\tau(\boldsymbol{\xi}). \tag{27}$$

In the weak formulation of Problem 3.1, we need the following bilinear form:

$$a(\boldsymbol{u},\boldsymbol{v})=a_0(\boldsymbol{u},\boldsymbol{v})+a_1(\boldsymbol{u},\boldsymbol{v}) \quad \forall\,\boldsymbol{u},\boldsymbol{v}\in V, \tag{28}$$

where

$$a_0(\boldsymbol{u},\boldsymbol{v})=2\mu\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx \quad \forall\,\boldsymbol{u},\boldsymbol{v}\in V, \tag{29}$$

$$a_1(\boldsymbol{u},\boldsymbol{v})=\int_\Omega(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\,dx \quad \forall\,\boldsymbol{u},\boldsymbol{v}\in V. \tag{30}$$

We also need the bilinear form

$$b(\boldsymbol{v},q)=-\int_\Omega q\,\mathrm{div}\,\boldsymbol{v}\,dx \quad \forall\,\boldsymbol{v}\in V,\ q\in Q. \tag{31}$$

The divergence-free condition on $\boldsymbol{b}$ stems from the fact that $\boldsymbol{b}$ represents a flow velocity field. The last condition in (24) means that the flow with the velocity field $\boldsymbol{b}$ can leak outward on Γ_1. We note that in [31], corresponding to our problem, the assumptions on $\boldsymbol{b}$ are: $\boldsymbol{b}\in H^1(\Omega;\mathbb{R}^d)$, $\mathrm{div}\,\boldsymbol{b}=0$ in Ω, $\boldsymbol{b}=\boldsymbol{0}$ on Γ_0, and $\boldsymbol{b}\cdot\boldsymbol{\nu}=0$ on Γ_1. The regularity condition $\boldsymbol{b}\in L^\infty(\Omega;\mathbb{R}^d)$ can be replaced by $\boldsymbol{b}\in H^1(\Omega;\mathbb{R}^d)$, or weakened to $\boldsymbol{b}\in L^r(\Omega;\mathbb{R}^d)$ for some $r<\infty$ for discussions below. Nevertheless, the assumption $\boldsymbol{b}\in L^\infty(\Omega;\mathbb{R}^d)$ allows us to avoid some technical details.

LEMMA 3.2. *Under the conditions* (24)*, the following inequality holds:*

$$a_1(\boldsymbol{v},\boldsymbol{v})\ge 0 \quad \forall\,\boldsymbol{v}\in V. \tag{32}$$

Proof. For $\boldsymbol{v}\in V$, we have

$$a_1(\boldsymbol{v},\boldsymbol{v})=\frac{1}{2}\int_{\partial\Omega}\boldsymbol{b}\cdot\boldsymbol{\nu}\,|\boldsymbol{v}|^2ds-\frac{1}{2}\int_\Omega \mathrm{div}\,\boldsymbol{b}\,|\boldsymbol{v}|^2dx.$$

Note that $\boldsymbol{v}=\boldsymbol{0}$ on Γ_0, $\mathrm{div}\,\boldsymbol{b}=0$ in Ω, and $\boldsymbol{b}\cdot\boldsymbol{\nu}\ge 0$ on Γ_1. We derive the inequality (32) from the previous equality. ∎

Through a standard procedure, the following mixed weak formulation can be derived for the problem (15)–(18).

PROBLEM 3.3. Find $(\boldsymbol{u},p)\in V\times Q$ such that

$$a(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},p)+\int_{\Gamma_1}\psi_\tau^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau)\,ds\ge\langle\boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in V, \tag{33}$$

$$b(\boldsymbol{u},q)=0 \quad \forall\,q\in Q. \tag{34}$$

If additionally, ψ_τ is convex, then Problem 3.3 is replaced by the following variational inequality.

PROBLEM 3.4. Find $(\boldsymbol{u},p) \in V \times Q$ such that

$$a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + b(\boldsymbol{v}-\boldsymbol{u},p) + \int_{\Gamma_1} \psi_\tau(\boldsymbol{v}_\tau)\,ds - \int_{\Gamma_1} \psi_\tau(\boldsymbol{u}_\tau)\,ds \ge \langle \boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}\rangle \quad \forall\, \boldsymbol{v} \in V, \tag{35}$$

$$b(\boldsymbol{u},q) = 0 \quad \forall\, q \in Q. \tag{36}$$

We can apply Theorem 2.4 for the well-posedness of Problem 3.3.

THEOREM 3.5. *Assume $H(\psi_\tau)$ and $\alpha_\psi < 2\,\mu\,\lambda_0$. Then for any $\boldsymbol{f} \in V^*$, Problem 3.3 has a unique solution $(\boldsymbol{u},p) \in V \times Q$. Moreover, the solution $(\boldsymbol{u},p) \in V \times Q$ depends Lipschitz continuously on $\boldsymbol{f} \in V^*$, and*

$$\|\boldsymbol{u}\|_V + \|p\|_Q \le c\,(1 + \|\boldsymbol{f}\|_{V^*})\,. \tag{37}$$

Proof. Let us verify the assumptions stated in Theorem 2.4. Since $\boldsymbol{b} \in L^\infty(\Omega;\mathbb{R}^d)$, it is obvious that the bilinear form $a(\cdot,\cdot)$ defined by (28) is bounded. Applying Lemma 3.2, we know that

$$a(\boldsymbol{v},\boldsymbol{v}) \ge 2\mu\,\|\boldsymbol{v}\|_V^2 \quad \forall\, \boldsymbol{v} \in V.$$

So $H(a)$ is satisfied with $m_a = 2\mu$ for the V-ellipticity (6). $H(b)$ is valid and (7) follows from the inf-sup condition ([40])

$$\sup_{\boldsymbol{v}\in V_0} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_V} \ge \alpha_b \|q\|_Q \quad \forall\, q \in Q, \tag{38}$$

where $V_0 = H_0^1(\Omega;\mathbb{R}^d)$. The assumption $H(\psi_\tau)$ implies $H(\psi)$, and the assumption (25) implies $H(f)$. The smallness condition (14) takes the form $\alpha_\psi < 2\,\mu\,\lambda_0$. Thus, by Theorem 2.4, Problem 3.3 has a solution $(\boldsymbol{u},p) \in V \times Q$, $\boldsymbol{u}$ is unique and depends on $\boldsymbol{f}$ Lipschitz continuously, and

$$\|\boldsymbol{u}\|_V \le c\,(1 + \|\boldsymbol{f}\|_{V^*})\,.$$

For the uniqueness of p, assume Problem 3.3 has two solutions $(\boldsymbol{u},p_1), (\boldsymbol{u},p_2) \in V \times Q$. Then,

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p_1) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in V_0,$$
$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p_2) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in V_0.$$

Hence,

$$b(\boldsymbol{v},p_1-p_2) = 0 \quad \forall\, \boldsymbol{v} \in V_0.$$

By (38),

$$\alpha_b \|p_1 - p_2\|_Q \le \sup_{\boldsymbol{v}\in V_0} \frac{b(\boldsymbol{v},p_1-p_2)}{\|\boldsymbol{v}\|_V} = 0.$$

Thus, $p_1 = p_2$ and the p component of the solution is unique.

For the Lipschitz continuous dependence of p on $\boldsymbol{f}$, let $\boldsymbol{f}_1, \boldsymbol{f}_2 \in V^*$ and denote by $(\boldsymbol{u}_1,p_1), (\boldsymbol{u}_2,p_2) \in V \times Q$ the solutions of Problem 3.3 for $\boldsymbol{f} = \boldsymbol{f}_1, \boldsymbol{f}_2$. Then,

$$a(\boldsymbol{u}_1,\boldsymbol{v}) + b(\boldsymbol{v},p_1) = \langle \boldsymbol{f}_1,\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in V_0,$$
$$a(\boldsymbol{u}_2,\boldsymbol{v}) + b(\boldsymbol{v},p_2) = \langle \boldsymbol{f}_2,\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in V_0,$$

and thus,

$$b(\boldsymbol{v},p_1-p_2) = \langle \boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{v}\rangle - a(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V_0.$$

By the boundedness of the bilinear form $a(\cdot,\cdot)$, we have

$$b(\boldsymbol{v}, p_1 - p_2) \le c\,(\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{V^*} + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V)\,\|\boldsymbol{v}\|_V \quad \forall\,\boldsymbol{v} \in V_0.$$

Applying the inf-sup condition (38), we derive from the above inequality that

$$\|p_1 - p_2\|_Q \le c\,(\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{V^*} + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V)\,.$$

This inequality implies the Lipschitz continuity of p on $\boldsymbol{f}$ since $\boldsymbol{u}$ depends on $\boldsymbol{f}$ Lipschitz continuously. Moreover, we have the bound (37). ∎

In the special case where ψ_τ is convex, $\alpha_\psi = 0$ and the smallness condition $\alpha_\psi < 2\,\mu\,\lambda_0$ is automatically valid. Then we have the next result on Problem 3.4.

THEOREM 3.6. *Assume $H(\psi_\tau)$ and ψ_τ is convex. Then for any $\boldsymbol{f} \in V^*$, Problem 3.4 has a unique solution $(\boldsymbol{u}, p) \in V \times Q$. Moreover, the solution $(\boldsymbol{u}, p) \in V \times Q$ depends Lipschitz continuously on $\boldsymbol{f} \in V^*$, and*

$$\|\boldsymbol{u}\|_V + \|p\|_Q \le c\,(1 + \|\boldsymbol{f}\|_{V^*})\,. \tag{39}$$

4. Mixed finite element methods. In this section, we consider the mixed finite element method to solve Problem 3.3. For simplicity, we assume Ω is a polygonal/polyhedral domain. Let $\{\mathcal{T}^h\}_h$ be a regular family of finite element partitions of $\overline{\Omega}$ that is compatible with the boundary splitting $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$ in the sense that if an element edge (in 2D) or element surface (in 3D) has a non-trivial overlap with $\overline{\Gamma_0}$ or $\overline{\Gamma_1}$, then it lies entirely on $\overline{\Gamma_0}$ or $\overline{\Gamma_1}$. The parameter h represents the mesh-size of the partition $\mathcal{T}^h$. Corresponding to the finite element partitions, we introduce finite element spaces $\{(V^h, Q^h)\}_h$, $V^h \subset V$ and $Q^h \subset Q$. Define $V_0^h = V^h \cap V_0$. We assume the discrete inf-sup condition, also known as the Babuška-Brezzi condition: for a constant $\beta > 0$ independent of h,

$$\beta\,\|q^h\|_Q \le \sup_{\boldsymbol{v}^h \in V_0^h} \frac{b(\boldsymbol{v}^h, q^h)}{\|\boldsymbol{v}^h\|_V} \quad \forall\,q^h \in Q. \tag{40}$$

The mixed finite element method for Problem 3.3 is the following.

PROBLEM 4.1. Find $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ such that

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^h) + \int_{\Gamma_1} \psi_\tau^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h)\,ds \ge \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall\,\boldsymbol{v}^h \in V^h, \tag{41}$$

$$b(\boldsymbol{u}^h, q^h) = 0 \quad \forall\,q^h \in Q^h. \tag{42}$$

Similar to the results on Problem 3.3, we have the next result on Problem 4.1.

THEOREM 4.2. *Assume $H(\psi_\tau)$, $\alpha_\psi < 2\,\mu\,\lambda_0$, and the discrete inf-sup condition* (40). *Then for any $\boldsymbol{f} \in V^*$, Problem 4.1 has a unique solution $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$. Moreover, the solution $(\boldsymbol{u}^h, p^h)$ depends Lipschitz continuously on $\boldsymbol{f} \in V^*$, and for a constant $c > 0$ independent of h,*

$$\|\boldsymbol{u}^h\|_V + \|p^h\|_Q \le c\,(1 + \|\boldsymbol{f}\|_{V^*})\,. \tag{43}$$

In the rest of the section, we focus on deriving an error estimate. For this purpose, we will keep the assumptions stated in Theorem 4.2: $H(\psi_\tau)$, $\alpha_\psi < 2\,\mu\,\lambda_0$, and the discrete inf-sup condition (40). We will apply the modified Cauchy-Schwarz inequality several times:

$$x\,y \le \epsilon\, x^2 + c\, y^2 \quad \forall\, x, y \in \mathbb{R}, \tag{44}$$

where $\epsilon > 0$ is arbitrarily small, and $c = c(\epsilon) = 1/(4\,\epsilon)$ is a constant depending on ϵ.

Frist, we present a Céa's inequality.

THEOREM 4.3. *Assume $H(\psi_\tau)$, $\alpha_\psi < 2\,\mu\,\lambda_0$, and the discrete inf-sup condition* (40). *Then,*

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_V^2 + \|p-p^h\|_Q^2 \le c\left(\|\boldsymbol{u}-\boldsymbol{v}^h\|_V^2 + \|\boldsymbol{u}_\tau - \boldsymbol{v}_\tau^h\|_{L^2(\Gamma_1;\mathbb{R}^d)} + \|p-q^h\|_Q^2\right) \tag{45}$$

for any $\boldsymbol{v}^h \in V^h$, $q^h \in Q^h$.

Proof. For an arbitrary $\boldsymbol{v}^h \in V^h$,

$$2\,\mu\,\|\boldsymbol{u}-\boldsymbol{u}^h\|_V^2 = a_0(\boldsymbol{u}-\boldsymbol{u}^h, \boldsymbol{u}-\boldsymbol{u}^h) = a_0(\boldsymbol{u}-\boldsymbol{u}^h, \boldsymbol{u}-\boldsymbol{v}^h) + a_0(\boldsymbol{u}-\boldsymbol{u}^h, \boldsymbol{v}^h-\boldsymbol{u}^h),$$

which is rewritten as

$$2\,\mu\,\|\boldsymbol{u}-\boldsymbol{u}^h\|_V^2 = a_0(\boldsymbol{u}-\boldsymbol{u}^h, \boldsymbol{u}-\boldsymbol{v}^h) + a_0(\boldsymbol{u}, \boldsymbol{v}^h-\boldsymbol{u}^h) + a_0(\boldsymbol{u}^h, \boldsymbol{u}^h-\boldsymbol{v}^h). \tag{46}$$

Take $\boldsymbol{v} = \boldsymbol{u}^h - \boldsymbol{v}^h$ in (33) to get

$$\begin{aligned} a_0(\boldsymbol{u}, \boldsymbol{v}^h-\boldsymbol{u}^h) \le a_1(\boldsymbol{u}, \boldsymbol{u}^h-\boldsymbol{v}^h) + b(\boldsymbol{u}^h-\boldsymbol{v}^h, p) + \int_{\Gamma_1} \psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{v}_\tau^h)\, ds \\ - \langle \boldsymbol{f}, \boldsymbol{u}^h-\boldsymbol{v}^h\rangle. \end{aligned}$$

Replace $\boldsymbol{v}^h$ by $\boldsymbol{v}^h - \boldsymbol{u}^h$ in (41) to get

$$\begin{aligned} a_0(\boldsymbol{u}^h, \boldsymbol{u}^h-\boldsymbol{v}^h) \le a_1(\boldsymbol{u}^h, \boldsymbol{v}^h-\boldsymbol{u}^h) + b(\boldsymbol{v}^h-\boldsymbol{u}^h, p^h) + \int_{\Gamma_1} \psi_\tau^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h)\, ds \\ - \langle \boldsymbol{f}, \boldsymbol{v}^h-\boldsymbol{u}^h\rangle. \end{aligned}$$

Use these two inequalities in (46),

$$2\,\mu\,\|\boldsymbol{u}-\boldsymbol{u}^h\|_V^2 \le a_0(\boldsymbol{u}-\boldsymbol{u}^h, \boldsymbol{u}-\boldsymbol{v}^h) + I_{a_1} + I_\psi + I_b, \tag{47}$$

where

$$\begin{aligned} I_{a_1} &= a_1(\boldsymbol{u}-\boldsymbol{u}^h, \boldsymbol{u}^h-\boldsymbol{v}^h), \\ I_\psi &= \int_{\Gamma_1} \left[\psi_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{u}_\tau^h - \boldsymbol{v}_\tau^h) + \psi_\tau^0(\boldsymbol{u}_\tau^h; \boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h)\right] ds, \\ I_b &= b(\boldsymbol{u}^h-\boldsymbol{v}^h, p-p^h). \end{aligned}$$

Let us bound each of the terms on the right hand side of (47). First, by the boundedness of the bilinear form a_0 and the modified Cauchy-Schwarz inequality (44), for any $\epsilon > 0$, there is a constant $c > 0$ depending on ϵ such that

$$a_0(\boldsymbol{u}-\boldsymbol{u}^h, \boldsymbol{u}-\boldsymbol{v}^h) \le 2\mu\,\|\boldsymbol{u}-\boldsymbol{u}^h\|_V \|\boldsymbol{u}-\boldsymbol{v}^h\|_V \le \epsilon\,\|\boldsymbol{u}-\boldsymbol{u}^h\|_V^2 + c\,\|\boldsymbol{u}-\boldsymbol{v}^h\|_V^2. \tag{48}$$

Write

$$I_{a_1} = -a_1(\boldsymbol{u}-\boldsymbol{u}^h, \boldsymbol{u}-\boldsymbol{u}^h) + a_1(\boldsymbol{u}-\boldsymbol{u}^h, \boldsymbol{u}-\boldsymbol{v}^h).$$

By Lemma 3.2, $a_1(\boldsymbol{u}-\boldsymbol{u}^h,\boldsymbol{u}-\boldsymbol{u}^h)\ge 0$. Hence,

$$I_{a_1} \le \int_\Omega (\boldsymbol{b}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{u}^h)\cdot(\boldsymbol{u}-\boldsymbol{v}^h)\,dx \le c\,\|\boldsymbol{u}-\boldsymbol{u}^h\|_V\|\boldsymbol{u}-\boldsymbol{v}^h\|_V.$$

By an application of the modified Cauchy-Schwarz inequality (44),

$$I_{a_1} \le \epsilon\,\|\boldsymbol{u}-\boldsymbol{u}^h\|_V^2 + c\,\|\boldsymbol{u}-\boldsymbol{v}^h\|_V^2. \tag{49}$$

For the integrand of the term I_ψ, by the subadditivity of the generalized directional derivative (cf. Proposition 2.2),

$$\begin{aligned}\psi_\tau^0(\boldsymbol{u}_\tau;\boldsymbol{u}_\tau^h-\boldsymbol{v}_\tau^h)+\psi_\tau^0(\boldsymbol{u}_\tau^h;\boldsymbol{v}_\tau^h-\boldsymbol{u}_\tau^h) \le{}& \left[\psi_\tau^0(\boldsymbol{u}_\tau;\boldsymbol{u}_\tau^h-\boldsymbol{u}_\tau)+\psi_\tau^0(\boldsymbol{u}_\tau^h;\boldsymbol{u}_\tau-\boldsymbol{u}_\tau^h)\right]\\ &+\left[\psi_\tau^0(\boldsymbol{u}_\tau;\boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h)+\psi_\tau^0(\boldsymbol{u}_\tau^h;\boldsymbol{v}_\tau^h-\boldsymbol{u}_\tau)\right].\end{aligned}$$

Then by $H(\psi_\tau)$ and (27),

$$\psi_\tau^0(\boldsymbol{u}_\tau;\boldsymbol{u}_\tau^h-\boldsymbol{v}_\tau^h)+\psi_\tau^0(\boldsymbol{u}_\tau^h;\boldsymbol{v}_\tau^h-\boldsymbol{u}_\tau^h) \le \alpha_\psi|\boldsymbol{u}_\tau-\boldsymbol{u}_\tau^h|^2 + c\left(1+|\boldsymbol{u}_\tau|+|\boldsymbol{u}_\tau^h|\right)|\boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h|.$$

Hence,

$$I_\psi \le \int_{\Gamma_1}\left[\alpha_\psi|\boldsymbol{u}_\tau-\boldsymbol{u}_\tau^h|^2 + c\left(1+|\boldsymbol{u}_\tau|+|\boldsymbol{u}_\tau^h|\right)|\boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h|\right]ds,$$

and then

$$I_\psi \le \alpha_\psi\lambda_0^{-1}\|\boldsymbol{u}-\boldsymbol{u}^h\|_V^2 + c\left(1+\|\boldsymbol{u}_\tau\|_{L^2(\Gamma_1;\mathbb{R}^d)}+\|\boldsymbol{u}_\tau^h\|_{L^2(\Gamma_1;\mathbb{R}^d)}\right)\|\boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h\|_{L^2(\Gamma_1;\mathbb{R}^d)}. \tag{50}$$

By making use of (34) and (42), we can write, for any $q^h\in Q^h$,

$$\begin{aligned}I_b &= b(\boldsymbol{u}^h-\boldsymbol{v}^h,p-p^h)\\ &= b(\boldsymbol{u}^h,p)-b(\boldsymbol{v}^h,p)+b(\boldsymbol{v}^h,p^h)\\ &= b(\boldsymbol{u}^h-\boldsymbol{u},p-q^h)+b(\boldsymbol{u}-\boldsymbol{v}^h,p)+b(\boldsymbol{v}^h-\boldsymbol{u},p^h)\\ &= b(\boldsymbol{u}^h-\boldsymbol{u},p-q^h)+b(\boldsymbol{u}-\boldsymbol{v}^h,p-p^h).\end{aligned}$$

Thus,

$$I_b \le c\left(\|\boldsymbol{u}-\boldsymbol{u}^h\|_V\|p-q^h\|_Q+\|\boldsymbol{u}-\boldsymbol{v}^h\|_V\|p-p^h\|_Q\right),$$

and by the modified Cauchy-Schwarz inequality (44),

$$I_b \le \epsilon\left(\|\boldsymbol{u}-\boldsymbol{u}^h\|_V^2+\|p-p^h\|_Q^2\right)+c\left(\|\boldsymbol{u}-\boldsymbol{v}^h\|_V^2+\|p-q^h\|_Q^2\right). \tag{51}$$

We use (48)–(51) in (47) to obtain, recalling the assumption $\alpha_\psi < 2\,\mu\,\lambda_0$, that for any sufficiently small $\epsilon>0$, with a constant c depending on ϵ,

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_V^2 \le c\left(\|\boldsymbol{u}-\boldsymbol{v}^h\|_V^2+\|\boldsymbol{u}_\tau-\boldsymbol{v}_\tau^h\|_{L^2(\Gamma_1;\mathbb{R}^d)}+\|p-q^h\|_Q^2\right)+\epsilon\,\|p-p^h\|_Q^2. \tag{52}$$

By the triangle inequality,

$$\|p-p^h\|_Q \le \|p-q^h\|_Q+\|p^h-q^h\|_Q. \tag{53}$$

By the discrete inf-sup condition (40),

$$\beta\,\|p^h-q^h\|_Q \le \sup_{\boldsymbol{v}^h\in V_0^h}\frac{b(\boldsymbol{v}^h,p^h-q^h)}{\|\boldsymbol{v}^h\|_V}. \tag{54}$$

Write

$$b(\boldsymbol{v}^h,p^h-q^h) = b(\boldsymbol{v}^h,p^h-p)+b(\boldsymbol{v}^h,p-q^h).$$

From (33) and (41), we derive that

$$a(\boldsymbol{u},\boldsymbol{v}^h)+b(\boldsymbol{v}^h,p)=\langle \boldsymbol{f},\boldsymbol{v}^h\rangle \quad \forall\,\boldsymbol{v}^h\in V_0,$$
$$a(\boldsymbol{u}^h,\boldsymbol{v}^h)+b(\boldsymbol{v}^h,p^h)=\langle \boldsymbol{f},\boldsymbol{v}^h\rangle \quad \forall\,\boldsymbol{v}^h\in V_0^h.$$

So

$$b(\boldsymbol{v}^h,p^h-p)=a(\boldsymbol{u}-\boldsymbol{u}^h,\boldsymbol{v}^h).$$

Note that

$$a(\boldsymbol{u}-\boldsymbol{u}^h,\boldsymbol{v}^h)\le c\,\|\boldsymbol{u}-\boldsymbol{u}^h\|_V\|\boldsymbol{v}^h\|_V.$$

Hence,

$$b(\boldsymbol{v}^h,p^h-p)\le c\,\|\boldsymbol{u}-\boldsymbol{u}^h\|_V\|\boldsymbol{v}^h\|_V,$$

and then

$$b(\boldsymbol{v}^h,p^h-q^h)\le c\left(\|\boldsymbol{u}-\boldsymbol{u}^h\|_V+\|p-q^h\|_Q\right)\|\boldsymbol{v}^h\|_V \quad \forall\,\boldsymbol{v}^h\in V_0.$$

Use this inequality in (54) to obtain

$$\|p^h-q^h\|_Q\le c\left(\|\boldsymbol{u}-\boldsymbol{u}^h\|_V+\|p-q^h\|_Q\right).$$

Summarizing, we can derive from (53) that

$$\|p-p^h\|_Q\le c\left(\|\boldsymbol{u}-\boldsymbol{u}^h\|_V+\|p-q^h\|_Q\right). \tag{55}$$

We combine (52) with a sufficiently small ϵ and (55) to get (45). ■

Céa's inequality is the basis for error estimates of the finite element solution $(\boldsymbol{u}^h,p^h)$. As an example, we consider the P1b/P1 element ([1]):

$$V^h=\left\{\boldsymbol{v}^h\in V\cap C^0(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T\in[P_1(T)\oplus B(T)]^d\ \forall\,T\in\mathcal{T}^h\right\},$$
$$Q^h=\left\{q^h\in Q\cap C^0(\overline{\Omega}) \mid q|_T\in P_1(T)\ \forall\,T\in\mathcal{T}^h\right\},$$

where $P_1(T)$ is the space of polynomials of a degree less than or equal to 1 on T, and $B(T)$ is the space of bubble functions on T. For this element, the discrete inf-sup condition (40) holds. To apply (45) to derive an error estimate, we express $\overline{\Gamma_1}$ as the union of a finite number of flat components:

$$\overline{\Gamma_1}=\bigcup_{l=1}^{l_0}\Gamma_{1,l},$$

and assume the following solution regularities:

$$\boldsymbol{u}\in H^2(\Omega;\mathbb{R}^d),\quad \boldsymbol{u}_\tau|_{\Gamma_{1,l}}\in H^2(\Gamma_{1,l};\mathbb{R}^d),\ 1\le l\le l_0,\quad p\in H^1(\Omega).$$

Then, applying the standard finite element interpolation error estimates ([2, 6, 3]), we can derive from (45) the following optimal order error estimate:

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_V+\|p-p^h\|_Q\le c\,h,$$

for a constant c depending the solution $(\boldsymbol{u},p)$.

5. Numerical results. We report some numerical results on a two-dimensional test problem. The super-potential function is taken to be

$$\psi_\tau(\boldsymbol{v}_\tau)=\int_0^{|\boldsymbol{v}_\tau|}\omega(t)\,dt,\quad \omega(t)=(a-b)e^{-\alpha t}+b.$$

The function $\psi_\tau \colon \mathbb{R} \to \mathbb{R}$ is non-convex and Lipschitz continuous. Its subdifferential is

$$\partial\psi_\tau(\mathbf{0}) = \omega(0)\, B_1(\mathbf{0}), \quad \partial\psi_\tau(\boldsymbol{v}_\tau) = \omega(|\boldsymbol{v}_\tau|)\, \boldsymbol{v}_\tau/|\boldsymbol{v}_\tau| \text{ if } \boldsymbol{v}_\tau \neq \mathbf{0},$$

where $B_1(\mathbf{0})$ is the unit ball centered at the origin in $\mathbb{R}^2$.

In solving the hemivariational inequality, we apply an iterative algorithm. For this purpose, we introduce a Lagrangian multiplier $\boldsymbol{\lambda} = -\boldsymbol{\sigma}/\omega(|\boldsymbol{u}_\tau|)$ on Γ_1. Then the boundary condition

$$-\boldsymbol{\sigma}_\tau \in \partial\psi_\tau(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_1$$

from (18) implies that $\boldsymbol{\lambda}$ belongs to the set

$$\boldsymbol{\Lambda} = \{\boldsymbol{\lambda} \in \boldsymbol{L}^\infty(\Gamma_1) \mid |\boldsymbol{\lambda}| \leq 1 \text{ a.e. on } \Gamma_1\}.$$

By making use of the Lagrangian multiplier $\boldsymbol{\lambda}$, we have another weak formulation for Problem 3.1:

PROBLEM 5.1. Find $\boldsymbol{u} \in V$, $p \in Q$ and $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + \int_{\Gamma_1} \boldsymbol{\lambda} \cdot \boldsymbol{v}_\tau \, ds = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in V, & \text{(56a)} \\ b(\boldsymbol{u}, q) = 0 \quad \forall\, q \in Q, & \text{(56b)} \\ \boldsymbol{\lambda} \cdot \boldsymbol{u}_\tau = |\boldsymbol{u}_\tau| \quad \text{a.e. on } \Gamma_1. & \text{(56c)} \end{cases}$$

Similarly, on the discrete level, we have the following variant of Problem 4.1:

PROBLEM 5.2. Find $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ and $\boldsymbol{\lambda}^h \in \boldsymbol{\Lambda}$ such that

$$\begin{cases} a(\boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^h) + \int_{\Gamma_1} \boldsymbol{\lambda}^h \cdot \boldsymbol{v}^h_\tau \, ds = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle \quad \forall\, \boldsymbol{v}^h \in V^h, & \text{(57a)} \\ b(\boldsymbol{u}^h, q^h) = 0 \quad \forall\, q^h \in Q^h, & \text{(57b)} \\ \boldsymbol{\lambda}^h \cdot \boldsymbol{u}^h_\tau = |\boldsymbol{u}^h_\tau| \quad \text{a.e. on } \Gamma_1. & \text{(57c)} \end{cases}$$

This form of the finite element system motivates us to introduce the following Uzawa algorithm.

Algorithm 1 Uzawa algorithm

Initialization. Choose $\boldsymbol{u}^h_0$, $\boldsymbol{\lambda}^h_1$, $\rho > 0$, ϵ_{tol}, and set $n = 1$.
Iteration. For $n \geq 1$, with $\boldsymbol{\lambda}^h_n \in \boldsymbol{\Lambda}$ known, find $(\boldsymbol{u}^h_n, p^h_n) \in V^h \times Q^h$ by solving

$$a(\boldsymbol{u}^h_n, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^h_n) = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle - \int_{\Gamma_1} \boldsymbol{\lambda}^h_n \cdot \boldsymbol{v}^h_\tau \, ds \quad \forall\, \boldsymbol{v}^h \in V^h,$$

$$b(\boldsymbol{u}^h_n, q^h) = 0 \quad \forall\, q^h \in Q^h$$

and compute

$$\boldsymbol{\lambda}^h_{n+1} = P_\Lambda(\boldsymbol{\lambda}^h_n + \rho\, \boldsymbol{u}^h_{n,\tau}).$$

Continue the iteration until $\|\boldsymbol{u}^h_n - \boldsymbol{u}^h_{n-1}\|_{L^2(\Omega)}/\|\boldsymbol{u}^h_n\|_{L^2(\Omega)} < \epsilon_{tol}$.

In the iteration step, P_Λ denotes the projection onto the unit ball. In the implementation, the integral on Γ_1 is computed by the trapezoid rule corresponding to a partition of Γ_1 induced by the finite element mesh of the domain. Consequently, only the values of $\boldsymbol{\lambda}^h_n$

at the mesh points on Γ_1 are involved in the algorithm. In the examples, we choose $\boldsymbol{u}_0^h = \mathbf{0}$, $\boldsymbol{\lambda}_1^h = \mathbf{1}$, $\epsilon_{tol} = 10^{-6}$, and test different values of the parameter ρ. To avoid an infinite loop, in the implementation of the Uzawa algorithm, we stop the iteration when a maximal iteration number is reached. For the numerical results reported in this section, however, it is always the case that the iteration stops with $\|\boldsymbol{u}_n^h - \boldsymbol{u}_{n-1}^h\|_{L^2(\Omega)}/\|\boldsymbol{u}_n^h\|_{L^2(\Omega)} < \epsilon_{tol}$.

For the data of the examples, we let $\Omega = (0,1) \times (0,1)$ be the unit square, $\Gamma_1 = (0,1) \times \{0\}$ the bottom of the square, and impose the homogeneous Dirichlet boundary condition along the rest of the boundary. We take $\mu = 1$, and define the source function by

$$\boldsymbol{f}_0 = -\mathrm{div}(2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_0)) + (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}_0 + \nabla p_0$$

with (a generic point in $\mathbb{R}^2$ is denoted by (x,y))

$$\boldsymbol{u}_0(x,y) = \begin{pmatrix} 20x^2(1-x)^2y(1-y)(1-2y) \\ -20x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix},$$

$$p_0(x,y) = 10(2x-1)(2y-1).$$

We note that a similar source function is used in the context of variational inequalities for the Navier-Stokes equations in [16] and for Stokes equations with leak boundary conditions in [26].

For the convective coefficient function $\boldsymbol{b}$, we consider the following two choices, both satisfying (24):

$$\boldsymbol{b}_1(x,y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

$$\boldsymbol{b}_2(x,y) = \begin{pmatrix} x^3 + x^2y + x^2 - 3xy^2 - 2xy + x + 1 \\ -3x^2y - xy^2 - 2xy + y^3 + y^2 - y - 1 \end{pmatrix}.$$

For comparison, we also consider the choice

$$\boldsymbol{b}_3 = \boldsymbol{u}.$$

In this case, the Oseen equations are actually the Navier-Stokes equations, and in the iteration step of the Uzawa algorithm, we replace $a(\boldsymbol{u}_n^h, \boldsymbol{v}^h)$ by $a_0(\boldsymbol{u}_n^h, \boldsymbol{v}^h) + a_1(\boldsymbol{u}_{n-1}^h; \boldsymbol{u}_n^h, \boldsymbol{v}^h)$.

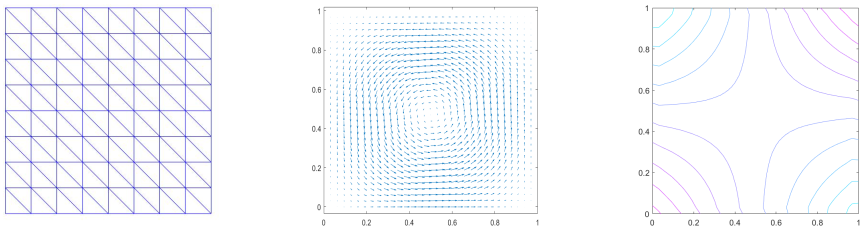
This problem is approximated by the $P1b/P1$ finite element pair on triangular meshes. We partition the domain $\overline{\Omega}$ by a sequence of uniform triangular meshes (cf. Fig. 1 (a)) with the interval $[0,1]$ being split into h^{-1} equal sub-intervals for $h = 1/4, 1/8, \cdots$, and use the $P1b/P1$ finite element pair ([1]). The discrete inf-sup condition is satisfied. In computing the numerical solution errors, we take $\boldsymbol{u}^* = \boldsymbol{u}^{1/256}$ and $p^* = p^{1/256}$ as the reference solution.

The numerical convergence order of the numerical solution is computed by

$$\text{Order} = \log(e^h/e^{h/2})/\log(2),$$

where $e^h = \|\boldsymbol{u}^h - \boldsymbol{u}^*\|_{L^2(\Omega)}$ for $E^{\boldsymbol{u}}_{L^2}(h)$, $e^h = |\boldsymbol{u}^h - \boldsymbol{u}^*|_{H^1(\Omega)}$ for $E^{\boldsymbol{u}}_{H^1}(h)$, and $e^h = \|p^h - p^*\|_{L^2(\Omega)}$ for $E^p_{L^2}(h)$.

In the numerical experiments, we let $\alpha = 10$ and consider three pairs of a and b: (C1) $a = 0.255$, $b = 0.25$; (C2) $a = 0.85$, $b = 0.8$; (C3) $a = 5.01$ $b = 5.0$. The velocity field and the pressure counters corresponding to (C1) are drawn in Fig. 1. Numerical errors and numerical convergence orders of the finite element approximations are shown in Tables 1–3 and Fig. 2 through Fig. 4. The tangential component of velocity $\boldsymbol{u}_\tau$ and the tangential component of stress tensor $\boldsymbol{\sigma}_\tau$ along the slip boundary Γ_1, the value of velocity in Ω are displayed in Fig. 5. In Fig. 1 (b)–(c) and Fig. 5, the results are obtained for $h = 1/32$. The numerical convergence orders match the theoretical convergence orders predicted in Theorem 4.3 even though it is not clear if the solution regularity assumptions stated in Theorem 4.3 are satisfied. The slip on boundary is observed to convince us that the superpotential function directly influences the occurrence of the slippage.



(a) mesh (b) velocity field (c) pressure isobars

Fig. 1. Mesh, velocity field and pressure isobars

Table 1. Numerical errors, $\boldsymbol{b} = \boldsymbol{b}_1$

	$a = 0.255$	$b = 0.25$	$(\rho = 1)$	$a = 0.85$	$b = 0.8$	$(\rho = 1)$	$a = 5.01$	$b = 5.0$	$(\rho = 1)$
h	$E^{\boldsymbol{u}}_{L^2}$	$E^{\boldsymbol{u}}_{H^1}$	$E^{p}_{L^2}$	$E^{\boldsymbol{u}}_{L^2}$	$E^{\boldsymbol{u}}_{H^1}$	$E^{p}_{L^2}$	$E^{\boldsymbol{u}}_{L^2}$	$E^{\boldsymbol{u}}_{H^1}$	$E^{p}_{L^2}$
1/4	2.05e-02	2.14e-01	3.75e-01	1.98e-02	2.11e-01	3.76e-01	3.65e-02	3.67e-01	3.89e-01
1/8	7.12e-03	9.06e-02	1.61e-01	6.91e-03	8.94e-02	1.61e-01	1.18e-02	1.94e-01	1.77e-01
1/16	1.88e-03	3.57e-02	5.96e-02	1.82e-03	3.52e-02	5.91e-02	3.05e-03	9.59e-02	6.38e-02
1/32	4.61e-04	1.47e-02	2.09e-02	4.47e-04	1.47e-02	2.056e-02	7.59e-04	4.75e-02	2.21e-02
1/64	1.08e-04	6.45e-03	7.13e-03	1.05e-04	6.39e-03	7.00e-03	1.88e-04	2.36e-02	7.73e-03
order	2.09	1.21	1.55	2.09	1.20	1.56	2.01	1.00	1.52

Table 2. Numerical errors, $\boldsymbol{b} = \boldsymbol{b}_2$

	$a = 0.255$	$b = 0.25$	$(\rho = 1)$	$a = 0.85$	$b = 0.8$	$(\rho = 1)$	$a = 5.01$	$b = 5.0$	$(\rho = 1)$
h	$E^{\boldsymbol{u}}_{L^2}$	$E^{\boldsymbol{u}}_{H^1}$	$E^{p}_{L^2}$	$E^{\boldsymbol{u}}_{L^2}$	$E^{\boldsymbol{u}}_{H^1}$	$E^{p}_{L^2}$	$E^{\boldsymbol{u}}_{L^2}$	$E^{\boldsymbol{u}}_{H^1}$	$E^{p}_{L^2}$
1/4	2.04e-02	2.11e-01	3.76e-01	1.98e-02	2.08e-01	3.77e-01	3.65e-02	3.67e-01	3.89e-01
1/8	7.04e-03	8.92e-02	1.62e-01	6.84e-03	8.81e-02	1.62e-01	1.18e-02	1.94e-01	1.77e-01
1/16	1.84e-03	3.53e-02	5.96e-02	1.79e-03	3.49e-02	5.91e-02	3.05e-03	9.59e-02	6.38e-02
1/32	4.52e-04	1.48e-02	2.09e-02	4.38e-04	1.47e-02	2.06e-02	7.59e-04	4.75e-02	2.21e-02
1/64	1.06e-04	6.45e-03	7.13e-03	1.03e-04	6.39e-03	7.00e-03	1.88e-04	2.36e-02	7.73e-03
order	2.09	1.20	1.55	2.09	1.20	1.57	2.01	1.01	1.52

Table 3. Numerical errors, $\boldsymbol{b} = \boldsymbol{b}_3$

	$a = 0.255$	$b = 0.25$	$(\rho = 1)$	$a = 0.85$	$b = 0.8$	$(\rho = 1)$	$a = 5.01$	$b = 5.0$	$(\rho = 1)$
h	$E^{\boldsymbol{u}}_{L^2}$	$E^{\boldsymbol{u}}_{H^1}$	$E^{p}_{L^2}$	$E^{\boldsymbol{u}}_{L^2}$	$E^{\boldsymbol{u}}_{H^1}$	$E^{p}_{L^2}$	$E^{\boldsymbol{u}}_{L^2}$	$E^{\boldsymbol{u}}_{H^1}$	$E^{p}_{L^2}$
1/4	2.10e-02	2.19e-01	3.77e-01	2.02e-02	2.15e-01	3.78e-01	3.65e-02	3.67e-01	3.89e-01
1/8	7.31e-03	9.32e-02	1.63e-01	7.05e-03	9.17e-02	1.63e-01	1.18e-02	1.94e-01	1.76e-01
1/16	1.93e-03	3.67e-02	6.10e-02	1.86e-03	3.61e-02	6.03e-02	3.05e-03	9.58e-02	6.37e-02
1/32	4.74e-04	1.53e-02	2.15e-02	4.57e-04	1.51e-02	2.11e-02	7.60e-04	4.75e-02	2.21e-02
1/64	1.12e-04	6.63e-03	7.37e-03	1.07e-04	6.56e-03	7.21e-03	1.89e-04	2.36e-02	7.72e-03
order	2.09	1.21	1.55	2.09	1.20	1.55	2.01	1.00	1.52

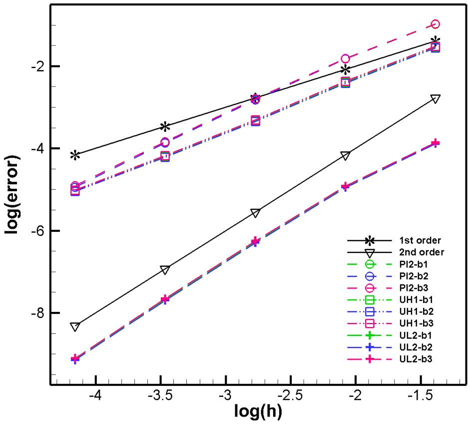


Fig. 2. Numerical convergence orders for velocity and pressure, (C1)

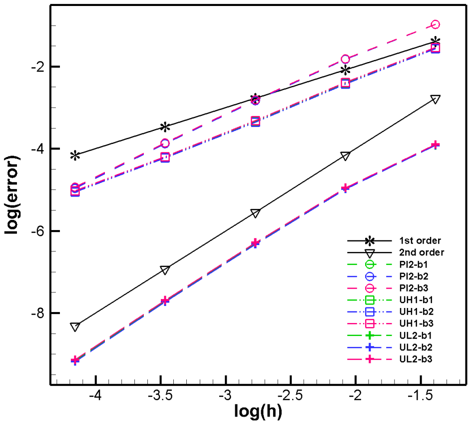


Fig. 3. Numerical convergence orders for velocity and pressure, (C2)

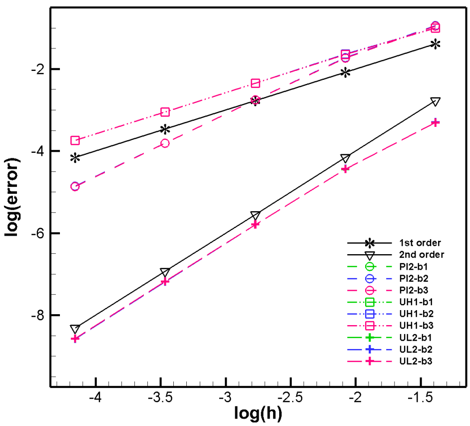
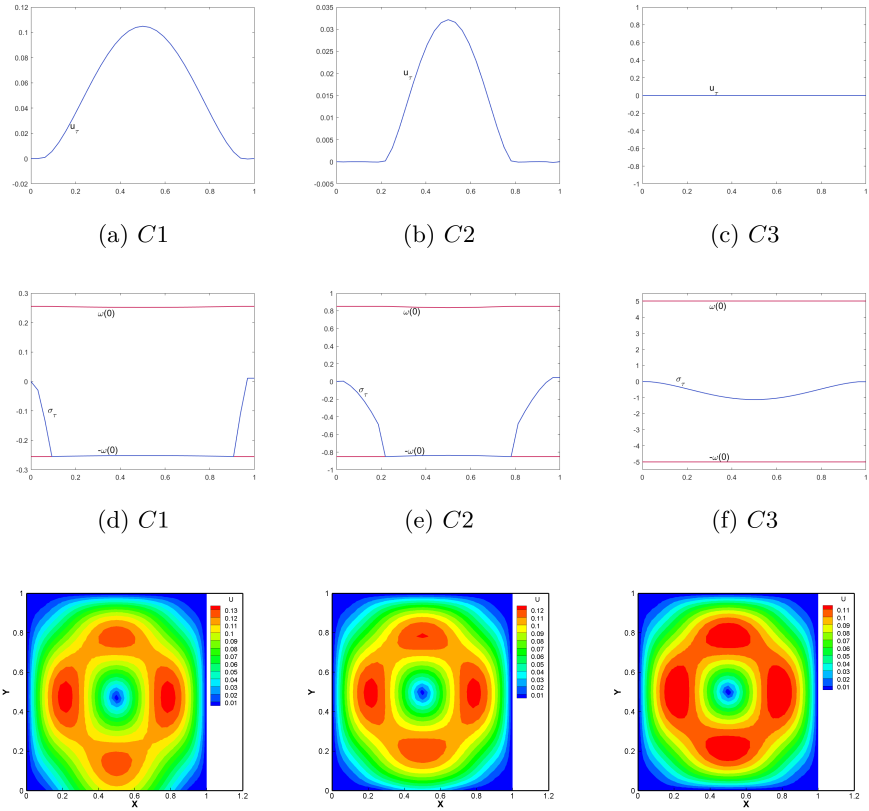


Fig. 4. Numerical convergence orders for velocity and pressure, (C3)



(a) $C1$ (b) $C2$ (c) $C3$

(d) $C1$ (e) $C2$ (f) $C3$

(g) $C1$ (h) $C2$ (i) $C3$

Fig. 5. Tangential components $\boldsymbol{u}_\tau$ and $\boldsymbol{\sigma}_\tau$, and $|\boldsymbol{u}|$

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