

Well-posedness and Numerical Analysis of Mixed Variational-hemivariational Inequalities

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Abstract. The paper is devoted to well-posedness analysis and the numerical solution of a family of general elliptic mixed variational-hemivariational inequalities. Various mixed variational equations, mixed variational inequalities and mixed hemivariational inequalities found in the literature are special cases of the mixed variational-hemivariational inequalities. Well-posedness of the mixed variational-hemivariational inequalities and their numerical approximations is studied via the projection iteration technique. This approach in a Hilbert space eliminates the need of abstract theories of pseudomonotone operators used in many references on variational-hemivariational inequalities. Moreover, uniqueness of both unknown variables is established through a more refined setting of the mixed variational-hemivariational inequalities. Error analysis of the numerical methods is presented. The results are applied to the study of a variational-hemivariational inequality of the Stokes equations for incompressible fluid flows subject to slip conditions of frictional type, both monotone and non-monotone. Optimal order error estimates are derived for the use of some stable finite element space pairs under certain solution regularity assumptions. Numerical results are reported demonstrating the theoretical prediction of convergence orders.

Keywords. Mixed variational-hemivariational inequality, well-posedness, projection iteration, numerical method, error estimation, Stokes equations

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1 Introduction

Mixed formulations arise frequently for application problems involving constraints. A well-known example of mixed formulations occurs in the treatment of the incompressibility constraint for fluid

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flow problems. Moreover, mixed formulations are useful in developing efficient numerical methods for the computation of physical quantities other than the original unknown variable of the underlying partial differential equations. To solve mixed problems from applications, numerical methods are needed. There is a large number of publications on the numerical solution of mixed problems. The book [4] is a standard reference on mixed finite element methods for solving a variety of boundary value problems through their mixed formulations. The majority of publications on mixed numerical methods are devoted to solving mixed variational equations. Nevertheless, some references are available on mixed finite element methods for solving mixed variational inequalities, e.g., [6, 13, 22, 33].

While mixed variational inequalities describe mathematical models of non-smooth phenomena in which the non-smooth functional is assumed to be convex, mixed hemivariational inequalities are mathematical models for non-smooth problems in which the non-smooth functional can be non-convex. A number of papers can be found on well-posedness analysis of mixed hemivariational inequalities, or that of more general mixed variational-hemivariational inequalities. In [2, 3, 26, 31], solution existence results for mixed variational-hemivariational inequalities are proved by applying abstract fixed-point principles for set-valued mappings. In contrast, in [18], for a particular mixed variational-hemivariational inequality of rank $(1, 1)$ (following the nomenclature in [16, Section 5.1, Chapter 7], cf. Section 2), a minimax principle is established and the inequality problem is studied through an equivalent saddle-point formulation. In [19], a combination of the results established in [18] and the Banach fixed-point argument leads to well-posedness results of general mixed variational-hemivariational inequalities of rank $(1, 1)$ and rank $(2, 1)$. In [35], a projection iteration technique is developed in the study of a mixed hemivariational inequality, which can be termed a mixed variational-hemivariational inequality of rank $(0, 1)$ following the nomenclature in [16]. In this paper, we extend the projection iteration technique for the well-posedness analysis of general mixed variational-hemivariational inequalities of rank $(1, 1)$ and rank $(2, 1)$. Compared to the results in [18, 19], uniqueness of both unknown variables (uniqueness of both velocity and pressure in applications of incompressible fluid flows) can be established in the abstract level, at the expense of a finer setting of the mixed variational-hemivariational inequalities. We then proceed to perform numerical analysis of the general mixed variational-hemivariational inequalities.

Hemivariational inequalities are useful for applications involving non-smooth, non-monotone and set-valued relations among physical quantities. Since the pioneering work of Panagiotopoulos four decades ago (cf. [32]), modeling, analysis, numerical solution and applications of hemivariational inequalities have attracted more and more attention from the research community. One has witnessed a substantial increase in the number of publications related to hemivariational inequalities in recent years. As representative recent references for well-posedness analysis results of variational-hemivariational inequalities, one is referred to [34] based on applications of abstract surjectivity results on pseudomonotone operators and to [16] for an accessible approach based on basic knowledge from Functional Analysis. One is also referred to [20, 17] for recent surveys on numerical analysis of variational-hemivariational inequalities.

In the study of hemivariational inequalities, we need the notions of the generalized directional derivative and generalized subdifferential in the sense of Clarke (cf. [8]). Let $\Psi: V \rightarrow \mathbb{R}$ be a locally Lipschitz continuous functional defined on a real Banach space V . Then its generalized (Clarke)

directional derivative at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u; v) := \limsup_{w \rightarrow u, \lambda \downarrow 0} \frac{\Psi(w + \lambda v) - \Psi(w)}{\lambda},$$

and the generalized subdifferential of Ψ at $u \in V$ is defined by

$$\partial\Psi(u) := \{\eta \in V^* \mid \Psi^0(u; v) \geq \langle \eta, v \rangle \ \forall v \in V\}.$$

In the special case where $\Psi: V \rightarrow \mathbb{R}$ is locally Lipschitz continuous and convex, the subdifferential $\partial\Psi(u)$ at any $u \in V$ in the sense of Clarke coincides with the convex subdifferential $\partial\Psi(u)$. Hence, the notion of the Clarke subdifferential can be viewed as a generalization of that of the convex subdifferential. For all $\lambda \in \mathbb{R}$ and all $u \in V$, we have

$$\partial(\lambda \Psi)(u) = \lambda \partial\Psi(u).$$

Moreover, for locally Lipschitz functions $\Psi_1, \Psi_2: V \rightarrow \mathbb{R}$, the inclusion

$$\partial(\Psi_1 + \Psi_2)(u) \subset \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall u \in V \quad (1.1)$$

holds, which is equivalent to the inequality

$$(\Psi_1 + \Psi_2)^0(u; v) \leq \Psi_1^0(u; v) + \Psi_2^0(u; v) \quad \forall u, v \in V. \quad (1.2)$$

Detailed discussions of the generalized directional derivative and the generalized subdifferential for locally Lipschitz continuous functionals, including their properties, can be found in several references, e.g. [8, 30].

The salient features of the paper are that (1) the mixed variational-hemivariational inequalities studied are rather general, and they include a variety of mixed formulations found in the literature as special cases (cf. the comments right after the statement of Problem 2.1 in Section 2); (2) analysis of well-posedness of the mixed variational-hemivariational inequalities and their numerical approximations is carried out by an easily accessible approach, without the need of abstract theories of pseudomonotone operators used in many references on hemivariational inequalities or variational-hemivariational inequalities.

The rest of the paper is organized as follows. In Section 2, we address the well-posedness of mixed variational-hemivariational inequalities, first for those of rank $(2, 1)$, and then for those of rank $(1, 1)$. In Section 3, we provide numerical analysis of the mixed variational-hemivariational inequalities. In Section 4, we apply the results presented in Sections 2 and 3 in the study of a variational-hemivariational inequality for the Stokes equations subject to non-leak slip boundary conditions of friction type. In Section 5, we report computer simulation results on some numerical examples.

2 Well-posedness of mixed variational-hemivariational inequalities

From now on, we let V and Q be two real Hilbert spaces. Their dual spaces are denoted by V^* and Q^* . The symbol $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V^* and V , or between Q^* and Q ; it

should be clear from the context which duality pairing is meant by $\langle \cdot, \cdot \rangle$. Let $K_V \subset V$ and $K_Q \subset Q$. Let $A: V \rightarrow V^*$, $b: V \times Q \rightarrow \mathbb{R}$, $\Phi: V \times V \rightarrow \mathbb{R}$, $\Psi: V \rightarrow \mathbb{R}$, and $f \in V^*$ be given operators and functionals. We consider a general mixed variational-hemivariational inequality.

Problem 2.1 Find $(u, p) \in K_V \times K_Q$ such that

$$\langle Au, v - u \rangle + b(v - u, p) + \Phi(u, v) - \Phi(u, u) + \Psi^0(u; v - u) \geq \langle f, v - u \rangle \quad \forall v \in K_V, \quad (2.1)$$

$$b(u, q - p) \leq 0 \quad \forall q \in K_Q. \quad (2.2)$$

We comment that the form of Problem 2.1 is rather general, and it includes a variety of mixed problems found in the literature as special cases. E.g., the mixed problem studied in [33] corresponds to the choice $\Phi \equiv 0$, $\Psi \equiv 0$, and A linear (for the mixed problem in [33], the right hand side of (2.2) is replaced by a continuous linear functional, which does not cause any complication in the analysis and numerical solution of the problem). For the mixed problem studied in [21] and the Stokes variational inequality studied in [9], A is linear, $\Psi \equiv 0$, $K_V = V$ and $K_Q = Q$, the latter having the implication that (2.2) is reduced to a simpler equation. The problem studied in [23] is simpler than that in [21] in the sense that Φ depends on only one variable. For the problem studied in [12] and [25], A is linear, $\Phi \equiv 0$, $K_V = V$ and $K_Q = Q$. In the very special case where $\Phi \equiv 0$, $\Psi \equiv 0$, $K_V = V$, $K_Q = Q$, and A linear, Problem 2.1 reduces to the commonly seen mixed variational equations extensively studied in the literature (cf. [4]).

In the study of Problem 2.1, we will use the following conditions on the problem data.

- $H(K_V)$ V is a real Hilbert space, $K_V \subset V$ is non-empty, closed and convex, and it contains a subspace V_0 .
- $H(K_Q)$ Q is a real Hilbert space, $K_Q \subset Q$ is non-empty, closed and convex.
- $H(A)$ $A: V \rightarrow V^*$ is Lipschitz continuous and strongly monotone.
- $H(b)$ $b: V \times Q \rightarrow \mathbb{R}$ is bilinear and bounded. Moreover, the inf-sup condition holds: there exists a positive constant $\alpha_b > 0$ such that

$$\alpha_b \|q\|_Q \leq \sup_{w \in V_0} \frac{b(w, q)}{\|w\|_V} \quad \forall q \in Q. \quad (2.3)$$

- $H(\Phi)_2$ $\Phi: V \times V \rightarrow \mathbb{R}$; for any $u \in V$, $\Phi(u, \cdot): V \rightarrow \mathbb{R}$ is convex and continuous; and there exists a constant $\alpha_\Phi \geq 0$ such that

$$\Phi(u_1, v_2) - \Phi(u_1, v_1) + \Phi(u_2, v_1) - \Phi(u_2, v_2) \leq \alpha_\Phi \|u_1 - u_2\| \|v_1 - v_2\| \quad \forall u_1, u_2, v_1, v_2 \in V. \quad (2.4)$$

Moreover,

$$\Phi(u, v + w) = \Phi(u, v) \quad \forall u, v \in K_V, w \in V_0 \quad (2.5)$$

- $H(\Psi)$ $\Psi: V \rightarrow \mathbb{R}$ is locally Lipschitz continuous, and there exists a constant $\alpha_\Psi \geq 0$ such that

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall v_1, v_2 \in V. \quad (2.6)$$

Moreover,

$$\Psi^0(v; w) = 0 \quad \forall v \in K_V, w \in V_0. \quad (2.7)$$

- $H(f)$ $f \in V^*$.

Following the nomenclature used in [16], Problem 2.1 is a mixed variational-hemivariational inequality of rank $(2, 1)$ since the function Φ depends on two variables and the function Ψ depends on one variable. When $\Psi \equiv 0$, Problem 2.1 is reduced to a mixed variational inequality. When $\Phi \equiv 0$, Problem 2.1 can be viewed as a “pure” mixed hemivariational inequality.

We comment that under the assumption $H(K_V)$, $v + w \in K_V$ for any $v \in K_V$ and any $w \in V_0$ (cf. [33, Lemma 2.1]).

Related to the condition $H(A)$, we will use $M_A > 0$ for the Lipschitz constant:

$$\|Av_1 - Av_2\|_{V^*} \leq M_A \|v_1 - v_2\|_V \quad \forall v_1, v_2 \in V, \quad (2.8)$$

and use $m_A > 0$ for the strong monotonicity constant:

$$\langle Av_1 - Av_2, v_1 - v_2 \rangle \geq m_A \|v_1 - v_2\|_V^2 \quad \forall v_1, v_2 \in V. \quad (2.9)$$

Related to the condition $H(b)$, we will use $M_b > 0$ for the boundedness constant:

$$|b(v, q)| \leq M_b \|v\|_V \|q\|_Q \quad \forall v \in V, q \in Q. \quad (2.10)$$

Assumption $H(b)$ allows us to define an operator $B \in \mathcal{L}(V; Q)$ by the relation

$$(Bv, q)_Q = b(v, q) \quad \forall v \in V, q \in Q.$$

The inequality (2.10) is equivalent to

$$\|B\| \leq M_b.$$

Note that the convex function $\Phi(u, \cdot): V \rightarrow \mathbb{R}$ is assumed to be continuous, instead of l.s.c. As is explained in [14, 15] or [16, Section 5.1], there is no loss of generality with the stronger assumption of continuity for a vast majority of applications. The subscript 2 in $H(\Phi)_2$ reminds the reader that this is a condition for the case where Φ depends on two variables. The constant $\alpha_\Psi \geq 0$ in (2.6) measures the strength of non-convexity of Ψ : the smaller the value of α_Ψ , the weaker the non-convexity of Ψ ; in particular, in the degenerate case where Ψ is convex, (2.6) holds with $\alpha_\Psi = 0$.

The first step in the analysis of Problem 2.1 is to convert the inequality (2.2) in the form of an equality. Let $P_{K_Q}: Q \rightarrow K_Q$ be the projection operator from Q to K_Q .

Lemma 2.2 *Under the assumption $H(b)$, (2.2) is equivalent to*

$$p = P_{K_Q}(p + \rho Bu) \quad \forall \rho > 0. \quad (2.11)$$

Proof. By making use of the operator B , we can rewrite (2.2) as

$$(Bu, q - p)_Q \leq 0 \quad \forall q \in Q.$$

For any $\rho > 0$, this is equivalent to

$$((p + \rho Bu) - p, q - p)_Q \leq 0 \quad \forall q \in Q,$$

i.e., (2.11) holds. ■

Theorem 2.3 Assume $H(K_V)$, $H(K_Q)$, $H(A)$, $H(b)$, $H(\Phi)_2$, $H(\Psi)$, $H(f)$, and

$$\alpha_\Phi + \alpha_\Psi < m_A. \quad (2.12)$$

Then, Problem 2.1 has a unique solution $(u, p) \in K_V \times K_Q$.

Proof. Let $p_0 \in V_Q$ be arbitrary but fixed. The parameter $\rho > 0$ will be chosen sufficiently small later in the proof. Define a sequence $\{(u_n, p_n)\}_{n \geq 0} \subset K_V \times K_Q$ by

$$\begin{aligned} \langle Au_n, v - u_n \rangle + \Phi(u_n, v) - \Phi(u_n, u_n) + \Psi^0(u_n; v - u_n) \\ \geq \langle f, v - u_n \rangle - b(v - u_n, p_n) \quad \forall v \in K_V, \end{aligned} \quad (2.13)$$

$$p_{n+1} = P_{K_Q}(p_n + \rho Bu_n). \quad (2.14)$$

Under the stated assumptions, given $p_n \in K_Q$, the variational-hemivariational inequality (2.13) has a unique solution $u_n \in K_V$ (cf. e.g., [16, Theorem 5.9]). We take $v = u_{n-1}$ in (2.13) and take $v = u_n$ in (2.13) with n replaced by $(n - 1)$:

$$\begin{aligned} \langle Au_n, u_{n-1} - u_n \rangle + \Phi(u_n, u_{n-1}) - \Phi(u_n, u_n) + \Psi^0(u_n; u_{n-1} - u_n) \\ \geq \langle f, u_{n-1} - u_n \rangle - b(u_{n-1} - u_n, p_n) \\ \langle Au_{n-1}, u_n - u_{n-1} \rangle + \Phi(u_{n-1}, u_n) - \Phi(u_{n-1}, u_{n-1}) + \Psi^0(u_{n-1}; u_n - u_{n-1}) \\ \geq \langle f, u_n - u_{n-1} \rangle - b(u_n - u_{n-1}, p_{n-1}). \end{aligned}$$

Add the two inequalities to get

$$\begin{aligned} \langle Au_n - Au_{n-1}, u_n - u_{n-1} \rangle &\leq \Phi(u_n, u_{n-1}) - \Phi(u_n, u_n) + \Phi(u_{n-1}, u_n) - \Phi(u_{n-1}, u_{n-1}) \\ &\quad + \Psi^0(u_n; u_{n-1} - u_n) + \Psi^0(u_{n-1}; u_n - u_{n-1}) \\ &\quad - b(u_n - u_{n-1}, p_n - p_{n-1}). \end{aligned}$$

Applying $H(A)$, $H(\Phi)_2$ and $H(\Psi)$,

$$m_A \|u_n - u_{n-1}\|_V^2 \leq \alpha_\Phi \|u_n - u_{n-1}\|_V^2 + \alpha_\Psi \|u_n - u_{n-1}\|_V^2 - b(u_n - u_{n-1}, p_n - p_{n-1}).$$

Hence,

$$(m_A - \alpha_\Phi - \alpha_\Psi) \|u_n - u_{n-1}\|_V^2 \leq -b(u_n - u_{n-1}, p_n - p_{n-1}). \quad (2.15)$$

We derive from (2.15) that

$$(m_A - \alpha_\Phi - \alpha_\Psi) \|u_n - u_{n-1}\|_V^2 \leq M_b \|u_n - u_{n-1}\|_V \|p_n - p_{n-1}\|_Q,$$

i.e.,

$$\|u_n - u_{n-1}\|_V \leq c_0 \|p_n - p_{n-1}\|_Q, \quad c_0 := \frac{M_b}{m_A - \alpha_\Phi - \alpha_\Psi}. \quad (2.16)$$

Next, from the definition (2.14) for p_{n+1} and p_n , and the non-expansiveness of P_{K_Q} , we have

$$\begin{aligned} \|p_{n+1} - p_n\|_Q^2 &\leq \|(p_n + \rho B u_n) - (p_{n-1} + \rho B u_{n-1})\|_Q^2 \\ &= \|p_n - p_{n-1}\|_Q^2 + \rho^2 \|B(u_n - u_{n-1})\|_Q^2 + 2\rho b(u_n - u_{n-1}, p_n - p_{n-1}). \end{aligned}$$

From (2.15),

$$b(u_n - u_{n-1}, p_n - p_{n-1}) \leq -(m_A - \alpha_\Phi - \alpha_\Psi) \|u_n - u_{n-1}\|_V^2.$$

Also,

$$\|B(u_n - u_{n-1})\|_Q \leq M_b \|u_n - u_{n-1}\|_V.$$

Thus,

$$\begin{aligned} \|p_{n+1} - p_n\|_Q^2 &\leq \|p_n - p_{n-1}\|_Q^2 + \rho^2 M_b^2 \|u_n - u_{n-1}\|_V^2 - 2\rho (m_A - \alpha_\Phi - \alpha_\Psi) \|u_n - u_{n-1}\|_V^2 \\ &= \|p_n - p_{n-1}\|_Q^2 - \theta \|u_n - u_{n-1}\|_V^2, \end{aligned}$$

where

$$\theta = 2\rho (m_A - \alpha_\Phi - \alpha_\Psi) - \rho^2 M_b^2.$$

Choose $\rho > 0$ sufficiently small so that $\theta > 0$. Then,

$$\|p_{n+1} - p_n\|_Q^2 \leq \|p_n - p_{n-1}\|_Q^2 - \theta \|u_n - u_{n-1}\|_V^2. \quad (2.17)$$

By making use of the assumptions (2.5) and (2.7), we derive from (2.13) the following equality

$$\langle A u_n, w \rangle + b(w, p_n) = \langle f, w \rangle \quad \forall w \in V_0. \quad (2.18)$$

Hence,

$$b(w, p_n - p_{n-1}) = -\langle A u_n - A u_{n-1}, w \rangle \quad \forall w \in V_0.$$

Applying the inf-sup condition (2.3),

$$\alpha_b \|p_n - p_{n-1}\|_Q \leq \sup_{w \in V_0} \frac{b(w, p_n - p_{n-1})}{\|w\|_V} = \sup_{w \in V_0} \frac{-\langle A u_n - A u_{n-1}, w \rangle}{\|w\|_V} \leq M_A \|u_n - u_{n-1}\|_V,$$

or,

$$\|u_n - u_{n-1}\|_V \geq \frac{\alpha_b}{M_A} \|p_n - p_{n-1}\|_Q. \quad (2.19)$$

Combining (2.17) and (2.19),

$$\|p_{n+1} - p_n\|_Q \leq \kappa \|p_n - p_{n-1}\|_Q,$$

where $\kappa := (1 - \theta \alpha_b^2 / M_A^2)^{1/2}$ is well-defined for $\rho > 0$ sufficiently small, and $\kappa < 1$. Then,

$$\|p_{n+1} - p_n\|_Q \leq \kappa^n \|p_1 - p_0\|_Q. \quad (2.20)$$

For $m > n$, we apply (2.20) repeatedly to derive

$$\begin{aligned} \|p_m - p_n\|_Q &\leq \|p_m - p_{m-1}\|_Q + \cdots + \|p_{n+1} - p_n\|_Q \\ &\leq (\kappa^{m-1} + \cdots + \kappa^n) \|p_1 - p_0\|_Q \\ &\leq \frac{\kappa^n}{1 - \kappa} \|p_1 - p_0\|_Q. \end{aligned}$$

Since $\kappa < 1$, we have

$$\|p_m - p_n\|_Q \rightarrow 0 \quad \text{as } m, n \rightarrow \infty,$$

i.e., $\{p_n\}$ is a Cauchy sequence. Since Q is complete, there exists an element $p \in Q$ such that

$$p_n \rightarrow p \quad \text{in } Q.$$

Moreover, since K_Q is closed, the limit $p \in K_Q$.

By (2.16) and (2.20),

$$\|u_n - u_{n-1}\|_V \leq c_0 \kappa^{n-1} \|p_1 - p_0\|_Q.$$

Hence, for $m > n$,

$$\|u_m - u_n\|_V \leq \|u_m - u_{m-1}\|_V + \cdots + \|u_{n+1} - u_n\|_V \leq \frac{c_0 \kappa^n}{1 - \kappa} \|p_1 - p_0\|_Q.$$

So $\{u_n\}$ is a Cauchy sequence in the complete space V , and thus has a limit $u \in V$. Since $u_n \in K_V$ and K_V is closed, $u \in K_V$.

Note that

$$\begin{aligned} &|(\Phi(u_n, v) - \Phi(u_n, u_n)) - (\Phi(u, v) - \Phi(u, u))| \\ &\leq |\Phi(u_n, v) - \Phi(u_n, u_n) - \Phi(u, v) + \Phi(u, u_n)| + |\Phi(u, u) - \Phi(u, u_n)| \\ &\leq \alpha_\Phi \|u_n - u\|_V \|v - u_n\|_V + |\Phi(u, u) - \Phi(u, u_n)| \\ &\rightarrow 0, \end{aligned}$$

and (cf. [30, Proposition 3.23 (ii)])

$$\limsup_{n \rightarrow \infty} \Psi^0(u_n; v - u_n) \leq \Psi^0(u; v - u).$$

We take the upper limit of both sides of (2.13) to obtain (2.1).

Due to the continuity of P_{K_Q} , we take the limit in (2.14) to obtain

$$p = P_{K_Q}(p + \rho Bu).$$

By Lemma 2.2, this implies (2.2).

Hence, the limit $(u, p) \in K_V \times K_Q$ is a solution of Problem 2.1.

For uniqueness, let $(u_1, p_1) \in K_V \times K_Q$ and $(u_2, p_2) \in K_V \times K_Q$ be two solutions of Problem 2.1. Then,

$$\langle Au_1, u_2 - u_1 \rangle + b(u_2 - u_1, p_1) + \Phi(u_1, u_2) - \Phi(u_1, u_1) + \Psi^0(u_1; u_2 - u_1) \geq \langle f, u_2 - u_1 \rangle, \quad (2.21)$$

$$b(u_1, p_2 - p_1) \leq 0 \quad (2.22)$$

and

$$\langle Au_2, u_1 - u_2 \rangle + b(u_1 - u_2, p_2) + \Phi(u_2, u_1) - \Phi(u_2, u_2) + \Psi^0(u_2; u_1 - u_2) \geq \langle f, u_1 - u_2 \rangle, \quad (2.23)$$

$$b(u_2, p_1 - p_2) \leq 0. \quad (2.24)$$

Add (2.22) and (2.24) to get

$$-b(u_1 - u_2, p_1 - p_2) \leq 0. \quad (2.25)$$

Then, add (2.21) and (2.23) to get

$$\begin{aligned} \langle Au_1 - Au_2, u_1 - u_2 \rangle &\leq \Phi(u_1, u_2) - \Phi(u_1, u_1) + \Phi(u_2, u_1) - \Phi(u_2, u_2) \\ &\quad + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2) - b(u_1 - u_2, p_1 - p_2). \end{aligned} \quad (2.26)$$

Hence, by making use of $H(A)$, $H(\Phi)_2$, $H(\Psi)$ and (2.25),

$$m_A \|u_1 - u_2\|_V^2 \leq \alpha_\Phi \|u_1 - u_2\|_V^2 + \alpha_\Psi \|u_1 - u_2\|_V^2, \quad (2.27)$$

i.e.,

$$(m_A - \alpha_\Phi - \alpha_\Psi) \|u_1 - u_2\|_V^2 \leq 0.$$

By the smallness condition (2.12), we deduce that $\|u_1 - u_2\|_V = 0$ and $u_1 = u_2$.

Similar to (2.19), we have

$$\|p_1 - p_2\|_Q \leq \frac{\alpha_b}{M_A} \|u_1 - u_2\|_V = 0,$$

implying the uniqueness of p : $p_1 = p_2$. ■

Now we show Lipschitz continuity of the solution with respect to the right side.

Theorem 2.4 *Keep the assumptions stated in Theorem 2.3. Then, there exists a constant $c > 0$ such that for solutions $(u_1, p_1), (u_2, p_2) \in K_V \times K_Q$ of Problem 2.1 with $f = f_1, f_2 \in V^*$, the following inequality holds:*

$$\|u_1 - u_2\|_V + \|p_1 - p_2\|_Q \leq c \|f_1 - f_2\|_{V^*}. \quad (2.28)$$

Proof. We start with (2.21)–(2.24) with f replaced by f_1 in (2.21) and by f_2 in (2.23). Then, (2.26) is modified to

$$\begin{aligned} \langle Au_1 - Au_2, u_1 - u_2 \rangle &\leq \Phi(u_1, u_2) - \Phi(u_1, u_1) + \Phi(u_2, u_1) - \Phi(u_2, u_2) \\ &\quad + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2) \\ &\quad - b(u_1 - u_2, p_1 - p_2) + \langle f_1 - f_2, u_1 - u_2 \rangle \end{aligned}$$

whereas (2.27) is modified to

$$m_A \|u_1 - u_2\|_V^2 \leq \alpha_\Phi \|u_1 - u_2\|_V^2 + \alpha_\Psi \|u_1 - u_2\|_V^2 + \|f_1 - f_2\|_{V^*} \|u_1 - u_2\|_V.$$

Then

$$\|u_1 - u_2\|_V \leq \frac{1}{m_A - \alpha_\Phi - \alpha_\Psi} \|f_1 - f_2\|_{V^*}. \quad (2.29)$$

Similar to (2.18), we have

$$\begin{aligned} \langle Au_1, w \rangle + b(w, p_1) &= \langle f_1, w \rangle \quad \forall w \in V_0, \\ \langle Au_2, w \rangle + b(w, p_2) &= \langle f_2, w \rangle \quad \forall w \in V_0. \end{aligned}$$

Subtract the two equations to get

$$b(w, p_1 - p_2) = \langle f_1 - f_2, w \rangle - \langle Au_1 - Au_2, w \rangle \quad \forall w \in V_0.$$

By the inf-sup condition (2.3),

$$\alpha_b \|p_1 - p_2\|_Q \leq \sup_{w \in V_0} \frac{1}{\|w\|_V} [\langle f_1 - f_2, w \rangle - \langle Au_1 - Au_2, w \rangle] \leq \|f_1 - f_2\|_{V^*} + M_A \|u_1 - u_2\|_V.$$

This inequality and (2.29) together imply (2.28). ■

We now introduce a variant of Problem 2.1 which is directly related to applications. To simplify the notation, for Δ a subset of Ω or a subset of its boundary, we use $I_\Delta(g)$ to denote the integral of an integrable function g on Δ . For application problems, very often, the mixed variational-hemivariational inequality of rank $(2, 1)$ is in the form of Problem 2.1 in which $\Psi^0(u; v - u)$ is replaced by an integral over Δ : $I_\Delta(\psi^0(\gamma u; \gamma v - \gamma u))$. Here, for a positive integer m , $\psi: \mathbb{R}^m \rightarrow \mathbb{R}$ is locally Lipschitz continuous and for a function space V_ψ on Δ , the operator $\gamma: V \rightarrow V_\psi$ is such that $\psi^0(\gamma u; \gamma v) \in L^1(\Delta)$ for all $u, v \in V$. In other words, for application problems, instead of Problem 2.1, we have its following variant.

Problem 2.5 Find $(u, p) \in K_V \times K_Q$ such that

$$\begin{aligned} \langle Au, v - u \rangle + b(v - u, p) + \Phi(u, v) - \Phi(u, u) + I_\Delta(\psi^0(\gamma u; \gamma v - \gamma u)) \\ \geq \langle f, v - u \rangle \quad \forall v \in K_V, \end{aligned} \quad (2.30)$$

$$b(u, q - p) \leq 0 \quad \forall q \in K_Q. \quad (2.31)$$

For mixed variational-hemivariational inequalities in applications, associated with second-order PDEs, we let $V_\psi = L^2(\Delta)$, and denote by $c_\Delta > 0$ the smallest constant in the inequality

$$I_\Delta(|\gamma v|_{\mathbb{R}^m}^2) \leq c_\Delta^2 \|v\|_V^2 \quad \forall v \in V. \quad (2.32)$$

We replace $H(\Psi)$ by the following assumptions on ψ .

- $H(\psi)$ For an integer $m \geq 1$, $\psi: \mathbb{R}^m \rightarrow \mathbb{R}$ is locally Lipschitz continuous, and there exist constants $c_\psi \geq 0$ and $\alpha_\psi \geq 0$ such that

$$|\partial\psi(z)|_{\mathbb{R}^m} \leq c_\psi (1 + |z|_{\mathbb{R}^m}) \quad \forall z \in \mathbb{R}^m, \quad (2.33)$$

$$\psi^0(z_1; z_2 - z_1) + \psi^0(z_2; z_1 - z_2) \leq \alpha_\psi |z_1 - z_2|_{\mathbb{R}^m}^2 \quad \forall z_1, z_2 \in \mathbb{R}^m. \quad (2.34)$$

We deduce from Theorems 2.3 and 2.4 the next result on Problem 2.5.

Theorem 2.6 *Assume $H(K_V)$, $H(K_Q)$, $H(A)$, $H(b)$, $H(\Phi)_2$, $H(\psi)$, $H(f)$, and*

$$\alpha_\Phi + \alpha_\psi c_\Delta^2 < m_A. \quad (2.35)$$

Then, Problem 2.5 has a unique solution $(u, p) \in K_V \times K_Q$. Moreover, there exists a constant $c > 0$ such that for solutions $(u_1, p_1), (u_2, p_2) \in K_V \times K_Q$ of Problem 2.5 with $f = f_1, f_2 \in V^*$, the following inequality holds:

$$\|u_1 - u_2\|_V + \|p_1 - p_2\|_Q \leq c \|f_1 - f_2\|_{V^*}.$$

Proof. Introduce an auxiliary functional $\Psi: V \rightarrow \mathbb{R}$ by

$$\Psi(v) = I_\Delta(\psi(\gamma v)), \quad v \in V.$$

Then, by [30, Section 3.3], Ψ is well-defined and is locally Lipschitz continuous on V . Moreover,

$$\Psi^0(u; v) \leq I_\Delta(\psi^0(\gamma u; \gamma v)) \quad \forall u, v \in V, \quad (2.36)$$

and

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \alpha_\psi c_\Delta^2 \|v_1 - v_2\|_V^2 \quad \forall v_1, v_2 \in V.$$

Hence, $H(\psi)$ implies $H(\Psi)$ with $\alpha_\Psi = \alpha_\psi c_\Delta^2$. From Theorems 2.3 and 2.4, we know that Problem 2.1 has a unique solution $(u, p) \in K_V \times K_Q$ and the solution depends Lipschitz continuously on f . Thanks to the property (2.36), (u, p) is also a solution of Problem 2.5. Moreover, the solution uniqueness of Problem 2.5 can be shown by slightly modifying the argument of the solution uniqueness of Problem 2.1 in the last part of the proof of Theorem 2.3. Consequently, the statements of Theorem 2.6 hold regarding Problem 2.5. $\blacksquare$

Finally, we consider degenerate special cases of Problem 2.1 and Problem 2.5 where Φ depends on only one variable. Let $\Phi: V \rightarrow \mathbb{R}$.

Problem 2.7 *Find $(u, p) \in K_V \times K_Q$ such that*

$$\langle Au, v - u \rangle + b(v - u, p) + \Phi(v) - \Phi(u) + \Psi^0(u; v - u) \geq \langle f, v - u \rangle \quad \forall v \in K_V, \quad (2.37)$$

$$b(u, q - p) \leq 0 \quad \forall q \in K_Q. \quad (2.38)$$

By the nomenclature used in [16], Problem 2.7 is called a mixed variational-hemivariational inequality of rank $(1, 1)$ since both the functions Φ and Ψ depend on one variable. In the study of Problem 2.7, we replace the assumption $H(\Phi)_2$ by $H(\Phi)_1$.

- $H(\Phi)_1$ $\Phi: V \rightarrow \mathbb{R}$ is convex and continuous, and

$$\Phi(v + w) = \Phi(v) \quad \forall v \in K_V, w \in V_0. \quad (2.39)$$

As a corollary of Theorems 2.3 and 2.4, we have the next result on Problem 2.7.

Theorem 2.8 *Assume $H(K_V)$, $H(K_Q)$, $H(A)$, $H(b)$, $H(\Phi)_1$, $H(\Psi)$, $H(f)$, and $\alpha_\Psi < m_A$. Then, Problem 2.7 has a unique solution $(u, p) \in K_V \times K_Q$. Moreover, there exists a constant $c > 0$ such that for solutions $(u_1, p_1), (u_2, p_2) \in K_V \times K_Q$ of Problem 2.7 with $f = f_1, f_2 \in V^*$, the following inequality holds:*

$$\|u_1 - u_2\|_V + \|p_1 - p_2\|_Q \leq c \|f_1 - f_2\|_{V^*}.$$

Similar to Problem 2.5, we introduce a variant of Problem 2.7.

Problem 2.9 *Find $(u, p) \in K_V \times K_Q$ such that*

$$\langle Au, v - u \rangle + b(v - u, p) + \Phi(v) - \Phi(u) + I_\Delta(\psi^0(\gamma u; \gamma v - \gamma u)) \geq \langle f, v - u \rangle \quad \forall v \in K_V, \quad (2.40)$$

$$b(u, q - p) \leq 0 \quad \forall q \in K_Q. \quad (2.41)$$

Theorem 2.10 *Assume $H(K_V)$, $H(K_Q)$, $H(A)$, $H(b)$, $H(\Phi)_1$, $H(\psi)$, $H(f)$, and $\alpha_\psi c_\Delta^2 < m_A$. Then, Problem 2.9 has a unique solution $(u, p) \in K_V \times K_Q$. Moreover, there exists a constant $c > 0$ such that for solutions $(u_1, p_1), (u_2, p_2) \in K_V \times K_Q$ of Problem 2.5 with $f = f_1, f_2 \in V^*$, the following inequality holds:*

$$\|u_1 - u_2\|_V + \|p_1 - p_2\|_Q \leq c \|f_1 - f_2\|_{V^*}.$$

3 Numerical analysis

In this section, we consider the numerical solution of Problem 2.5. We keep the assumptions stated in Theorem 2.6 so that Problem 2.5 has a unique solution. Let $K_V^h \subset K_V$ and $K_Q^h \subset K_Q$ be finite dimensional approximations of K_V and K_Q . Then, the numerical method is the following.

Problem 3.1 *Find $(u^h, p^h) \in K_V^h \times K_Q^h$ such that*

$$\begin{aligned} \langle Au^h, v^h - u^h \rangle + b(v^h - u^h, p^h) + \Phi(u^h, v^h) - \Phi(u^h, u^h) + I_\Delta(\psi^0(\gamma u^h; \gamma v^h - \gamma u^h)) \\ \geq \langle f, v^h - u^h \rangle \quad \forall v^h \in K_V^h, \end{aligned} \quad (3.1)$$

$$b(u^h, q^h - p^h) \leq 0 \quad \forall q^h \in K_Q^h. \quad (3.2)$$

Denote

$$V_0^h = V_0 \cap K_V^h$$

and assume the finite element space pair $V^h \times Q^h$ satisfies the discrete inf-sup condition (Babuška-Brezzi condition, or simply BB condition)

$$\alpha_b \|q^h\|_Q \leq \sup_{w^h \in V_0^h} \frac{b(w^h, q^h)}{\|w^h\|_V}, \quad (3.3)$$

where $\alpha_b > 0$ is a constant independent of h .

By a discrete analogue of Theorem 2.6, we know that under the assumptions stated in Theorem 2.6 and (3.3), Problem 3.1 has a unique solution $(u^h, p^h) \in K_V^h \times K_Q^h$.

The rest of the section is devoted to an error analysis of the numerical method. We will use the Young inequality several times:

$$a b \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2 \quad \forall \epsilon > 0, \quad a, b \in \mathbb{R}. \quad (3.4)$$

Theorem 3.2 *Keep the assumptions stated in Theorem 2.6 and (3.3), and let $(u^h, p^h) \in K_V^h \times K_Q^h$ be the unique solution of Problem 3.1. For any $(v^h, q^h) \in K_V^h \times K_Q^h$, we have the error bound*

$$\|u - u^h\|_V^2 + \|p - p^h\|_Q^2 \leq c_1 I_R + c (\|u - v^h\|_V^2 + \|\gamma(u - v^h)\|_{V_\psi} + \|p - q^h\|_Q^2 + \|p - q^h\|_Q). \quad (3.5)$$

Furthermore, if K_Q and K_Q^h are subspaces, for any $(v^h, q^h) \in K_V^h \times K_Q^h$, we have the error bound

$$\|u - u^h\|_V^2 + \|p - p^h\|_Q^2 \leq c_1 I_R + c (\|u - v^h\|_V^2 + \|\gamma(u - v^h)\|_{V_\psi} + \|p - q^h\|_Q^2). \quad (3.6)$$

Proof. Let $(v^h, q^h) \in K_V^h \times K_Q^h$ be arbitrary. Take $v = u^h$ in (2.1) and $q = p^h$ in (2.2) to get

$$\langle Au, u^h - u \rangle + b(u^h - u, p) + \Phi(u, u^h) - \Phi(u, u) + I_\Delta(\psi^0(\gamma u; \gamma u^h - \gamma u)) \geq \langle f, u^h - u \rangle, \quad (3.7)$$

$$b(u, p^h - p) \leq 0. \quad (3.8)$$

Write

$$\langle Au - Au^h, u - u^h \rangle = \langle Au, u - u^h \rangle + \langle Au^h, u^h - v^h \rangle + \langle Au, v^h - u \rangle.$$

Use (3.7) and (3.1) to obtain

$$\langle Au - Au^h, u - u^h \rangle = I_A + I_R + I_b + I_\Phi + I_\psi, \quad (3.9)$$

where

$$I_A = \langle Au - Au^h, u - v^h \rangle, \quad (3.10)$$

$$I_R = \langle Au, v^h - u \rangle + b(v^h - u, p) + \Phi(u, v^h) - \Phi(u, u) + I_\Delta(\psi^0(\gamma u; \gamma v^h - \gamma u)) - \langle f, v^h - u \rangle, \quad (3.11)$$

$$I_b = b(u^h - v^h, p - p^h), \quad (3.12)$$

$$I_\Phi = \Phi(u, u^h) - \Phi(u, v^h) + \Phi(u^h, v^h) - \Phi(u^h, u^h), \quad (3.13)$$

$$I_\psi = I_\Delta(\psi^0(\gamma u; \gamma u^h - \gamma u) + \psi^0(\gamma u^h; \gamma u - \gamma u^h) + \psi^0(\gamma u; \gamma u - \gamma v^h) + \psi^0(\gamma u^h; \gamma v^h - \gamma u)). \quad (3.14)$$

Note that I_R is a residual-type quantity.

We now bound the terms on the right side of (3.9). For I_A , we use the Lipschitz continuity of A :

$$I_A \leq M_A \|u - u^h\|_V \|u - v^h\|_V.$$

For I_Φ , apply (2.4) followed by the use of the triangle inequality:

$$I_\Phi \leq \alpha_\Phi \|u - u^h\|_V \|u^h - v^h\|_V \leq \alpha_\Phi \|u - u^h\|_V (\|u - u^h\|_V + \|u - v^h\|_V).$$

Then, use (3.4),

$$\alpha_\Phi \|u - u^h\|_V \|u - v^h\|_V \leq \epsilon \|u - u^h\|_V^2 + c \|u - v^h\|_V^2$$

for a constant c depending on ϵ . Hence,

$$I_\Phi \leq (\alpha_\Phi + \epsilon) \|u - u^h\|_V^2 + c_\epsilon \|u - v^h\|_V^2.$$

For I_ψ , we use $H(\psi)$ and (2.32) to get

$$\begin{aligned} I_\psi &\leq \alpha_\psi I_\Delta(|\gamma(u - u^h)|_{V_\psi}^2) + c(1 + \|\gamma u\|_{V_\psi} + \|\gamma u^h\|_{V_\psi}) \|\gamma(u - v^h)\|_{V_\psi} \\ &\leq \alpha_\psi c_\Delta^2 \|u - u^h\|_V^2 + c(1 + \|\gamma u\|_{V_\psi} + \|\gamma(u - u^h)\|_{V_\psi}) \|\gamma(u - v^h)\|_{V_\psi} \\ &\leq \alpha_\psi c_\Delta^2 \|u - u^h\|_V^2 + c(1 + \|\gamma u\|_{V_\psi}) \|\gamma(u - v^h)\|_{V_\psi} + c \|u - u^h\|_V \|u - v^h\|_V. \end{aligned}$$

Apply the modified Cauchy-Schwarz inequality (3.4) to get

$$c \|u - u^h\|_V \|u - v^h\|_V \leq \epsilon \|u - u^h\|_V^2 + c \|u - v^h\|_V$$

for a constant c depending on ϵ on the right side of the inequality. Thus,

$$I_\psi \leq (\alpha_\psi c_\Delta^2 + \epsilon) \|u - u^h\|_V^2 + c(1 + \|\gamma u\|_{V_\psi}) \|\gamma(u - v^h)\|_{V_\psi}.$$

Regarding I_b , write

$$I_b = b(u^h - u, p - q^h) + b(u - v^h, p - p^h) + b(u, p - q^h) + b(u^h, q^h - p^h) + b(u, p^h - p).$$

From (2.2),

$$b(u, p^h - p) \leq 0.$$

From (3.2),

$$b(u^h, q^h - p^h) \leq 0.$$

Hence,

$$I_b \leq b(u^h - u, p - q^h) + b(u - v^h, p - p^h) + b(u, p - q^h), \quad (3.15)$$

from which,

$$I_b \leq M_b (\|u^h - u\|_V \|p - q^h\|_Q + \|u - v^h\|_V \|p - p^h\|_Q + \|u\|_V \|p - q^h\|_Q). \quad (3.16)$$

Applying (3.4), we then have

$$I_b \leq \epsilon (\|u - u^h\|_V^2 + \|p - p^h\|_Q^2) + c (\|u - v^h\|_V^2 + \|p - q^h\|_Q^2 + \|p - q^h\|_Q).$$

Summarizing, from (3.9) and the above bounds, we derive that

$$(m_A - \alpha_\Phi - 3\epsilon) \|u - u^h\|_V^2 \leq \epsilon \|p - p^h\|_Q^2 + I_R + c (\|u - v^h\|_V^2 + \|\gamma(u - v^h)\|_{V_\psi} + \|p - q^h\|_Q^2 + \|p - q^h\|_Q). \quad (3.17)$$

Turning to an error estimate on $p - p^h$, for any $q^h \in Q^h$, by the triangle inequality,

$$\|p - p^h\|_Q \leq \|p - q^h\|_Q + \|q^h - p^h\|_Q.$$

Thus,

$$\|p - p^h\|_Q^2 \leq 2 \|p - q^h\|_Q^2 + 2 \|q^h - p^h\|_Q^2. \quad (3.18)$$

Apply the BB condition (3.3),

$$\alpha_b \|q^h - p^h\|_Q \leq \sup_{w^h \in V_0^h} \frac{b(w^h, q^h - p^h)}{\|w^h\|_V}.$$

Similar to (2.18), due to the assumptions (2.5) and (2.7), we have the equality

$$\langle Au, w \rangle + b(w, p) = \langle f, w \rangle \quad \forall w \in V_0.$$

Moreover, we derive from (3.1) that

$$\langle Au^h, w^h \rangle + b(w^h, p^h) = \langle f, w^h \rangle \quad \forall w^h \in V_0^h.$$

From the last two equalities, we find that

$$b(w^h, p - p^h) = \langle Au^h - Au, w^h \rangle \quad \forall w^h \in V_0^h.$$

Now,

$$b(w^h, q^h - p^h) = b(w^h, q^h - p) + b(w^h, p - p^h) = b(w^h, q^h - p) + \langle Au^h - Au, w^h \rangle.$$

Hence,

$$\alpha_b \|q^h - p^h\|_Q \leq \sup_{w^h \in V_0^h} \frac{b(w^h, q^h - p) + \langle Au^h - Au, w^h \rangle}{\|w^h\|_V} \leq M_A \|u - u^h\|_V + M_b \|p - q^h\|_Q.$$

This implies

$$\|q^h - p^h\|_Q^2 \leq [(M_A/\alpha_b)^2 + 1] \|u - u^h\|_V^2 + c \|p - q^h\|_Q^2.$$

From the above inequality and (3.18),

$$\|p - p^h\|_Q^2 \leq 2 [(M_A/\alpha_b)^2 + 1] \|u - u^h\|_V^2 + c \|p - q^h\|_Q^2.$$

Together with (3.17), when choosing $\epsilon > 0$ small enough, we have two positive constants $c_1, c > 0$ such that (3.5) holds.

Next, consider the special case where K_Q and K_Q^h are subspaces. Then, (2.2) is equivalent to

$$b(u, q) = 0 \quad \forall q \in K_Q,$$

and (3.2) is equivalent to

$$b(u^h, q^h) = 0 \quad \forall q^h \in K_Q^h.$$

Write

$$I_b = b(u^h - u, p - p^h) + b(u - v^h, p - p^h).$$

Then,

$$b(u^h - u, p - p^h) = b(u^h, p - p^h) = b(u^h, p - q^h) = b(u^h - u, p - q^h),$$

and so instead of (3.15),

$$\begin{aligned} I_b &= b(u^h - u, p - q^h) + b(u - v^h, p - p^h) \\ &\leq M_b (\|u - u^h\|_V \|p - q^h\|_Q + \|u - v^h\|_V \|p - p^h\|_Q), \end{aligned}$$

and instead of (3.16), we have

$$I_b \leq M_b (\|u^h - u\|_V \|p - q^h\|_Q + \|u - v^h\|_V \|p - p^h\|_Q). \quad (3.19)$$

Eventually, (3.5) is improved to (3.6). ■

A crude bound on the term I_R is

$$|I_R| \leq c(1 + \|u\|_Q + \|p\|_Q + \|f\|_{V^*}) \|u - v^h\|_V + c |\Phi(u, v^h) - \Phi(u, u)|.$$

Then we deduce convergence of the numerical solutions from either (3.5) or (3.6).

Corollary 3.3 *Keep the assumptions stated in Theorem 2.6 and (3.3). If $\cup_{h>0} K_V^h$ is dense in K_V with respect to the $\|\cdot\|_V$ -norm and $\cup_{h>0} K_Q^h$ is dense in K_Q with respect to the $\|\cdot\|_Q$ -norm, then we have the convergence*

$$\|u - u^h\|_V + \|p - p^h\|_Q \rightarrow 0 \quad \forall h \rightarrow 0.$$

4 A Stokes variational-hemivariational inequality

In this section, we consider a variational-hemivariational inequality of the Stokes equations for an incompressible fluid flow occupying the domain Ω subject to non-leaking slip boundary conditions of friction types on parts of the boundary. Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be an open bounded Lipschitz domain. Its boundary $\Gamma = \partial\Omega$ is split into three parts: Γ_D , $\Gamma_{S,1}$ and $\Gamma_{S,2}$. The unit outward normal vector exists a.e. on Γ and is denoted by $\boldsymbol{\nu} = (\nu_1, \dots, \nu_d)^T$. For a vector $\mathbf{v}$, its normal component and tangential component on Γ are $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$ and $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$.

Let $\mu > 0$ be the viscosity of the fluid, and denote by $\mathbf{u}$ and p for the unknown velocity and pressure fields of the fluid. Denote

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$$

for the rate of deformation tensor and

$$\boldsymbol{\sigma} = 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) - p \mathbf{I}$$

for the fluid stress tensor, $\mathbf{I}$ being the identity tensor. The normal component and the tangential component of the stress on the boundary are $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$.

Given a source density function $\mathbf{f}: \Omega \rightarrow \mathbb{R}^d$, a convex function $\phi: \Gamma_{S,1} \rightarrow \mathbb{R}$, and a non-convex function $\psi: \Gamma_{S,2} \rightarrow \mathbb{R}$, the pointwise formulation of the problem is the following.

Problem 4.1 Find a velocity field $\mathbf{u}: \Omega \rightarrow \mathbb{R}^d$ and a pressure field $p: \Omega \rightarrow \mathbb{R}$ such that

$$-\operatorname{div}(2\mu \boldsymbol{\varepsilon}(\mathbf{u})) + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (4.1)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (4.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad (4.3)$$

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\phi(\mathbf{u}_\tau) \quad \text{on } \Gamma_{S,1}, \quad (4.4)$$

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial\psi(\mathbf{u}_\tau) \quad \text{on } \Gamma_{S,2}. \quad (4.5)$$

Problem 4.1 will be studied through its formulation. For this purpose, we introduce the following function spaces:

$$V = \{\mathbf{v} \in H^1(\Omega)^d \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D, v_\nu = 0 \text{ on } \Gamma_S\}, \quad (4.6)$$

$$V_{\operatorname{div}} = \{\mathbf{v} \in V \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\}, \quad (4.7)$$

$$V_0 = H_0^1(\Omega)^d, \quad (4.8)$$

$$Q = \{q \in L^2(\Omega) \mid (q, 1)_\Omega = 0\}, \quad (4.9)$$

where $\Gamma_S = \Gamma_{S,1} \cup \Gamma_{S,2}$, and $(q, 1)_\Omega$ stands for the integral of q over Ω . Over the space V , we use the norm $\|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{L^2(\Omega)^{d \times d}}$, which is equivalent to the standard $H^1(\Omega)^d$ norm over V .

The reduced weak formulation of Problem 4.1 is the following.

Problem 4.2 Find $\mathbf{u} \in V_{\operatorname{div}}$ such that

$$\begin{aligned} & \int_\Omega 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}) \, dx + \int_{\Gamma_{S,1}} [\phi(\mathbf{v}_\tau) - \phi(\mathbf{u}_\tau)] \, ds + \int_{\Gamma_{S,2}} \psi^0(\mathbf{u}_\tau; \mathbf{v}_\tau - \mathbf{u}_\tau) \, ds \\ & \geq (\mathbf{f}, \mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in V_{\operatorname{div}}. \end{aligned} \quad (4.10)$$

The full weak formulation of Problem 4.1 that includes the pressure unknown p , is the following.

Problem 4.3 Find $(\mathbf{u}, p) \in V \times Q$ such that

$$\begin{aligned} & \int_\Omega 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}) \, dx - \int_\Omega p \operatorname{div}(\mathbf{v} - \mathbf{u}) \, dx + \int_{\Gamma_{S,1}} [\phi(\mathbf{v}_\tau) - \phi(\mathbf{u}_\tau)] \, ds \\ & + \int_{\Gamma_{S,2}} \psi^0(\mathbf{u}_\tau; \mathbf{v}_\tau - \mathbf{u}_\tau) \, ds \geq (\mathbf{f}, \mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in V, \end{aligned} \quad (4.11)$$

$$\int_\Omega q \operatorname{div} \mathbf{u} \, dx = 0 \quad \forall q \in Q. \quad (4.12)$$

We will apply the theoretical results in the last two sections to study Problem 4.3 and its numerical approximation. Let $K_V = V$, $K_Q = Q$, and define for $\mathbf{u}, \mathbf{v} \in V$ and $q \in Q$,

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx, \\ b(\mathbf{v}, q) &= - \int_{\Omega} q \operatorname{div} \mathbf{v} \, dx, \\ \Phi(\mathbf{v}) &= \int_{\Gamma_{S,1}} \phi(\mathbf{v}_{\tau}) \, ds, \\ (\mathbf{f}, \mathbf{v}) &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx. \end{aligned}$$

It is well known that the inf-sup condition (2.3) is satisfied.

For the functions ϕ and ψ , we introduce the following assumptions.

- $H(\phi)$ $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$ is convex and Lipschitz continuous.
- $H(\psi)$ $\psi: \mathbb{R}^d \rightarrow \mathbb{R}$ is locally Lipschitz continuous and for some constants $c_{\psi} \geq 0$ and $\alpha_{\psi} \geq 0$,

$$|\partial\psi(\mathbf{z})|_{\mathbb{R}^d} \leq c_{\psi} (1 + |\mathbf{z}|_{\mathbb{R}^d}) \quad \forall \mathbf{z} \in \mathbb{R}^d, \quad (4.13)$$

$$\psi^0(\mathbf{z}_1; \mathbf{z}_2 - \mathbf{z}_1) + \psi^0(\mathbf{z}_2; \mathbf{z}_1 - \mathbf{z}_2) \leq \alpha_{\psi} |\mathbf{z}_1 - \mathbf{z}_2|_{\mathbb{R}^d}^2 \quad \forall \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^d. \quad (4.14)$$

Theorem 4.4 *Assume $H(\phi)$, $H(\psi)$, and $\alpha_{\psi} < 2\mu\lambda_0$, where $\lambda_0 > 0$ is the smallest eigenvalue of the eigenvalue problem*

$$\mathbf{u} \in V, \quad \int_{\Omega} 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx = \lambda \int_{\Gamma_{S,2}} \mathbf{u}_{\tau} \cdot \mathbf{v}_{\tau} \, ds \quad \forall \mathbf{v} \in V.$$

Then for any $\mathbf{f} \in L^2(\Omega)^d$, both Problem 4.2 and Problem 4.3 admit a unique solution and the two problems are equivalent in the sense that if $\mathbf{u}$ is a solution of Problem 4.2, then there exists p such that $(\mathbf{u}, p)$ solves Problem 4.3, and conversely, if $(\mathbf{u}, p)$ is a solution of Problem 4.3, then $\mathbf{u}$ solves Problem 4.2.

Proof. First, we apply Theorem 2.6 to know that Problem 4.3 has a unique solution. Note that $c_{\Delta}^2 = \lambda_0^{-1}$. By [16, Theorem 5.9], we know Problem 4.2 has a unique solution.

Suppose $(\mathbf{u}, p) \in V \times Q$ is the unique solution of Problem 4.3. Then we take $q = \operatorname{div} \mathbf{u} \in Q$ in (4.12) to see that $\mathbf{u} \in V_{\operatorname{div}}$. By restricting $\mathbf{v}$ to V_{div} , we derive (4.10) from (4.11).

Conversely, suppose $\mathbf{u}$ is the unique solution of Problem 4.2. Then from the previous paragraph, $\mathbf{u}$ is the first component of the unique solution $(\mathbf{u}, p) \in V \times Q$ to Problem 4.3. $\blacksquare$

Turn now to discussion of a finite element approximation of Problem 4.3. For simplicity, assume Ω is a polygonal/polyhedral domain. Consider a regular family of finite element partitions $\{\mathcal{T}^h\}_h$ of $\overline{\Omega}$ into triangular elements (if $d = 2$) or tetrahedral elements (if $d = 3$). We assume that each

partition is compatible to the boundary splitting according to the boundary condition types, i.e., if a side or a face of an element has a nontrivial intersection with Γ_D or $\Gamma_{S,1}$ or $\Gamma_{S,2}$, then the side or the face lies in $\overline{\Gamma_D}$ or $\overline{\Gamma_{S,1}}$ or $\overline{\Gamma_{S,2}}$. Let $V^h \times Q^h \subset V \times Q$ be a stable pair of finite element spaces corresponding to the partition $\mathcal{T}^h$, i.e., for a constant $\beta_b > 0$, independent of h , such that

$$\beta_b \|q^h\|_Q \leq \sup_{\mathbf{v}^h \in V_0^h} \frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_V} \quad \forall q^h \in Q^h, \quad (4.15)$$

where

$$V_0^h = V^h \cap V_0.$$

Then the finite element method for solving Problem 4.3 is the following.

Problem 4.5 Find $(\mathbf{u}^h, p^h) \in V^h \times Q^h$ such that

$$\begin{aligned} & \int_{\Omega} 2\mu \boldsymbol{\varepsilon}(\mathbf{u}^h) : \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{u}^h) dx - \int_{\Omega} p^h \operatorname{div}(\mathbf{v}^h - \mathbf{u}^h) dx + \int_{\Gamma_{S,1}} [\phi(\mathbf{v}_{\tau}^h) - \phi(\mathbf{u}_{\tau}^h)] ds \\ & + \int_{\Gamma_{S,2}} \psi^0(\mathbf{u}_{\tau}^h; \mathbf{v}_{\tau}^h - \mathbf{u}_{\tau}^h) ds \geq (\mathbf{f}, \mathbf{v}^h - \mathbf{u}^h) \quad \forall \mathbf{v}^h \in V^h, \end{aligned} \quad (4.16)$$

$$\int_{\Omega} q^h \operatorname{div} \mathbf{u}^h dx = 0 \quad \forall q^h \in Q^h. \quad (4.17)$$

Under the assumptions stated in Theorem 4.4, we can apply Theorem 2.10 to conclude that Problem 4.5 has a unique solution. The rest of the section focuses on an error analysis of the numerical method. By (3.6),

$$\|\mathbf{u} - \mathbf{u}^h\|_V^2 + \|p - p^h\|_Q^2 \leq c_1 I_R + c (\|\mathbf{u} - \mathbf{v}^h\|_V^2 + \|\mathbf{u}_{\tau} - \mathbf{v}_{\tau}^h\|_{L^2(\Gamma_S)^2} + \|p - q^h\|_Q^2), \quad (4.18)$$

where from (3.11),

$$I_R = a(\mathbf{u}, \mathbf{v}^h - \mathbf{u}) + b(\mathbf{v}^h - \mathbf{u}, p) + I_{\Gamma_{S,1}}(\phi(\mathbf{v}_{\tau}^h) - \phi(\mathbf{u}_{\tau})) + I_{\Gamma_{S,2}}(\psi^0(\mathbf{u}_{\tau}; \mathbf{v}_{\tau}^h - \mathbf{u}_{\tau})) - (\mathbf{f}, \mathbf{v}^h - \mathbf{u}).$$

Let $\{\Gamma_i\}_{1 \leq i \leq i_0}$ be the plat components of $\Gamma_S = \Gamma_{S,1} \cup \Gamma_{S,2}$. Assume the solution regularities:

$$\mathbf{u} \in H^2(\Omega)^d, \quad \mathbf{u}_{\tau}|_{\Gamma_i} \in H^2(\Gamma_i)^d, \quad 1 \leq i \leq i_0, \quad p \in H^1(\Omega). \quad (4.19)$$

As in the proof of Theorem 4.5 in [12], it can be shown that the solution $(\mathbf{u}, p) \in V \times Q$ of Problem 4.3 satisfies the following pointwise relations:

$$-\operatorname{div}(2\mu \boldsymbol{\varepsilon}(\mathbf{u})) + \nabla p = \mathbf{f} \quad \text{a.e. in } \Omega, \quad (4.20)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{a.e. in } \Omega. \quad (4.21)$$

Also, note the boundary values from $\mathbf{u} \in V$ and the definition of the space V :

$$\mathbf{u} = \mathbf{0} \quad \text{a.e. on } \Gamma_D, \quad (4.22)$$

$$u_{\nu} = 0 \quad \text{a.e. on } \Gamma_{S,1} \cup \Gamma_{S,2}. \quad (4.23)$$

Performing an integration by parts and using the above pointwise relations, we have

$$\begin{aligned}
I_R &= \int_{\Gamma} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot (\mathbf{v}^h - \mathbf{u}) \, ds + \int_{\Omega} [-\operatorname{div}(2\mu \boldsymbol{\varepsilon}(\mathbf{u})) + \nabla p - \mathbf{f}] \, dx \\
&\quad + I_{\Gamma_{S,1}}(\phi(\mathbf{v}_{\tau}^h) - \phi(\mathbf{u}_{\tau})) + I_{\Gamma_{S,2}}(\psi^0(\mathbf{u}_{\tau}; \mathbf{v}_{\tau}^h - \mathbf{u}_{\tau})) \\
&= \int_{\Gamma_S} \boldsymbol{\sigma}_{\tau} \cdot (\mathbf{v}_{\tau}^h - \mathbf{u}_{\tau}) \, ds + I_{\Gamma_{S,1}}(\phi(\mathbf{v}_{\tau}^h) - \phi(\mathbf{u}_{\tau})) + I_{\Gamma_{S,2}}(\psi^0(\mathbf{u}_{\tau}; \mathbf{v}_{\tau}^h - \mathbf{u}_{\tau})).
\end{aligned}$$

Then,

$$I_R \leq c \|\mathbf{u}_{\tau} - \mathbf{v}_{\tau}^h\|_{L^2(\Gamma_S)^d}.$$

In conclusion, we derive from (4.18) that

$$\|\mathbf{u} - \mathbf{u}^h\|_V^2 + \|p - p^h\|_Q^2 \leq c (\|\mathbf{u} - \mathbf{v}^h\|_V^2 + \|\mathbf{u}_{\tau} - \mathbf{v}_{\tau}^h\|_{L^2(\Gamma_S)^2}^2 + \|p - q^h\|_Q^2). \quad (4.24)$$

As an example, if we use the P1b/P1 element pair proposed and analyzed in [1], by using the standard finite element approximation theory (cf. [5, 7, 11]), under the solution regularity assumptions (4.19), we obtain from (4.24) the optimal-order error estimate

$$\|\mathbf{u} - \mathbf{u}^h\|_V^2 + \|p - p^h\|_Q^2 \leq c h^2. \quad (4.25)$$

5 Numerical examples

In the numerical examples, we let

$$\phi(\mathbf{v}_{\tau}) = g |\mathbf{v}_{\tau}|,$$

where g is a positive constant, and let

$$\psi(\mathbf{z}) = \int_0^{|\mathbf{z}|} \omega(t) \, dt, \quad \mathbf{z} \in \mathbb{R}^d, \quad \omega(t) = (a - b)e^{-\beta t} + b,$$

where a, b, β are all positive constants, and $a > b$. Since ω is decreasing on $[0, +\infty)$, ψ is not convex. The slip boundary condition on $\Gamma_{S,2}$ is equivalent to

$$|\boldsymbol{\sigma}_{\tau}| \leq \omega(0) \text{ if } \mathbf{u}_{\tau} = \mathbf{0}, \quad -\boldsymbol{\sigma}_{\tau} = \omega(|\mathbf{u}_{\tau}|) \frac{\mathbf{u}_{\tau}}{|\mathbf{u}_{\tau}|} \text{ if } \mathbf{u}_{\tau} \neq \mathbf{0}.$$

Define two Lagrangian multipliers $\boldsymbol{\lambda}_1 = -\boldsymbol{\sigma}_{\tau}/g$, $\boldsymbol{\lambda}_2 = -\boldsymbol{\sigma}_{\tau}/\omega(|\mathbf{u}_{\tau}|)$, elements of the sets

$$\begin{aligned}
\Lambda_1 &= \{\boldsymbol{\lambda}_1 \in L^2(\Gamma_{S,1})^d \mid |\boldsymbol{\lambda}_1| \leq 1 \text{ a.e. on } \Gamma_{S,1}\}, \\
\Lambda_2 &= \{\boldsymbol{\lambda}_2 \in L^2(\Gamma_{S,2})^d \mid |\boldsymbol{\lambda}_2| \leq 1 \text{ a.e. on } \Gamma_{S,2}\},
\end{aligned}$$

respectively, and introduce the following form of the weak formulation of the problem.

Problem 5.1 Find $(\mathbf{u}, p, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \in \mathcal{W} \times \mathcal{Q} \times \Lambda_1 \times \Lambda_2$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) + \int_{\Gamma_{S,1}} g \boldsymbol{\lambda}_1 \cdot \mathbf{v}_\tau ds + \int_{\Gamma_{S,2}} \omega(|\mathbf{u}_\tau|) \boldsymbol{\lambda}_2 \cdot \mathbf{v}_\tau ds &= \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= 0 \quad \forall q \in Q, \\ \boldsymbol{\lambda}_1 \cdot \mathbf{u}_\tau &= |\mathbf{u}_\tau| \quad \text{a.e. on } \Gamma_{S,1}, \\ \boldsymbol{\lambda}_2 \cdot \mathbf{u}_\tau &= |\mathbf{u}_\tau| \quad \text{a.e. on } \Gamma_{S,2}. \end{aligned}$$

We use the P1b/P1 finite element pair for the spatial discretization. It is known that the discrete inf-sup condition is satisfied (cf. [1]). Let $\mathbf{P} : \mathbb{R}^d \rightarrow \overline{B(\mathbf{0}; 1)}$ be the orthogonal projection operator onto the closed unit ball in $\mathbb{R}^d$. We implement the following Uzawa algorithm for the numerical solution of Problem 5.1.

Initialization. Choose a parameter $\rho > 0$, a maximal iteration number l_0 and error tolerance $\varepsilon > 0$, let the approximate initial velocity $\mathbf{u}_0^h = \mathbf{0}$, and set $\boldsymbol{\lambda}_{1,0}^h = \boldsymbol{\lambda}_{2,0}^h = \mathbf{0}$.

Iteration. For $l \geq 1$, find $(\mathbf{u}_l^h, p_l^h) \in \mathbf{V}^h \times Q^h$ such that

$$\begin{aligned} a(\mathbf{u}_l^h, \mathbf{v}^h) - b(\mathbf{v}^h, p_l^h) &= \langle \mathbf{f}, \mathbf{v}^h \rangle - \int_{\Gamma_{S,1}} g \boldsymbol{\lambda}_{1,l-1}^h \cdot \mathbf{v}_\tau^h ds - \int_{\Gamma_{S,2}} \omega(|\mathbf{u}_{l-1,\tau}^h|) \boldsymbol{\lambda}_{2,l-1}^h \cdot \mathbf{v}_\tau^h ds \quad \forall \mathbf{v}^h \in \mathbf{V}^h, \\ b(\mathbf{u}_l^h, q^h) &= 0 \quad \forall q^h \in Q^h \end{aligned}$$

and update

$$\begin{aligned} \boldsymbol{\lambda}_{1,l}^h &= \mathbf{P}(\boldsymbol{\lambda}_{1,l-1}^h + \rho \mathbf{u}_{l,\tau}^h) \quad \text{on } \Gamma_{S,1}, \\ \boldsymbol{\lambda}_{2,l}^h &= \mathbf{P}(\boldsymbol{\lambda}_{2,l-1}^h + \rho \mathbf{u}_{l,\tau}^h) \quad \text{on } \Gamma_{S,2} \end{aligned}$$

until $l = l_0$ or $\|\mathbf{u}_l^h - \mathbf{u}_{l-1}^h\|_{L^2(\Omega)^d} / \|\mathbf{u}_l^h\|_{L^2(\Omega)^d} < \varepsilon$.

The algorithm is implemented in Python with the FEniCS package ([24]). In the numerical examples, we let $\mu = 1$ for the viscosity coefficient, $d = 2$ for the spatial dimension, $\rho = 10$. For the function $\omega(t)$, use $\beta = 1$, $a = 0.35$, $b = 0.25$. For the function $\phi(t)$, use $g = 0.2$. The source function is defined by

$$\mathbf{f}(x, y) = 2 \begin{pmatrix} -10(1 - 6x + 6x^2)y(1 - y)(1 - 2y) + 30x^2(1 - x)^2(1 - 2y) - (1 - 2y) \\ 10(1 - 6y + 6y^2)x(1 - x)(1 - 2x) - 30y^2(1 - y)^2(1 - 2x) - (1 - 2x) \end{pmatrix}.$$

For the stopping criteria, let $l_0 = 200$ and $\varepsilon = 10^{-6}$.

Example 5.1. In the first example, we use uniform triangulations of the domain $\Omega = (0, 1) \times (0, 1)$. The unit interval $[0, 1]$ is split into $1/h$ ($h = 1/8, 1/16, \dots$) equal length sub-intervals. The triangulation of the unit square with $h = 1/16$ is shown in Figure 1. Its boundary is split into three

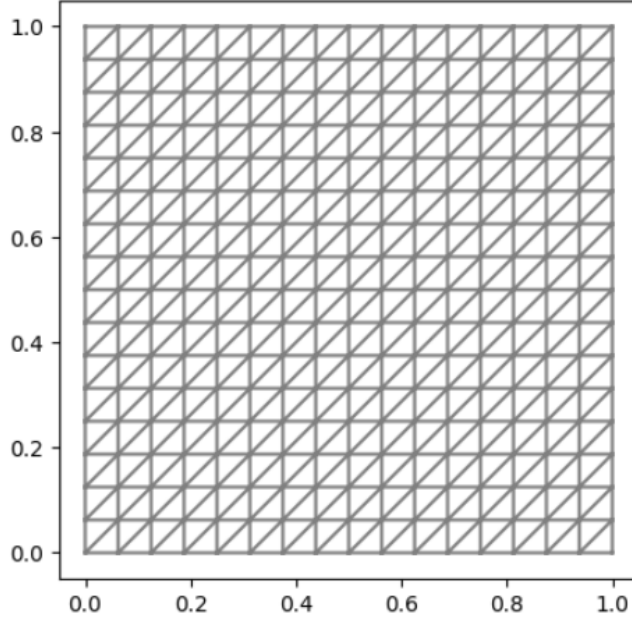


Figure 1: Triangulation for Example 5.1

parts Γ_D , $\Gamma_{S,1}$, and $\Gamma_{S,2}$; Γ_D is the union of the left and right sides, $\Gamma_{S,1}$ is the top side, and $\Gamma_{S,2}$ is the bottom side.

To calculate numerical errors, we use the numerical solution with $h = 1/256$ as the reference solution $(\mathbf{u}^*, \boldsymbol{\lambda}_1^*, \boldsymbol{\lambda}_2^*)$. We report numerical convergence orders of $\mathbf{u}^h - \mathbf{u}^*$ in the $L^2(\Omega)^2$ -norm and the $H^1(\Omega)^2$ -norm, that of $p^h - p^*$ in the $L^2(\Omega)$ -norm, that of $\boldsymbol{\lambda}_1^h - \boldsymbol{\lambda}_1^*$ in the $L^2(\Gamma_{S,1})^2$ -norm, and that of $\boldsymbol{\lambda}_2^h - \boldsymbol{\lambda}_2^*$ in the $(L^2(\Gamma_{S,2}))^2$ -norm.

Tables 1 and 2 show numerical convergence orders with respect to h . We also report the number of iterations required for the Uzawa algorithm. Values of the $L^2(\Omega)^2$ -norm and $H^1(\Omega)^2$ -norm of $\mathbf{u}^*$, the $L^2(\Omega)$ -norm of p^* , the $L^2(\Gamma_{S,1})^2$ -norm of $\boldsymbol{\lambda}_1^*$, and the $L^2(\Gamma_{S,2})^2$ -norm of $\boldsymbol{\lambda}_2^*$ for the reference solution are

$$\begin{aligned} \|\mathbf{u}^*\|_{L^2(\Omega)^2} &\doteq 4.619 \times 10^{-2}, & \|\mathbf{u}^*\|_{H^1(\Omega)^2} &\doteq 3.047 \times 10^{-1}, & \|p^*\|_{L^2(\Omega)} &\doteq 2.340 \times 10^{-1}, \\ \|\boldsymbol{\lambda}_1^*\|_{L^2(\Gamma_{S,1})^2} &\doteq 9.261 \times 10^{-1}, & \|\boldsymbol{\lambda}_2^*\|_{L^2(\Gamma_{S,2})^2} &\doteq 8.533 \times 10^{-1}. \end{aligned}$$

Numerical results for $h = 1/16$ are shown in Figures 2, 3 and 4. To illustrate the friction effect, in Figures 5, 6, 7 and 8, we show $\mathbf{u}_\tau$ and $\boldsymbol{\sigma}_\tau$ along the slip boundary $\Gamma_{S,1}$, and $\Gamma_{S,2}$ for the reference solution, respectively. The top and bottom sides of the domain are straight and parallel to the x -axis. Therefore, the tangential velocity $\mathbf{u}_\tau$ on the side can be equated with its first component, and similarly for the friction force $\boldsymbol{\sigma}_\tau$.

Example 5.2. In the second example, the domain is a curved variant of the unit square, with the bottom side replaced by a curve defined by

$$y = \frac{1}{10} \sin(2\pi x), \quad x \in [0, 1],$$

Table 1: Numerical convergence orders of $\mathbf{u}$ and p for Example 5.1

h	$\ \mathbf{u}^h - \mathbf{u}^*\ _{L^2}$	Order	$\ \mathbf{u}^h - \mathbf{u}^*\ _{H^1}$	Order	$\ p^h - p^*\ _{L^2}$	Order
1/8	6.505e-03	-	9.472e-02	-	7.661e-02	-
1/16	1.860e-03	1.81	4.750e-02	1.00	2.774e-02	1.47
1/32	4.877e-04	1.93	2.342e-02	1.02	8.712e-03	1.67
1/64	1.191e-04	2.03	1.149e-02	1.03	2.568e-03	1.76

 Table 2: Numerical convergence orders of λ_1 and λ_2 for Example 5.1

h	$\ \lambda_1^h - \lambda_1^*\ _{L^2(\Gamma_{S,1})}$	Order	$\ \lambda_2^h - \lambda_2^*\ _{L^2(\Gamma_{S,2})}$	Order	Iterations
1/8	3.395e-01	-	3.320e-01	-	89
1/16	2.125e-01	0.68	1.313e-01	1.34	149
1/32	7.304e-02	1.54	4.589e-02	1.52	144
1/64	2.489e-02	1.55	1.500e-02	1.61	159

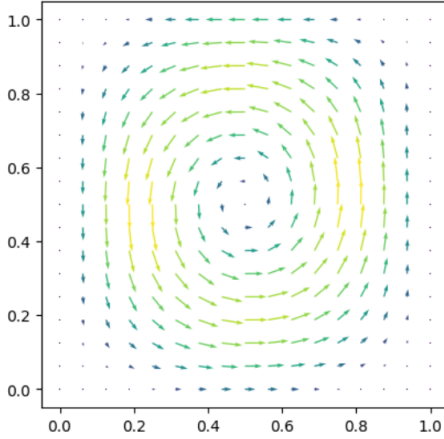


Figure 2: Velocity field for Example 5.1

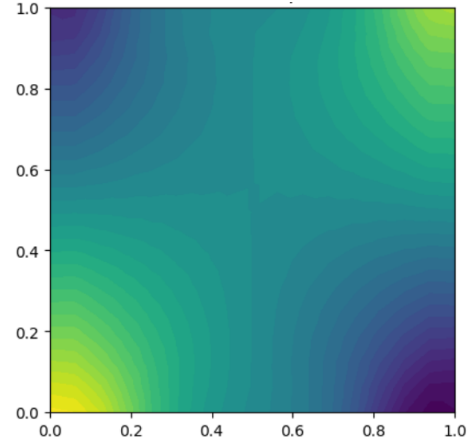


Figure 3: Pressure isobars for Example 5.1

We remove a disk of radius 0.2 centered at $(0.5, 0.6)^T$ from the domain and choose the circle to be $\Gamma_{S,1}$. Moreover, Γ_D is the union of the top, left, and right sides, and $\Gamma_{S,2}$ is the bottom side. We create the mesh by applying Gmsh, which is a 3D finite element mesh generator, cf. the website <https://gmsh.info/> developed and managed by C. Geuzaine and J.-F. Remacle. In Gmsh, let the element size factor be 0.4, 0.26, 0.181, 0.1264, 0.06285, and set the side length $h = 1/8, 1/16, 1/32, 1/64, 1/256$ on the left, top, and right boundaries. As a result, we get the total number of cells $N_{cells} = 128, 511, 2055, 8187, 131179$, respectively. The triangulations of the regions with $h = 1/16$ are shown in Figure 9.

As in Example 5.1, we use the numerical solution with $h = 1/256$ as the reference solution $(\mathbf{u}^*, \lambda_1^*, \lambda_2^*)$ in computing errors of the numerical solutions on coarse meshes. For the function $\omega(t)$, use $\beta = 1$, $a = 0.2$, $b = 0.1$. Table 3 and Table 4 show the numerical convergence orders as h decreases. Values of the $L^2(\Omega)^2$ -norm and $H^1(\Omega)^2$ -norm of $\mathbf{u}^*$, the $L^2(\Omega)$ -norm of p^* , and the

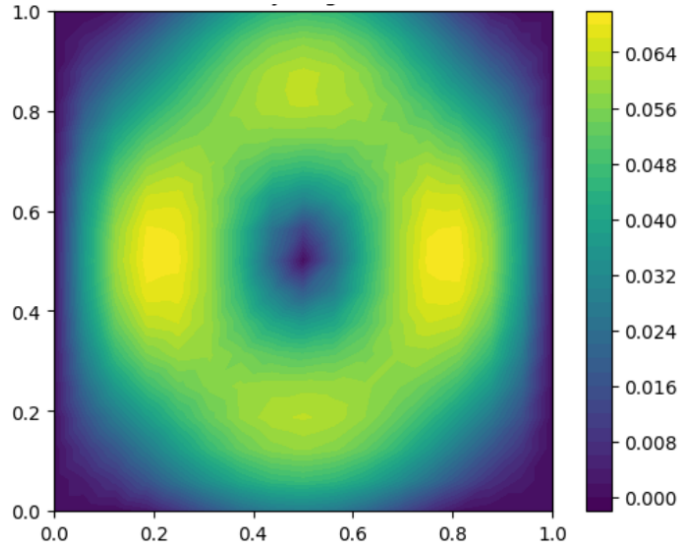


Figure 4: Magnitude of the velocity $|\mathbf{u}^h|$ for Example 5.1

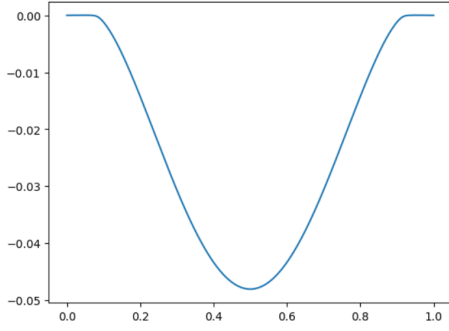


Figure 5: $\mathbf{u}_\tau$ along $\Gamma_{S,1}$, Example 5.1

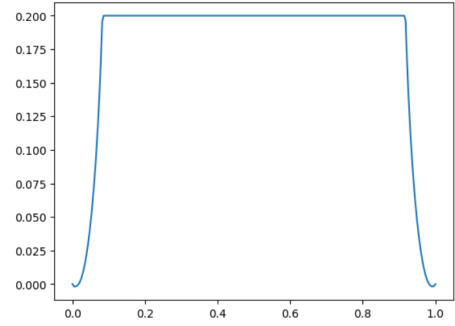


Figure 6: σ_τ along $\Gamma_{S,1}$, Example 5.1

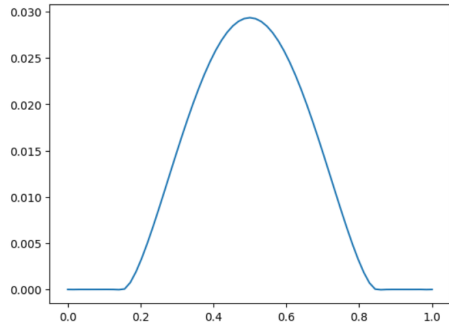


Figure 7: $\mathbf{u}_\tau$ along $\Gamma_{S,2}$, Example 5.1

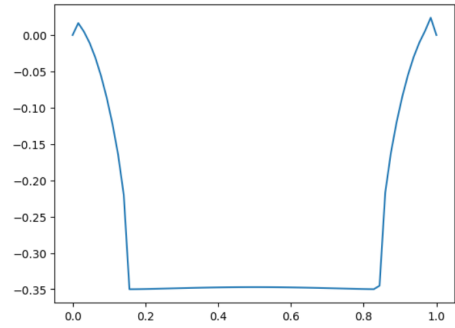


Figure 8: σ_τ along $\Gamma_{S,2}$, Example 5.1

$L^2(\Gamma_S)^2$ -norm of $\boldsymbol{\lambda}_1^*$ and $\boldsymbol{\lambda}_2^*$ for the reference solution:

$$\begin{aligned} \|\mathbf{u}^*\|_{L^2(\Omega)^2} &\doteq 2.216 \times 10^{-2}, \quad \|\mathbf{u}^*\|_{H^1(\Omega)^2} \doteq 1.914 \times 10^{-1}, \quad \|p^*\|_{L^2(\Omega)} \doteq 5.499 \times 10^{-1}, \\ \|\boldsymbol{\lambda}_1^*\|_{L^2(\Gamma_{S,1})^2} &\doteq 1.104, \quad \|\boldsymbol{\lambda}_2^*\|_{L^2(\Gamma_{S,2})^2} \doteq 7.975 \times 10^{-1}. \end{aligned}$$

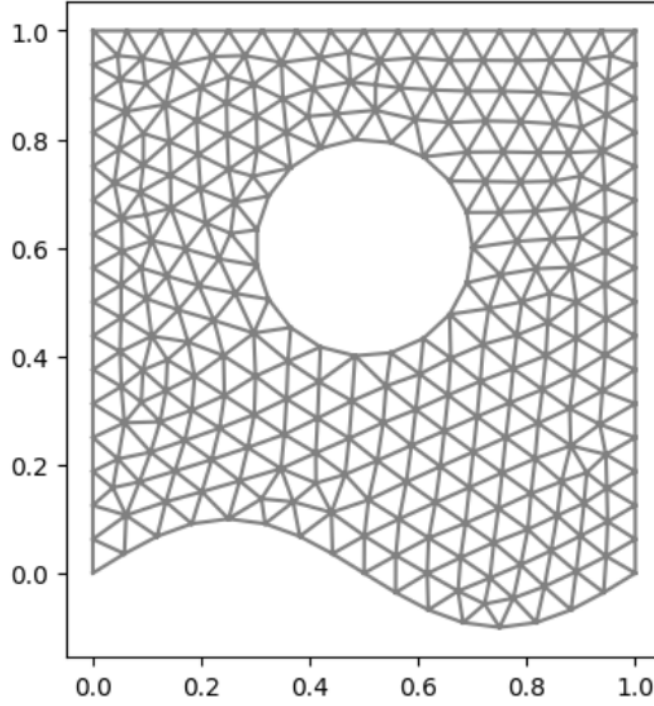


Figure 9: Triangulation for Example 5.2

Table 3: Numerical convergence orders of $\mathbf{u}$ and p for Example 5.2

h	$\ \mathbf{u}^h - \mathbf{u}^*\ _{L^2}$	Order	$\ \mathbf{u}^h - \mathbf{u}^*\ _{H^1}$	Order	$\ p^h - p^*\ _{L^2}$	Order
1/8	5.637e-03	-	8.980e-02	-	1.536e-01	-
1/16	2.053e-03	1.46	4.100e-02	1.13	5.337e-02	1.53
1/32	8.234e-04	1.32	2.017e-02	1.02	2.085e-02	1.36
1/64	3.236e-04	1.35	9.802e-03	1.04	8.486e-03	1.30

Table 4: Numerical convergence orders of $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ for Example 5.2

h	$\ \boldsymbol{\lambda}_1^h - \boldsymbol{\lambda}_1^*\ _{L^2(\Gamma_{S,1})}$	Order	$\ \boldsymbol{\lambda}_2^h - \boldsymbol{\lambda}_2^*\ _{L^2(\Gamma_{S,2})}$	Order	Iterations
1/8	1.090e-01	-	6.712e-01	-	145
1/16	2.667e-02	2.03	4.032e-01	0.74	184
1/32	1.732e-02	0.62	2.058e-01	0.97	205
1/64	8.235e-02	1.07	1.005e-01	1.03	202

The numerical results when $h = 1/16$ are shown in Figures 10, 11, and 12. The magnitudes of the tangential velocity and the friction force on the slip boundary for the reference solution are shown in Figures 13, 14, 15, and 16. In Figure 13, the upper curve corresponds to $|\mathbf{u}_\tau|$ on the top semicircular hole, while the lower curve corresponds to $|\mathbf{u}_\tau|$ on the bottom semicircular hole. From Figures 13 and 14, we observe that on the lower portion of the hole, for $x \in (0.495076, 0.600568)$, we

have $|\mathbf{u}_\tau| = 0$, corresponding to $|\boldsymbol{\sigma}_\tau| < 0.2$. On the remaining part of the hole, we have $|\boldsymbol{\sigma}_\tau| = 0.2$ and consequently $|\mathbf{u}_\tau| \neq 0$.

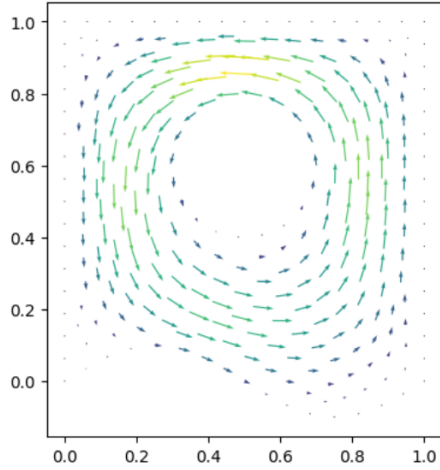


Figure 10: Velocity field for Example 5.2

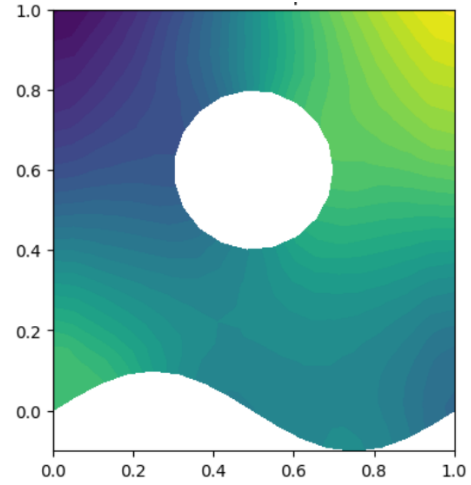


Figure 11: Pressure isobars for Example 5.2

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References

- [1] D. N. Arnold, F. Brezzi, and M. Fortin, A stable finite element for the Stokes equations, *Calcolo* **21** (1984), 337–344.
- [2] Y. Bai, S. Migórski, and S. Zeng, Well-posedness of a class of generalized mixed hemivariational-variational inequalities, *Nonlinear Analysis: Real World Applications* **48** (2019), 424–444.
- [3] Y. Bai, S. Migórski, and S. Zeng, A class of generalized mixed variational-hemivariational inequalities I: Existence and uniqueness results, *Computers and Mathematics with Applications* **79** (2020), 2897–2911.
- [4] D. Boffi, F. Brezzi, and M. Fortin, *Mixed Finite Element Methods and Applications*, Springer, Heidelberg, 2013.
- [5] S. C. Brenner and R. L. Scott, *The Mathematical Theory of Finite Element Methods*, third edition, Springer-Verlag, 2008.
- [6] F. Brezzi, W. Hager, and P. A. Raviart, Error estimates for the finite element solution of variational inequalities, Part II. *Numer. Math.* **31** (1978), 1–16.
- [7] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

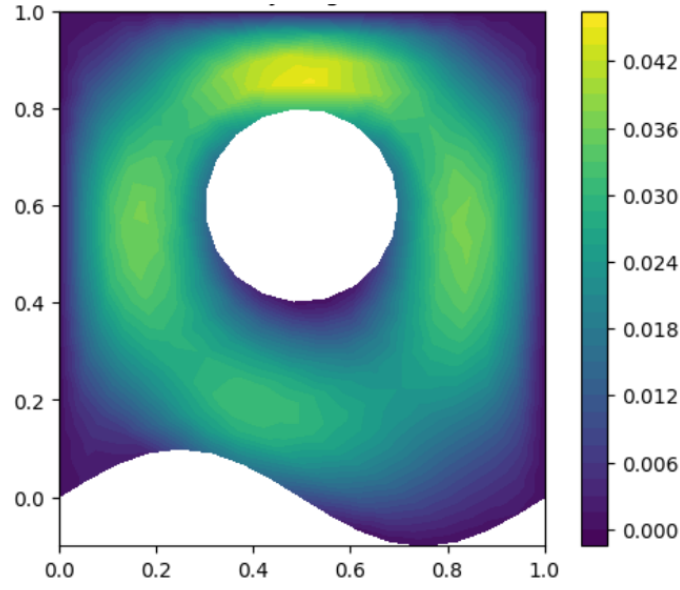


Figure 12: Magnitude of the velocity $|\mathbf{u}^h|$ for Example 5.2

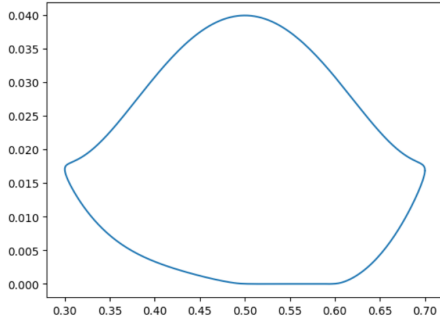


Figure 13: $|\mathbf{u}_\tau|$ along $\Gamma_{S,1}$, Example 5.2

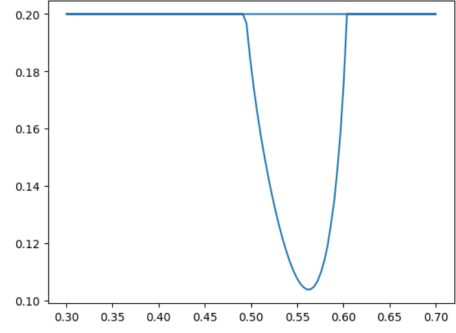


Figure 14: $|\boldsymbol{\sigma}_\tau|$ along $\Gamma_{S,1}$, Example 5.2

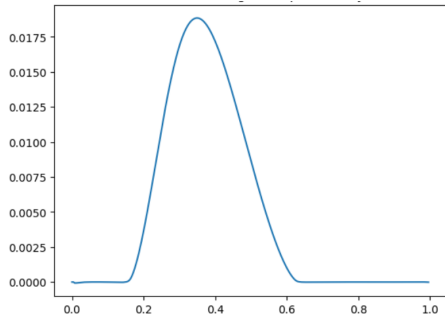


Figure 15: $|\mathbf{u}_\tau|$ along the slip boundary $\Gamma_{S,2}$

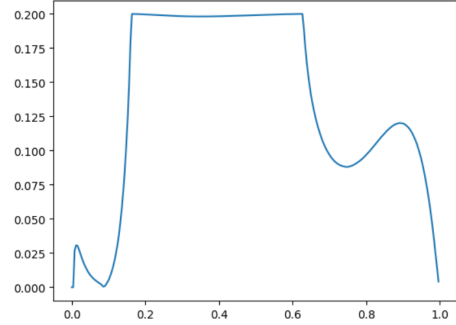


Figure 16: $|\boldsymbol{\sigma}_\tau|$ along $\Gamma_{S,2}$, Example 5.2

[8] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[9] J. Djoko and J. Koko, Numerical methods for the Stokes and Navier-Stokes equations driven by

- threshold slip boundary conditions, *Computer Methods in Applied Mechanics and Engineering* **305** (2016), 936–958.
- [10] I. Ekeland and R. Témam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.
 - [11] A. Ern and J.-L. Guermond, *Finite Elements I: Approximation and Interpolation*, Springer, 2021.
 - [12] C. Fang, K. Czuprynski, W. Han, X.L. Cheng, and X. Dai, Finite element method for a stationary Stokes hemivariational inequality with slip boundary condition, *IMA J. Numer. Anal.* **40** (2020), 2696–2716.
 - [13] G.N. Gatica, M. Maischak, and E.P. Stephan, Numerical analysis of a transmission problem with Signorini contact using mixed-FEM and BEM, *ESAIM Math. Model. Numer. Anal.* **45** (2011), 779–802.
 - [14] W. Han, Minimization principles for elliptic hemivariational inequalities, *Nonlinear Analysis: Real World Applications* **54** (2020), 103114.
 - [15] W. Han, A revisit of elliptic variational-hemivariational inequalities, *Numerical Functional Analysis and Optimization* **42** (2021), 371–395.
 - [16] W. Han, *An Introduction to Theory and Applications of Stationary Variational-Hemivariational Inequalities*, Springer, New York, 2024.
 - [17] W. Han, F. Feng, F. Wang, and J. Huang, Numerical analysis of hemivariational inequalities with applications in contact mechanics, *Advances in Applied Mechanics* **60** (2025), 113–178.
 - [18] W. Han and A. Matei, Minimax principles for elliptic mixed hemivariational-variational inequalities, *Nonlinear Analysis: Real World Applications* **64** (2022), article number 103448.
 - [19] W. Han and A. Matei, Well-posedness of a general class of elliptic mixed hemivariational-variational inequalities, *Nonlinear Analysis: Real World Applications* **66** (2022), 103553.
 - [20] W. Han and M. Sofonea, Numerical analysis of hemivariational inequalities in contact mechanics, *Acta Numerica* **28** (2019), 175–286.
 - [21] J. Haslinger, R. Kučera, V. Šátek, and T. Sassi, Stokes system with solution-dependent threshold slip boundary conditions: Analysis, approximation and implementation, *Mathematics and Mechanics of Solids* **23** (2017), 294–307.
 - [22] I. Hlaváček, J. Haslinger, J. Nečas, and J. Lovíšek, *Solution of Variational Inequalities in Mechanics*, Springer-Verlag, New York, 1988.
 - [23] R. Kučera, J. Haslinger, V. Šátek, and M. Jarošová, Efficient methods for solving the Stokes problem with slip boundary conditions, *Mathematics and Computers in Simulation* **145** (2018), 114–124.

- [24] H. P. Langtangen and A. Logg, *Solving PDEs in Python: The FEniCS Tutorial I*, Springer, 2016.
- [25] M. Ling, W. Han, and S. Zeng, A pressure projection stabilized mixed finite element method for a Stokes hemivariational inequality, *Journal of Scientific Computing* **92** (2022), article number 13.
- [26] A. Matei, A mixed hemivariational-variational problem and applications, *Computers and Mathematics with Applications* **77** (2019), 2989–3000.
- [27] S. Migórski, Y. Chao, J. He, and S. Dudek, Analysis of quasi-variational-hemivariational inequalities with applications to Bingham-type fluids, *Commun. Nonlinear Sci. Numer. Simul.* **133** (2024), Paper No. 107968.
- [28] S. Migórski and S. Dudek, A class of variational-hemivariational inequalities for Bingham type fluids, *Appl. Math. Optim.* **85** (2022), Paper No. 16.
- [29] S. Migórski and S. Dudek, Well-posedness of steady-state Bingham type system by a quasi variational- hemivariational approach, *Contemp. Math.* **786** (2023), 185–213.
- [30] S. Migórski, A. Ochal, and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Springer, 2013.
- [31] S. Migórski, Y. Bai, and S. Zeng, A class of generalized mixed variational-hemivariational inequalities II: Applications, *Nonlinear Analysis: Real World Applications* **50** (2019), 633–650.
- [32] P. D. Panagiotopoulos, Nonconvex energy functions, hemivariational inequalities and stationary principles, *Acta Mechanica* **42** (1983), 160–183.
- [33] L. Slimane, A. Bendali, and P. Laborde, Mixed formulations for a class of variational inequalities, *ESAIM: Math. Model. Numer. Anal.* **38** (2004), 177–201.
- [34] M. Sofonea and S. Migórski, *Variational-Hemivariational Inequalities with Applications*, 2nd edition, CRC Press, Boca Raton, FL, 2025.
- [35] X. Tan and M. Zhou, A new method for the solvability of mixed hemivariational inequalities, *Mathematics and Mechanics of Solids* **30** (2025), 2275–2284.