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A virtual element method for a quasistatic frictional contact problem

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ABSTRACT

In this paper, we first consider the numerical solution of an abstract quasistatic variational inequality arising in the study of quasistatic physical processes. The temporal discretization is carried out by the backward Euler difference scheme, while the spatial discretization is based on the virtual element method. A general framework is provided for the spatially semidiscrete and the fully discrete approximations of the quasistatic variational inequality. Then, as an application of the theoretical results on the abstract problem, a quasistatic contact problem is studied and optimal order error estimates are derived for the lowest-order VEM, under appropriate solution regularity assumptions. Numerical examples are presented to show the performance of the proposed methods.

1. Introduction

Frictional contact phenomena for deformable bodies abound in industry and daily life [1–5]. Much research has been done on modeling, analysis and numerical simulations of contact processes. In particular, comprehensive presentations of studies of contact models for the quasistatic processes can be found in Han and Sofonea [1], Shillor et al. [4]. A unified approach, which can be applied to various quasistatic problems, including unilateral and bilateral contact with nonlocal friction, or normal compliance conditions, is given in Badea and Cocou [6], and quasistatic problems with Coulomb friction are analyzed in Cocou [7]. Quasistatic contact problems are naturally studied in the form of quasistatic variational inequalities. Such quasistatic variational inequalities also arise in the study of other applications in sciences and engineering, e.g., in elastoplasticity [8]. To solve the quasistatic variational inequalities, the finite element method and the discontinuous Galerkin (DG) method have been commonly applied, cf. Barboteu et al. [9], Han et al. [10] and Wang et al. [11], respectively. In this paper, we study the virtual element method (VEM) for the numerical solution of quasistatic variational inequalities.

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The VEM was first proposed and analyzed in Ahmad et al. [12], Beirao Da Veiga et al. [13,14]. It has been applied successfully to a wide variety of scientific and engineering problems because of its favorable features in handling problems with complex geometries or problems requiring high-regularity solutions. In recent years, this method has been applied to solve variational inequalities [15–23].

In this paper, we consider the numerical solution of a quasistatic variational inequality by the virtual element method for spatial discretization and the backward Euler difference scheme for temporal discretization. Optimal order error estimates are derived for both semi-discrete solutions and fully discrete solutions under appropriate solution regularity assumptions. The fully discrete problem can be converted into a convex programming and an adaptive semi-smooth Newton method [24] can be applied to solve the fully discrete variational inequality. Numerical results are reported to illustrate computational performance of the method in this paper.

The rest of this paper is organized as follows. Error analysis is given for both the spatially semidiscrete and the fully discrete schemes of the abstract quasistatic variational inequality in Section 2. We apply the virtual element method in spatial discretization and backward difference method in time discretization of a quasistatic contact problem. Then, optimal order error estimates are derived for both schemes under appropriate solution regularity assumptions in Section 3. In Section 4, we present numerical simulation results on a numerical example to illustrate the theoretically predicted convergence order.

2. A general framework for numerical solution of the quasistatic problem

Let V be a Hilbert space, which is equipped with the norm $\|\cdot\|_V$. The dual space of V is denoted by V' and the duality pairing between V' and V is denoted by $\langle\cdot,\cdot\rangle$. Let T be a positive number and let m be a non-negative integer. We denote by $C^m([0,T];V)$ the space of continuous functions $u:[0,T]\to V$ that have continuous derivatives of an order less than or equal to m, which is a Banach space endowed with the norm

$$\|v\|_{C^m([0,T],V)} = \sum_{k=0}^{m} \max_{0\le t\le T} \|v^{(k)}(t)\|_V .$$

For $1\le p<\infty$, the space $L^p(0,T;V)$ consists of all measurable functions v from $[0,T]$ to V for which

$$\|v\|_{L^p(0,T;V)} = \left(\int_0^T \|v(t)\|_V^p\,dt\right)^{1/p} < \infty.$$

For the case $p=\infty$, we denote the space by $L^\infty(0,T;V)$ with the norm

$$\|v\|_{L^\infty(0,T;V)} = \operatorname{ess\,sup}_{0\le t\le T}\|v(t)\|_V .$$

For an integer $m\ge 0$ and a real $p\ge 1$, we denote by $W^{m,p}(0,T;V)$ the space of functions $f\in L^p(0,T;V)$ such that $f^{(i)}\in L^p(0,T;V)$, $1\le i\le m$, with the norm

$$\|f\|_{W^{m,p}(0,T;V)} = \left\{\sum_{i=0}^{m}\left\|f^{(i)}\right\|^p_{L^p(0,T;V)}\right\}^{1/p}.$$

When $p=2$, we write $H^m(0,T;V)$ for $W^{m,p}(0,T;V)$.

Let $a: V\times V\to\mathbb{R}$ be a symmetric, bounded, and V-elliptic bilinear form, $j:V\to\mathbb{R}$ a continuous seminorm, and $\ell\in W^{1,\infty}(0,T;V')$. Let $u_0\in V$ such that

$$a(u_0,v)+j(v)\ge\langle\ell(0),v\rangle\quad\forall\,v\in V. \tag{2.1}$$

We consider the following abstract problem.

Problem 2.1. Find $u:[0,T]\to V$ such that

$$u(0)=u_0 \tag{2.2}$$

and for almost all $t\in(0,T)$, $\dot u(t)\in V$ and

$$a(u(t),v-\dot u(t))+j(v)-j(\dot u(t))\ge\langle\ell(t),v-\dot u(t)\rangle\quad\forall\,v\in V. \tag{2.3}$$

By Han and Sofonea [1, Theorem 4.16], under the stated assumptions on the data, there exists a unique solution $u\in W^{1,\infty}(0,T;V)$ to Problem 2.1.

Suppose the spatial domain associated with Problem 2.1 is a bounded polygon $\Omega\subset\mathbb{R}^2$. Let $\{\mathcal{T}_h\}_h$, $\mathcal{T}_h:=\{K\}_{K\in\mathcal{T}_h}$, be a sequence of decompositions of Ω into polygons. A generic element in $\mathcal{T}_h$ is denoted by K whose diameter is denoted by $h_K:=\operatorname{diam}(K)$. The mesh-size of $\mathcal{T}_h$ is $h:=\max_{K\in\mathcal{T}_h}h_K$. Corresponding to the mesh $\mathcal{T}_h$, we construct a finite dimensional subspace V^h of V. For a non-negative integer k and a bounded domain D in $\mathbb{R}$ or $\mathbb{R}^2$, denote by $\mathbb{P}_k(D)$ the set of all polynomials on D with a total degree no more than k. Assume the bilinear form $a(\cdot,\cdot)$ allows for the decomposition

$$a(v,w):=\sum_{K\in\mathcal{T}_h}a_K(v,w),\quad\forall\,v,w\in V, \tag{2.4}$$

where $a_K(\cdot,\cdot)$ is a symmetric bilinear form over the space $V_K:=V|_K$. For a function in V, we naturally view its restriction to K as a function in V_K. We equip the space V_K with a norm or semi-norm $\|\cdot\|_{V_K}$ such that

$$\|v\|_V^2=\sum_{K\in\mathcal{T}_h}\|v\|_{V_K}^2\quad\forall\,v\in V,$$

and assume for all $K \in \mathcal{T}_h$,

$$a_K(v,v) \lesssim \|v\|^2_{V_K} \quad \forall\, v \in V_K, \tag{2.5}$$

where and in what follows, for two quantities a and b, $\}\}a \lesssim b\varepsilon$ means $\}\}a \le Cb\varepsilon$, C being a generic constant independent of h_K or h, which may take different values at different occurrences. Assume $\ell^h \in W^{1,\infty}(0,T;(V^h)')$ is an approximation of ℓ such that

$$\langle \ell^h(0), v^h\rangle = \langle \ell(0), v^h\rangle \quad \forall\, v^h \in V^h. \tag{2.6}$$

Over the given finite dimensional space V^h, we approximate the bilinear form $a(\cdot,\cdot)$ by a modified bilinear form $a^h(\cdot,\cdot)$ constructed as follows:

$$a^h(u,v) := \sum_{K\in\mathcal{T}_h} a^h_K(u,v), \tag{2.7}$$

where $a^h_K(\cdot,\cdot)$ is an approximation of $a_K(\cdot,\cdot)$. We assume $a^h_K(\cdot,\cdot)$ is a symmetric bilinear form on K such that it is k-consistent and stable, for some given natural number k. The k-consistency refers to the property that

$$a^h_K(p,v) = a_K(p,v) \quad \forall\, p \in \mathbb{P}_k(K),\ v \in V^h_K := V^h|_K. \tag{2.8}$$

The stability means that there exist two positive constants $\alpha_\star$ and $\alpha^\star$, independent of h_K and K, such that

$$\alpha_\star a_K(v,v) \le a^h_K(v,v) \le \alpha^\star a_K(v,v) \quad \forall\, v \in V^h_K. \tag{2.9}$$

Denote $\|v^h\|_{a^h} := a^h(v^h,v^h)^{1/2}$ for $v^h \in V^h$. By (2.7), (2.9), (2.4) and the assumptions on $a(\cdot,\cdot)$, we can verify that $\|\cdot\|_{a^h}$ defines a norm on V^h and it is uniformly equivalent to $\|\cdot\|_V$ on V^h, i.e., the equivalence coefficients are independent of h.

For later use, we recall the following elementary implication:

$$x,a,b \ge 0 \text{ and } x^2 \le ax + b \Rightarrow x \le a + b^{\frac{1}{2}}. \tag{2.10}$$

We define the energy projection operator $\mathcal{P}^h : V \to V^h$ by

$$\mathcal{P}^h u \in V^h, \quad a^h(\mathcal{P}^h u, v^h) = a(u,v^h) \quad \forall\, v^h \in V^h. \tag{2.11}$$

By the stability of a^h in (2.9), we observe that $a^h(\cdot,\cdot)$ is continuous and coercive on V^h. Since the functional $v^h \mapsto a(u,v^h)$ is continuous on V^h, we can apply Lax–Milgram Lemma to conclude that $\mathcal{P}^h u \in V^h$ is uniquely defined by (2.11). By taking $v^h = \mathcal{P}^h u$ in (2.11) and making use of properties of a^h and a, we can prove the stability inequality

$$\|\mathcal{P}^h u\|_V \lesssim \|u\|_V. \tag{2.12}$$

A suitable discrete initial data can be chosen by $u^h_0 = \mathcal{P}^h u_0$ since

$$a^h(\mathcal{P}^h u_0, v^h) = a(u_0, v^h) \quad \forall\, v^h \in V^h.$$

Then the discrete analog of (2.1) holds:

$$a^h(\mathcal{P}^h u_0, v^h) + j(v^h) \ge \langle \ell^h(0), v^h\rangle \quad \forall\, v^h \in V^h. \tag{2.13}$$

In the rest of the section, we assume all the properties of the data a, ℓ, u_0, a^h and ℓ^h stated above are valid.

2.1. Spatially semi-discrete scheme

The spatially semi-discrete approximation of Problem 2.1 is as follows.

Problem 2.2. Find $u^h : [0,T] \to V^h$ such that

$$u^h_0 = \mathcal{P}^h u_0 \tag{2.14}$$

and for almost all $t \in (0,T)$, $\dot u^h(t) \in V^h$ and

$$a^h(u^h(t), v^h - \dot u^h(t)) + j(v^h) - j(\dot u^h(t)) \ge \langle \ell^h(t), v^h - \dot u^h(t)\rangle \quad \forall\, v^h \in V^h. \tag{2.15}$$

Applying Han and Sofonea [1, Theorem 4.16], we know there is a unique solution $u^h \in W^{1,\infty}(0,T;V^h)$ to Problem 2.2. We use the notation $L^1(0,T;V^h)$ for the subspace of functions v^h in $L^1(0,T;V)$ such that $v^h(t) \in V^h$ for a.a. $t \in (0,T)$, and $\|v^h(\cdot)\|_V \in L^1(0,T)$. The following result is useful in estimating the semi-discrete approximation error.

Theorem 2.3. *Let u and u^h be the solutions of Problems 2.1 and 2.2. Then for any $v^h \in L^1(0,T;V^h)$,*

$$\begin{aligned}\|u-u^h\|_{L^\infty(0,T;V)} \lesssim{}& \|u-\mathcal{P}^h u\|_{L^\infty(0,T;V)} + \|\mathcal{P}^h \dot u - \dot u\|_{L^1(0,T;V)} + \|\dot u - v^h\|_{L^1(0,T;V)} \\ &+ \|R(v^h;\cdot)\|^{1/2}_{L^1(0,T)} + \|\ell - \ell^h\|_{W^{1,1}(0,T;(V^h)')},\end{aligned} \tag{2.16}$$

where for $t \in (0,T)$,

$$\|\ell(t) - \ell^h(t)\|_{(V^h)'} := \sup_{v^h\in V^h} \frac{\langle \ell(t) - \ell^h(t), v^h\rangle}{\|v^h\|_V},$$

$$R(v^h;t) := a(u(t), v^h - \dot u(t)) + j(v^h) - j(\dot u(t)) - \langle \ell(t), v^h - \dot u(t)\rangle. \tag{2.17}$$

Proof. We will use the fact that $\|\cdot\|_{a^h}$ defines a norm that is uniformly equivalent to the norm $\|\cdot\|_V$ on V^h. For any $v^h \in L^1(0,T;V^h)$, we write

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{P}^h u(t)-u^h(t)\|_{a^h}^2 = a^h(\mathcal{P}^h u(t)-u^h(t), \mathcal{P}^h\dot u(t)-\dot u^h(t)) = T_1+T_2+T_3, \tag{2.18}$$

where

$$\begin{aligned} T_1 &:= a^h(\mathcal{P}^h u(t)-u^h(t), \mathcal{P}^h\dot u(t)-\dot u(t)),\\ T_2 &:= a^h(\mathcal{P}^h u(t)-u^h(t), \dot u(t)-v^h(t)),\\ T_3 &:= a^h(\mathcal{P}^h u(t)-u^h(t), v^h(t)-\dot u^h(t)). \end{aligned}$$

Then,

$$T_1 \lesssim \|\mathcal{P}^h u(t)-u^h(t)\|_{a^h}\|\mathcal{P}^h\dot u(t)-\dot u(t)\|_V, \tag{2.19}$$

$$T_2 \lesssim \|\mathcal{P}^h u(t)-u^h(t)\|_{a^h}\|\dot u(t)-v^h(t)\|_V. \tag{2.20}$$

Note that

$$T_3 = a^h(\mathcal{P}^h u(t), v^h-\dot u^h(t)) - a^h(u^h(t), v^h(t)-\dot u^h(t)). \tag{2.21}$$

In view of (2.11), we obtain

$$\begin{aligned} a^h(\mathcal{P}^h u(t), v^h(t)-\dot u^h(t)) &= a(u(t), v^h(t)-\dot u^h(t))\\ &= a(u(t), v^h(t)-\dot u(t)) + a(u(t), \dot u(t)-\dot u^h(t)). \end{aligned} \tag{2.22}$$

We take $v=\dot u^h(t)$ in (2.3),

$$a(u(t), \dot u^h(t)-\dot u(t)) + j(\dot u^h(t)) - j(\dot u(t)) \ge \langle \ell(t), \dot u^h(t)-\dot u(t)\rangle_V. \tag{2.23}$$

From (2.15),

$$-a^h(u^h(t), v^h(t)-\dot u^h(t)) \le j(v^h(t)) - j(\dot u^h(t)) - \langle \ell^h(t), v^h(t)-\dot u^h(t)\rangle. \tag{2.24}$$

Combining (2.21)–(2.24), we get

$$\begin{aligned} T_3 &\le a(u(t), v^h(t)-\dot u(t)) + j(v^h(t)) - j(\dot u(t)) - \langle \ell(t), \dot u^h(t)-\dot u(t)\rangle - \langle \ell^h(t), v^h(t)-\dot u^h(t)\rangle\\ &\le R(v^h(t);t) + \langle \ell(t)-\ell^h(t), v^h(t)-\dot u^h(t)\rangle, \end{aligned} \tag{2.25}$$

where $R(v^h(t);t)$ is defined in (2.17). Combining (2.18), (2.19), (2.20) and (2.25), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathcal{P}^h u(t)-u^h(t)\|_{a^h}^2 &\lesssim \|\mathcal{P}^h u(t)-u^h(t)\|_{a^h}\|\mathcal{P}^h\dot u(t)-\dot u(t)\|_V\\ &\quad + \|\mathcal{P}^h u(t)-u^h(t)\|_{a^h}\|\dot u(t)-v^h(t)\|_V + R(v^h(t);t)\\ &\quad + \langle \ell(t)-\ell^h(t), v^h(t)-\mathcal{P}^h\dot u(t)\rangle + \langle \ell(t)-\ell^h(t), \mathcal{P}^h\dot u(t)-\dot u^h(t)\rangle. \end{aligned}$$

For any $s\in[0,T]$, integrating the above inequality from $t=0$ to s, noting that $\mathcal{P}^h u_0 = u_0^h$, we obtain

$$\begin{aligned} \|\mathcal{P}^h u(s)-u^h(s)\|_{a^h}^2 &\lesssim \int_0^s \|\mathcal{P}^h u(t)-u^h(t)\|_{a^h}\left(\|\mathcal{P}^h\dot u(t)-\dot u(t)\|_V + \|\dot u(t)-v^h(t)\|_V\right)dt\\ &\quad + \int_0^s R(v^h(t);t)\,dt + \int_0^s \langle \ell(t)-\ell^h(t), v^h(t)-\mathcal{P}^h\dot u(t)\rangle\,dt\\ &\quad + \int_0^s \langle \ell(t)-\ell^h(t), \mathcal{P}^h\dot u(t)-\dot u^h(t)\rangle dt. \end{aligned}$$

We can replace the $\|\cdot\|_{a^h}$-norm by $\|\cdot\|_V$-norm to get

$$\begin{aligned} \|\mathcal{P}^h u(s)-u^h(s)\|_V^2 &\lesssim \int_0^s \|\mathcal{P}^h u(t)-u^h(t)\|_V\left(\|\mathcal{P}^h\dot u(t)-\dot u(t)\|_V + \|\dot u(t)-v^h(t)\|_V\right)dt\\ &\quad + \int_0^s R(v^h(t);t)\,dt + \int_0^s \langle \ell(t)-\ell^h(t), v^h(t)-\mathcal{P}^h\dot u(t)\rangle\,dt\\ &\quad + \int_0^s \langle \ell(t)-\ell^h(t), \mathcal{P}^h\dot u(t)-\dot u^h(t)\rangle dt. \end{aligned} \tag{2.26}$$

Perform an integration by parts,

$$\begin{aligned} \int_0^s \langle \ell(t)-\ell^h(t), \mathcal{P}^h\dot u(t)-\dot u^h(t)\rangle dt &= \langle \ell(s)-\ell^h(s), \mathcal{P}^h u(s)-u^h(s)\rangle\\ &\quad - \int_0^s \langle \dot\ell(t)-\dot\ell^h(t), \mathcal{P}^h u(t)-u^h(t)\rangle dt. \end{aligned} \tag{2.27}$$

According to the Sobolev embedding theorem, we have

$$W^{1,1}(0,T;(V^h)') \hookrightarrow C([0,T];(V^h)'). \tag{2.28}$$

Then,

$$\langle \ell(s) - \ell^h(s), \mathcal{P}^h u(s) - u^h(s)\rangle \le \|\ell - \ell^h\|_{W^{1,1}(0,T;(V^h)')}\|\mathcal{P}^h u - u^h\|_{L^\infty(0,T;V)}, \tag{2.29}$$

$$-\int_0^s \langle \dot{\ell}(t) - \dot{\ell}^h(t), \mathcal{P}^h u(t) - u^h(t)\rangle dt \le \|\ell - \ell^h\|_{W^{1,1}(0,T;(V^h)')}\|\mathcal{P}^h u - u^h\|_{L^\infty(0,T;V)}. \tag{2.30}$$

Furthermore,

$$\begin{aligned}\int_0^s \langle \ell(t) - \ell^h(t), v^h(t) - \mathcal{P}^h\dot{u}(t)\rangle dt &\lesssim \|\ell - \ell^h\|_{L^\infty(0,T;(V^h)')}\|v^h - \mathcal{P}^h\dot{u}\|_{L^1(0,T;V)} \\ &\lesssim \|\ell - \ell^h\|_{W^{1,1}(0,T;(V^h)')}\|v^h - \mathcal{P}^h\dot{u}\|_{L^1(0,T;V)}. \end{aligned}\tag{2.31}$$

Combining (2.26)–(2.31),

$$\begin{aligned}\|\mathcal{P}^h u - u^h\|^2_{L^\infty(0,T;V)} \lesssim{}& \|\mathcal{P}^h u - u^h\|_{L^\infty(0,T;V)}\big(\|\mathcal{P}^h\dot{u} - \dot{u}\|_{L^1(0,T;V)} + \|\dot{u} - v^h\|_{L^1(0,T;V)} \\ &+ \|\ell - \ell^h\|_{W^{1,1}(0,T;(V^h)')}\big) + \|R(v^h)\|_{L^1(0,T)} \\ &+ \|\ell - \ell^h\|_{W^{1,1}(0,T;(V^h)')}\|v^h - \mathcal{P}^h\dot{u}\|_{L^1(0,T;V)},\end{aligned}$$

where $\|R(v^h)\|_{L^1(0,T)}$ is a short-hand notation for $\|R(v^h;\cdot)\|_{L^1(0,T)}$. By (2.10), the Cauchy–Schwarz inequality, and

$$\|v^h - \mathcal{P}^h\dot{u}\|_{L^1(0,T;V)} \le \|\mathcal{P}^h\dot{u} - \dot{u}\|_{L^1(0,T;V)} + \|\dot{u} - v^h\|_{L^1(0,T;V)},$$

we get

$$\begin{aligned}\|\mathcal{P}^h u - u^h\|_{L^\infty(0,T;V)} \lesssim{}& \|\mathcal{P}^h\dot{u} - \dot{u}\|_{L^1(0,T;V)} + \|\dot{u} - v^h\|_{L^1(0,T;V)} \\ &+ \|R(v^h)\|^{1/2}_{L^1(0,T)} + \|\ell - \ell^h\|_{W^{1,1}(0,T;(V^h)')}.\end{aligned}\tag{2.32}$$

Finally, by the triangle inequality,

$$\|u - u^h\|_{L^\infty(0,T;V)} \le \|u - \mathcal{P}^h u\|_{L^\infty(0,T;V)} + \|\mathcal{P}^h u - u^h\|_{L^\infty(0,T;V)},$$

we obtain (2.16) from (2.32). □

2.2. Fully discrete scheme

For a full discretization, in addition to the finite dimensional subspace V^h of V, we need a partition of the time interval: $[0,T] = \bigcup_{n=1}^{N}[t_{n-1},t_n]$ with $0 = t_0 < t_1 < \ldots < t_N = T$. Denote the step-size by $k_n = t_n - t_{n-1}$, $1 \le n \le N$. Let $k = \max_n k_n$ and write $u_n = u(t_n)$ for a continuous function $u(t)$. We use the symbol $\Delta u_n = u_n - u_{n-1}$ for the backward difference and $\delta_n u_n = \Delta u_n / k_n$ for the backward divided difference. Then a fully discrete approximation of Problem 2.1 is as follows.

Problem 2.4. Find $u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset V^h$ such that

$$u_0^{hk} = \mathcal{P}^h u_0 \tag{2.33}$$

and for $n = 1, \ldots, N$,

$$a^h\big(u_n^{hk}, v^h - \delta_n u_n^{hk}\big) + j\big(v^h\big) - j\big(\delta_n u_n^{hk}\big) \ge \langle \ell_n^h, v^h - \delta_n u_n^{hk}\rangle_V \quad \forall\, v^h \in V^h. \tag{2.34}$$

Existence of a unique solution for Problem 2.4 can be obtained following the arguments used in proving [8, Theorem 11.7]. For the fully discrete scheme, we only need the approximations $\ell_n^h \approx \ell^h(t_n)$, $0 \le n \le N$. Nevertheless, for simplicity and without loss of generality, we continue to use ℓ^h as an approximation of ℓ on $[0,T]$, as in the case of spatially semi-discrete scheme. Let us show a stability result for the solution of Problem 2.4 with respect to the initial value u_0 and the right hand side ℓ^h.

Theorem 2.5. *For $\ell_1^h, \ell_2^h \in W^{1,1}(0,T;(V^h)')$ and initial data $\mathcal{P}^h u_{1,0}, \mathcal{P}^h u_{2,0} \in V^h$, the corresponding solutions $\{u_{1,n}^{hk}\}_{n=0}^N$ and $\{u_{2,n}^{hk}\}_{n=0}^N$ of Problem 2.4 satisfy the inequality*

$$\max_{0\le n\le N}\|u_{1,n}^{hk} - u_{2,n}^{hk}\|_V \lesssim \|\mathcal{P}^h u_{1,0} - \mathcal{P}^h u_{2,0}\|_V + \|\ell_1^h - \ell_2^h\|_{W^{1,1}(0,T;(V^h)')}. \tag{2.35}$$

Proof. From the defining inequality (2.34) for the solutions $\{u_{1,n}^{hk}\}_{n=0}^N$ and $\{u_{2,n}^{hk}\}_{n=0}^N$, we have

$$a^h(u_{1,n}^{hk}, v^h - \delta_n u_{1,n}^{hk}) + j(v^h) - j(\delta_n u_{1,n}^{hk}) \ge \langle \ell_{1,n}^h, v^h - \delta_n u_{1,n}^{hk}\rangle \quad \forall\, v^h \in V^h, \tag{2.36}$$

$$a^h(u_{2,n}^{hk}, v^h - \delta_n u_{2,n}^{hk}) + j(v^h) - j(\delta_n u_{2,n}^{hk}) \ge \langle \ell_{2,n}^h, v^h - \delta_n u_{2,n}^{hk}\rangle \quad \forall\, v^h \in V^h. \tag{2.37}$$

Denote $e_n = u^{hk}_{1,n} - u^{hk}_{2,n}$. Taking $v^h = \delta_n u^{hk}_{2,n}$ in (2.36) and $v^h = \delta_n u^{hk}_{1,n}$ in (2.37), adding the two inequalities, we get

$$A^h_n := a^h(e_n, \delta_n e_n) \le \langle \ell^h_{1,n} - \ell^h_{2,n}, \delta_n e_n\rangle.$$

Write

$$A^h_n = \frac{1}{k_n}\big(a^h(e_n, e_n) - a^h(e_n, e_{n-1})\big).$$

By the Cauchy–Schwarz inequality,

$$a^h(e_n, e_{n-1}) \le \frac{1}{2}\Big(\|e_n\|^2_{a^h} + \|e_{n-1}\|^2_{a^h}\Big).$$

Hence,

$$A^h_n \ge \frac{1}{2k_n}\big(\|e_n\|^2_{a^h} - \|e_{n-1}\|^2_{a^h}\big). \tag{2.38}$$

Then for $1 \le n \le N$, we obtain

$$\|e_n\|^2_{a^h} - \|e_{n-1}\|^2_{a^h} \le 2\langle \ell^h_{1,n} - \ell^h_{2,n}, e_n - e_{n-1}\rangle.$$

An induction argument yields

$$\begin{aligned}\|e_n\|^2_{a^h} &\le \|e_0\|^2_{a^h} + 2\sum_{i=1}^{n}\langle \ell^h_{1,i} - \ell^h_{2,i}, e_i - e_{i-1}\rangle\\ &= \|e_0\|^2_{a^h} + 2\langle \ell^h_{1,n} - \ell^h_{2,n}, e_n\rangle - 2\langle \ell^h_{1,1} - \ell^h_{2,1}, e_0\rangle\\ &\quad + 2\sum_{i=1}^{n-1}\Big\langle (\ell^h_{1,i} - \ell^h_{1,i+1}) - (\ell^h_{2,i} - \ell^h_{2,i+1}), e_i\Big\rangle.\end{aligned}$$

Then, since $\|\cdot\|_{a^h}$ is uniformly equivalent to $\|\cdot\|_V$,

$$\begin{aligned}\|e_n\|^2_V &\lesssim \|e_0\|^2_V + \langle \ell^h_{1,n} - \ell^h_{2,n}, e_n\rangle - \langle \ell^h_{1,1} - \ell^h_{2,1}, e_0\rangle\\ &\quad + \sum_{i=1}^{n-1}\Big\langle (\ell^h_{1,i} - \ell^h_{1,i+1}) - (\ell^h_{2,i} - \ell^h_{2,i+1}), e_i\Big\rangle.\end{aligned}$$

Denote $M = \max_{0\le n\le N}\|e_n\|_V$. We have, for some n between 0 and N,

$$\begin{aligned}M^2 &\lesssim \|e_0\|^2_V + M\big(\|\ell^h_{1,n} - \ell^h_{2,n}\|_{(V^h)'} + \|\ell^h_{1,1} - \ell^h_{2,1}\|_{(V^h)'}\\ &\quad + \sum_{i=1}^{n-1}\|(\ell^h_{1,i} - \ell^h_{1,i+1}) - (\ell^h_{2,i} - \ell^h_{2,i+1})\|_{(V^h)'}\big).\end{aligned}$$

Applying (2.10) and (2.28), we obtain

$$\|e_n\|_V \lesssim \|\mathcal{P}^h u_{1,0} - \mathcal{P}^h u_{2,0}\|_V + \|\ell^h_1 - \ell^h_2\|_{W^{1,1}(0,T;(V^h)')}.$$

Therefore, (2.35) holds. □

Now, we turn our attention to error analysis for the fully discrete scheme.

Theorem 2.6. *Let u and u^{hk}_n be the solutions of Problems 2.1 and 2.4. Then, for any $\{v^h_n\}_{n=1}^N \subset V^h$, we have*

$$\begin{aligned}\max_{1\le n\le N}\|u_n - u^{hk}_n\|_V &\lesssim \max_{1\le n\le N}\|u_n - \mathcal{P}^h u_n\|_V + \sum_{n=1}^{N} k_n\|\delta_n\mathcal{P}^h u_n - \dot u_n\|_V + \sum_{n=1}^{N} k_n\|\dot u_n - v^h_n\|_V\\ &\quad + \max_{1\le n\le N}\|\ell_n - \ell^h_n\|_{(V^h)'} + \sum_{n=1}^{N-1}\|(\ell_{n+1} - \ell_n) - (\ell^h_{n+1} - \ell^h_n)\|_{(V^h)'}\\ &\quad + \Big(\sum_{n=1}^{N} k_n R_n(v^h_n)\Big)^{1/2},\end{aligned} \tag{2.39}$$

where

$$\|\ell_n - \ell^h_n\|_{(V^h)'} := \sup_{v^h\in V^h}\frac{\langle \ell_n - \ell^h_n, v^h\rangle}{\|v^h\|_V},$$

$$R_n(v^h) := a(u_n, v^h - \dot u_n) + j(v^h) - j(\dot u_n) - \langle \ell_n, v^h - \dot u_n\rangle.$$

Proof. The quantity of interest is the error $e_n = u_n - u_n^{hk}$. Write

$$e_n = u_n - \mathcal{P}^h u_n + \mathcal{P}^h u_n - u_n^{hk}.$$

Then,

$$\|e_n\|_V \le \|u_n - \mathcal{P}^h u_n\|_V + \|\mathcal{P}^h u_n - u_n^{hk}\|_V. \tag{2.40}$$

Denote $e_n^h = \mathcal{P}^h u_n - u_n^{hk}$. Let $A_n^h = a^h(e_n^h, \delta_n e_n^h)$. By (2.38),

$$A_n^h \ge \frac{1}{2k_n}\left(\|e_n^h\|_{a^h}^2 - \|e_{n-1}^h\|_{a^h}^2\right). \tag{2.41}$$

Write

$$\delta_n e_n^h = \delta_n \mathcal{P}^h u_n - \delta_n u_n^{hk} = \left(\delta_n \mathcal{P}^h u_n - \dot{u}_n\right) + \left(\dot{u}_n - v^h\right) + \left(v^h - \delta_n u_n^{hk}\right).$$

Then,

$$A_n^h = a^h\left(e_n^h, \delta_n \mathcal{P}^h u_n - \dot{u}_n\right) + a^h\left(e_n^h, \dot{u}_n - v^h\right) + a^h\left(\mathcal{P}^h u_n - u_n^{hk}, v^h - \delta_n u_n^{hk}\right). \tag{2.42}$$

By the definition of $\mathcal{P}^h$ in (2.11), we have

$$a^h(\mathcal{P}^h u_n - u_n^{hk}, v^h - \delta_n u_n^{hk}) = a(u_n, v^h - \delta_n u_n^{hk}) - a^h(u_n^{hk}, v^h - \delta_n u_n^{hk}).$$

Let $v = \delta_n u_n^{hk}$ in (2.3) to get

$$a(u_n, \delta_n u_n^{hk} - \dot{u}_n) + j(\delta_n u_n^{hk}) - j(\dot{u}_n) \ge \langle \ell_n, \delta_n u_n^{hk} - \dot{u}_n\rangle. \tag{2.43}$$

Combining (2.34) with $v^h = v_n^h$ and (2.43), we have

$$\begin{aligned} -a^h(u_n^{hk}, v_n^h - \delta_n u_n^{hk}) \le{}& a(u_n, \delta_n u_n^{hk} - \dot{u}_n) + j(v_n^h) - j(\dot{u}_n) \\ & - \langle \ell_n, \delta_n u_n^{hk} - \dot{u}_n\rangle - \langle \ell_n^h, v_n^h - \delta_n u_n^{hk}\rangle. \end{aligned}$$

Then,

$$\begin{aligned} a^h(\mathcal{P}^h u_n - u_n^{hk}, v_n^h - \delta_n u_n^{hk}) \le{}& a(u_n, v_n^h - \delta_n u_n^{hk}) + a(u_n, \delta_n u_n^{hk} - \dot{u}_n) + j(v_n^h) \\ & - j(\dot{u}_n) - \langle \ell_n, \delta_n u_n^{hk} - \dot{u}_n\rangle - \langle \ell_n^h, v_n^h - \delta_n u_n^{hk}\rangle \\ ={}& R_n(v_n^h) + \langle \ell_n - \ell_n^h, v_n^h - \delta_n u_n^{hk}\rangle. \end{aligned} \tag{2.44}$$

In view of (2.41), (2.42) and (2.44), we have

$$\begin{aligned} \frac{1}{2k_n}\left(\|e_n^h\|_{a^h}^2 - \|e_{n-1}^h\|_{a^h}^2\right) \le{}& a^h(e_n^h, \delta_n \mathcal{P}^h u_n - \dot{u}_n) + a^h(e_n^h, \dot{u}_n - v_n^h) + R_n(v_n^h) \\ & + \langle \ell_n - \ell_n^h, v_n^h - \delta_n \mathcal{P}^h u_n\rangle + \frac{1}{k_n}\langle \ell_n - \ell_n^h, e_n^h - e_{n-1}^h\rangle. \end{aligned} \tag{2.45}$$

We change the index n to i in the inequality (2.45), multiply both sides by k_i and sum from $i = 1$ to n to obtain

$$\begin{aligned} \|e_n^h\|_{a^h}^2 \le{}& 2\sum_{i=1}^n k_i a^h(e_i^h, \delta_i \mathcal{P}^h u_i - \dot{u}_i) + 2\sum_{i=1}^n k_i a^h(e_i^h, \dot{u}_i - v_i^h) + 2\sum_{i=1}^n k_i R_i(v_i^h) \\ & + 2\sum_{i=1}^n k_i \langle \ell_i - \ell_i^h, v_i^h - \delta_i \mathcal{P}^h u_i\rangle + 2\sum_{i=1}^n \langle \ell_i - \ell_i^h, e_i^h - e_{i-1}^h\rangle, \end{aligned}$$

where we used the fact that

$$e_0^h = \mathcal{P}^h u_0 - u_0^h = 0.$$

Note that

$$\sum_{i=1}^n \langle \ell_i - \ell_i^h, e_i^h - e_{i-1}^h\rangle = \langle \ell_n - \ell_n^h, e_n^h\rangle - \sum_{i=1}^{n-1} \langle (\ell_{i+1} - \ell_i) - (\ell_{i+1}^h - \ell_i^h), e_i^h\rangle.$$

Denote $M = \max_{1\le n\le N} \|e_n^h\|_{a^h}$. By (2.7), (2.9), Cauchy–Schwarz inequality, we get

$$\begin{aligned} M^2 \lesssim{}& M\Big(\sum_{n=1}^N k_n \|\delta_n \mathcal{P}^h u_n - \dot{u}_n\|_V + \sum_{n=1}^N k_n \|\dot{u}_n - v_n^h\|_V \\ & + \max_{1\le n\le N} \|\ell_n - \ell_n^h\|_{(V^h)'} + \sum_{n=1}^{N-1} \|(\ell_{n+1} - \ell_n) - (\ell_{n+1}^h - \ell_n^h)\|_{(V^h)'}\Big) \\ & + \sum_{n=1}^N k_n R_n(v_n^h) + \sum_{n=1}^N k_n \|\ell_n - \ell_n^h\|_{(V^h)'} \|v_n^h - \delta_n \mathcal{P}^h u_n\|_V. \end{aligned}$$

Applying (2.10), since $\|\cdot\|_{a^h}$ is uniformly equivalent to $\|\cdot\|_V$ on V^h, we obtain

$$\begin{aligned}\max_{1\le n\le N}\|\mathcal{P}^h u_n-u_n^{hk}\|_V \lesssim{}& \sum_{n=1}^N k_n\|\delta_n\mathcal{P}^h u_n-\dot u_n\|_V+\sum_{n=1}^N k_n\|\dot u_n-v_n^h\|_V+\max_{1\le n\le N}\|\ell_n-\ell_n^h\|_{(V^h)'}\\ &+\sum_{n=1}^{N-1}\|(\ell_{n+1}-\ell_n)-(\ell_{n+1}^h-\ell_n^h)\|_{(V^h)'}+\Big(\sum_{n=1}^N k_nR_n(v_n^h)\Big)^{\frac12}\\ &+\Big(\sum_{n=1}^N k_n\|v_n^h-\delta_n\mathcal{P}^h u_n\|_V\|\ell_n-\ell_n^h\|_{(V^h)'}\Big)^{\frac12}.\end{aligned} \tag{2.46}$$

We notice that

$$\begin{aligned}&\Big(\sum_{n=1}^N k_n\|v_n^h-\delta_n\mathcal{P}^h u_n\|_V\|\ell_n-\ell_n^h\|_{(V^h)'}\Big)^{\frac12}\\ &\le \max_{1\le n\le N}\|\ell_n-\ell_n^h\|_{(V^h)'}^{1/2}\Big(\sum_{n=1}^N k_n\|v_n^h-\delta_n\mathcal{P}^h u_n\|_V\Big)^{\frac12}\\ &\lesssim \max_{1\le n\le N}\|\ell_n-\ell_n^h\|_{(V^h)'}+\sum_{n=1}^N k_n\|\delta_n\mathcal{P}^h u_n-\dot u_n\|_V+\sum_{n=1}^N k_n\|\dot u_n-v_n^h\|_V.\end{aligned}$$

Applying this relation in (2.46), and then combining with (2.40), we get (2.39). □

3. A quasistatic frictional contact problem

In this section, we consider the numerical solution of a quasistatic frictional contact problem and apply the results from the previous section to derive convergence order error estimates of the numerical solutions under appropriate solution regularity assumptions. The setting of the contact problem is as follows. We assume that the set of governing equations is posed on a bounded Lipschitz domain $\Omega\subset\mathbb{R}^2$ with boundary Γ, which is divided into three disjoint and measurable parts Γ_D, Γ_N and Γ_C such that $\mathrm{meas}(\Gamma_D)>0$. Let $T>0$ and $t\in[0,T]$ be the time variable. We assume the contact is bilateral and the friction obeys Tresca's friction law. The strain-displacement relation is $\boldsymbol{\varepsilon}(\boldsymbol{u})=\frac12(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T)$. For a vector $\boldsymbol{v}$, denote on the boundary by $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$ its normal component and $\boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$ the tangential component, respectively. Let $\mathbb{C}:\mathbb{S}^2\to\mathbb{S}^2$ be the fourth order elasticity tensor, which is bounded, symmetric and positive definite. Here, $\mathbb{S}^2$ denotes the set of all symmetric second order tensor in $\mathbb{R}^2$. We will focus on the case of an isotropic, nonhomogeneous linear elastic material:

$$\mathbb{C}\boldsymbol{\varepsilon}:=2\mu\boldsymbol{\varepsilon}+\lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I},$$

where λ and μ are the constant Lamé coefficients, $\boldsymbol{I}$ is the unit matrix. We define the normal component as $\sigma_\nu=(\boldsymbol{\sigma\nu})\cdot\boldsymbol{\nu}$ and tangential component as $\boldsymbol{\sigma}_\tau=\boldsymbol{\sigma\nu}-\sigma_\nu\boldsymbol{\nu}$. Then, the classical formulation of the mechanical problem in a quasistatic process can be described as follows.

Problem 3.1. Find a displacement field $\boldsymbol{u}:\Omega\times[0,T]\to\mathbb{R}^2$ satisfying the relations

$$\boldsymbol{\sigma}=\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\quad\text{in }\Omega\times(0,T), \tag{3.1}$$

$$-\mathrm{div}\boldsymbol{\sigma}=\boldsymbol{f}_1\quad\text{in }\Omega\times(0,T), \tag{3.2}$$

$$\boldsymbol{u}=\boldsymbol{0}\quad\text{on }\Gamma_D\times(0,T), \tag{3.3}$$

$$\boldsymbol{\sigma\nu}=\boldsymbol{f}_2\quad\text{on }\Gamma_N\times(0,T), \tag{3.4}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0\quad\text{in }\Omega \tag{3.5}$$

$$\left.\begin{aligned}&u_\nu=0,\ \ |\boldsymbol{\sigma}_\tau|\le g\\ &|\boldsymbol{\sigma}_\tau|<g\Rightarrow\ \dot{\boldsymbol{u}}_\tau=\boldsymbol{0}\\ &|\boldsymbol{\sigma}_\tau|=g\Rightarrow\exists\lambda\ge0\ s.t.\ \boldsymbol{\sigma}_\tau=-\lambda\dot{\boldsymbol{u}}_\tau\end{aligned}\right\}\quad\text{on }\Gamma_C\times(0,T). \tag{3.6}$$

In Problem 3.1, $\boldsymbol{f}_1$ is the density of the volume forces. Boundary condition (3.3) means that the body is clamped on Γ_D. Surface traction $\boldsymbol{f}_2$ acts on $\Gamma_N\times(0,T)$ in (3.4), $g\ge0$ represents a friction bound function.

We introduce a Hilbert space

$$\boldsymbol{V}=\{\boldsymbol{v}\in H^1(\Omega;\mathbb{R}^2)\,|\,\boldsymbol{v}|_{\Gamma_D}=\boldsymbol{0},\ v_\nu|_{\Gamma_C}=0\},$$

which is equipped with the inner product and the induced norm given by

$$(\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{V}}=\int_\Omega\boldsymbol{\varepsilon}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx,\quad\|\boldsymbol{v}\|_{\boldsymbol{V}}=(\boldsymbol{v},\boldsymbol{v})_{\boldsymbol{V}}^{1/2}.$$

Since meas$(\Gamma_D) > 0$, Korn's inequality holds, implying that $\|\cdot\|_{\boldsymbol{V}}$ is a norm on $\boldsymbol{V}$, equivalent to the standard $H^1(\Omega;\mathbb{R}^2)$ norm on $\boldsymbol{V}$. To simplify the notation, we write $(H^1(\Omega))^2$ for $H^1(\Omega;\mathbb{R}^2)$, $(L^2(\Omega))^2$ for $L^2(\Omega;\mathbb{R}^2)$ and so on. For the force densities, assume

$$\boldsymbol{f}_1 \in W^{1,\infty}(0,T;(L^2(\Omega))^2), \quad \boldsymbol{f}_2 \in W^{1,\infty}(0,T;(L^2(\Gamma_N))^2), \tag{3.7}$$

and for the friction bound function, we assume

$$g \in L^\infty(\Gamma_C), \quad g \geq 0 \text{ a.e. on } \Gamma_C.$$

We define a bilinear form $a(\cdot,\cdot)$ over $\boldsymbol{V}$ by

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx = 2\mu(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})) + \lambda(\mathrm{div}\boldsymbol{u},\mathrm{div}\boldsymbol{v}) \quad \forall\, \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}.$$

Let

$$\langle \boldsymbol{\ell}(t),\boldsymbol{v}\rangle := \int_\Omega \boldsymbol{f}_1(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_N} \boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,ds, \quad j(\boldsymbol{v}) := \int_{\Gamma_C} g|\boldsymbol{v}_\tau|\,ds.$$

Under the assumption (3.7), $\boldsymbol{\ell}(t) \in W^{1,\infty}(0,T;\boldsymbol{V}')$. Then the weak formulation of Problem 3.1 is as follows.

Problem 3.2. Find $\boldsymbol{u} : [0,T] \to \boldsymbol{V}$ such that

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \tag{3.8}$$

and for a.e. $t \in (0,T)$,

$$a(\boldsymbol{u}(t),\boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \geq \langle \boldsymbol{\ell}(t),\boldsymbol{v} - \dot{\boldsymbol{u}}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.9}$$

Assume the initial data satisfies

$$\boldsymbol{u}_0 \in \boldsymbol{V},\ a(\boldsymbol{u}_0,\boldsymbol{v}) + j(\boldsymbol{v}) \geq \langle \boldsymbol{\ell}(0),\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.10}$$

Then we can apply Han and Sofonea [1, Theorem 4.16] to conclude that Problem 3.2 has a unique solution $\boldsymbol{u} \in W^{1,\infty}(0,T;\boldsymbol{V})$. Note that (3.10) is trivially satisfied if $\boldsymbol{u}_0 = \boldsymbol{0}$ and $\boldsymbol{\ell}(0) = \boldsymbol{0}$. More generally, if the initial displacement $\boldsymbol{u}_0$ and the initial loading $\boldsymbol{\ell}(0)$ are in equilibrium in the sense that

$$a(\boldsymbol{u}_0,\boldsymbol{v}) = \langle \boldsymbol{\ell}(0),\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},$$

then (3.10) is valid.

Following [25], we make the following assumption for the mesh $\mathcal{T}_h$ that are compatible with the partition of the boundary Γ_C into its flat components:

$$\Gamma_C = \cup_{i=1}^{i_0}\Gamma_{C,i},$$

where for $1 \leq i \leq i_0$, $\Gamma_{C,i}$ is a line segment if $d = 2$ or a polygon if $d = 3$.

Assumption 3.3. For each $K \in \mathcal{T}_h$, there exists a "virtual triangulation" $\mathcal{T}_K$ of K such that $\mathcal{T}_K$ is uniformly shape regular and quasi-uniform. The corresponding mesh size of $\mathcal{T}_K$ is bounded below by a constant multiple of h_K. Each edge of K is a side of a triangle in $\mathcal{T}_K$.

Define a local virtual element space on the element K,

$$W_K^h := \{v \in H^1(K) : \Delta v = 0, v|_{\partial K} \in C^0(\partial K), v|_e \in \mathbb{P}_1(e)\ \forall\, e \subset \partial K\},$$

with the function values at the vertices of K as a set of degrees of freedom.

Let

$$\boldsymbol{W}_K^h := (W_K^h)^2. \tag{3.11}$$

Then the global virtual element space $\boldsymbol{V}^h$ is defined by

$$\boldsymbol{V}^h := \{\boldsymbol{v} \in \boldsymbol{V} : \boldsymbol{v}|_K \in \boldsymbol{W}_K^h \quad \forall\, K \in \mathcal{T}_h\}.$$

Denote $\boldsymbol{V}_K^h = \boldsymbol{V}^h|_K$. Corresponding to the local degrees of freedom on each element K, we choose function values at the vertices as global degrees of freedom for functions in $\boldsymbol{V}^h$. For a continuous function $\boldsymbol{v}$, the global nodal interpolation operator is denoted by $\boldsymbol{I}^h\boldsymbol{v}$, which is equal to $\boldsymbol{I}_K\boldsymbol{v}$ on K. The following interpolation error estimate is shown in Chen and Huang [25].

Lemma 3.4. *For the nodal interpolation operator $\boldsymbol{I}_K : (H^2(K))^2 \to \boldsymbol{V}_K^h$, there holds*

$$\|\boldsymbol{v} - \boldsymbol{I}_K\boldsymbol{v}\|_{(L^2(K))^2} + h_K\|\boldsymbol{v} - \boldsymbol{I}_K\boldsymbol{v}\|_{\boldsymbol{V}_K} \lesssim h_K^2|\boldsymbol{v}|_{(H^2(K))^2}, \quad \boldsymbol{v} \in (H^2(K))^2 \cap \boldsymbol{V}_K. \tag{3.12}$$

To construct $a^h(\cdot,\cdot)$, as in Beirao Da Veiga et al. [26], Zhang et al. [27], we define a projection operator $\Pi_K : \boldsymbol{W}_K^h \to (\mathbb{P}_1(K))^2$ by the relations

$$\begin{cases}(\boldsymbol{\varepsilon}(\Pi_K \boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{q}))_K = (\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{q}))_K \quad \forall \boldsymbol{q} \in (\mathbb{P}_1(K))^2,\\ \displaystyle\int_{\partial K} \Pi_K \boldsymbol{v}\, ds = \int_{\partial K} \boldsymbol{v}\, ds,\\ \displaystyle\int_K \nabla \times \Pi_K \boldsymbol{v}\, dx = \int_K \nabla \times \boldsymbol{v}\, dx.\end{cases}$$

At this point, we construct the bilinear form $a_K^h(\cdot,\cdot)$ on each polygon $K \in \mathcal{T}_h$ as follows:

$$\begin{aligned} a_K^h(\boldsymbol{u}^h, \boldsymbol{v}^h) &= 2\mu\,(\boldsymbol{\varepsilon}(\boldsymbol{\Pi}_K \boldsymbol{u}^h), \boldsymbol{\varepsilon}(\boldsymbol{\Pi}_K \boldsymbol{v}^h))_K + \lambda\big(\Pi_0^0 \mathrm{div} \boldsymbol{u}^h, \Pi_0^0 \mathrm{div} \boldsymbol{v}^h\big)_K \\ &\quad + S_K(\boldsymbol{u}^h - \boldsymbol{\Pi}_K \boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{\Pi}_K \boldsymbol{v}^h) \quad \forall \boldsymbol{u}^h, \boldsymbol{v}^h \in \boldsymbol{V}_K^h, \end{aligned} \tag{3.13}$$

where Π_0^0 is the L^2-projection on K to $\mathbb{P}_0(K)$ and $S_K(\boldsymbol{v}, \boldsymbol{w}) = \sum_{i=1}^{N_K} \chi_i(\boldsymbol{v})\chi_i(\boldsymbol{w})$, χ_i being the ith local degree of freedom on K, $1 \le i \le N_K := \dim \boldsymbol{V}_K^h$.

To define an approximation of the right-hand side $\boldsymbol{\ell}(t)$, as in Beirao Da Veiga et al. [26], we introduce

$$\langle \boldsymbol{\ell}^h(t), \boldsymbol{v}^h \rangle := \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\Pi}_0^0 \boldsymbol{f}_1(t) \cdot \hat{\boldsymbol{v}}^h\, dx + \int_{\Gamma_N} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}^h\, ds \quad \forall \boldsymbol{v}^h \in \boldsymbol{V}^h,$$

where $\boldsymbol{\Pi}_0^0$ is the projection operator in $(L^2(K))^2$ norm onto the space $(\mathbb{P}_0(K))^2$ and $\hat{\boldsymbol{v}}^h$ is the average value of the function $\boldsymbol{v}^h$ over all vertices of K. According to Chen and Huang [25], Beirao Da Veiga et al. [26], for all $t \in [0,T]$, we have

$$(\boldsymbol{\ell}(t), \boldsymbol{v}^h) - \langle \boldsymbol{\ell}^h(t), \boldsymbol{v}^h \rangle \lesssim h \|\boldsymbol{f}_1(t)\|_{(L^2(\Omega))^2} \|\boldsymbol{v}^h\|_{\boldsymbol{V}},$$

which implies

$$\|\boldsymbol{\ell}(t) - \boldsymbol{\ell}^h(t)\|_{(\boldsymbol{V}^h)'} \lesssim h \|\boldsymbol{f}_1(t)\|_{(L^2(\Omega))^2}. \tag{3.14}$$

Now we consider a semi-discrete approximation of Problem 3.2 by the VEM.

Problem 3.5. Find $\boldsymbol{u}^h : [0,T] \mapsto \boldsymbol{V}^h$ such that

$$\boldsymbol{u}^h(0) = \mathcal{P}^h \boldsymbol{u}_0 \tag{3.15}$$

and for $t \in (0,T)$, $\dot{\boldsymbol{u}}^h(t) \in \boldsymbol{V}^h$ with

$$a^h(\boldsymbol{u}^h(t), \boldsymbol{v}^h - \dot{\boldsymbol{u}}^h(t)) + j(\boldsymbol{v}^h) - j(\dot{\boldsymbol{u}}^h(t)) \ge \langle \boldsymbol{\ell}^h(t), \boldsymbol{v}^h - \dot{\boldsymbol{u}}^h(t) \rangle \quad \forall \boldsymbol{v}^h \in \boldsymbol{V}^h. \tag{3.16}$$

Similar to Problem 3.2, under the stated conditions on the data, Problem 3.5 has a unique solution $\boldsymbol{u}^h \in W^{1,\infty}(0,T;\boldsymbol{V}^h)$.

According to the definition of $a_K^h(\cdot,\cdot)$ in (3.13), it satisfies the condition (2.8). Similar to the result in Beirao Da Veiga et al. [26], Brenner [28], the stability condition (2.9) can be obtained by the norm equivalence in Chen and Huang [25]. Applying the general results in Chen and Huang [25], Brenner et al. [29], Brenner and Scott [30] for the virtual element space (3.11) with a degree $k = 1$, due to Assumption 3.3, we get the following result. We denote $\boldsymbol{V}|_K$ by $\boldsymbol{V}_K$.

Lemma 3.6. *For every $\boldsymbol{v} \in (H^2(K))^2$, there exists a function $\boldsymbol{v}_\pi \in (\mathbb{P}_1(K))^2$ such that*

$$\|\boldsymbol{v} - \boldsymbol{v}_\pi\|_{(L^2(K))^2} + h_K \|\boldsymbol{v} - \boldsymbol{v}_\pi\|_{\boldsymbol{V}_K} \lesssim h_K^2 |\boldsymbol{v}|_{(H^2(K))^2}. \tag{3.17}$$

Lemma 3.7. *Let $\mathcal{P}^h : \boldsymbol{V} \to \boldsymbol{V}^h$ be defined by (2.11). Then for all $\boldsymbol{u} \in \boldsymbol{V} \cap (H^2(\Omega))^2$,*

$$\|\mathcal{P}^h \boldsymbol{u} - \boldsymbol{u}\|_{\boldsymbol{V}} \lesssim h\, |\boldsymbol{u}|_{(H^2(\Omega))^2}. \tag{3.18}$$

Proof. Since $\mathrm{meas}(\Gamma_D) > 0$, by the Poincaré-Friedrichs inequality, we have

$$\|\mathcal{P}^h \boldsymbol{u} - \boldsymbol{u}\|_{(L^2(\Omega))^2} \lesssim |\mathcal{P}^h \boldsymbol{u} - \boldsymbol{u}|_{\boldsymbol{V}}. \tag{3.19}$$

Denote $\boldsymbol{w}_h := \mathcal{P}^h \boldsymbol{u} - \boldsymbol{u}_I$. Then

$$\alpha_\star |\boldsymbol{w}_h|_{\boldsymbol{V}}^2 \le a^h(\boldsymbol{w}_h, \boldsymbol{w}_h) = a^h(\mathcal{P}^h \boldsymbol{u}, \boldsymbol{w}_h) - a^h(\boldsymbol{u}_I, \boldsymbol{w}_h). \tag{3.20}$$

By the definition (2.11),

$$a^h(\mathcal{P}^h \boldsymbol{u}, \boldsymbol{w}_h) = a^h(\boldsymbol{u}, \boldsymbol{w}_h).$$

Now

$$a^h(\boldsymbol{u}_I, \boldsymbol{w}_h) = \sum_{K \in \mathcal{T}_h} a_K^h(\boldsymbol{u}_I, \boldsymbol{w}_h) = \sum_{K \in \mathcal{T}_h} \left[a_K^h(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{w}_h) + a_K^h(\boldsymbol{u}_\pi, \boldsymbol{w}_h)\right].$$

By the vector-valued function version of (2.8),

$$a_K^h(\boldsymbol{u}_\pi, \boldsymbol{w}_h) = a_K(\boldsymbol{u}_\pi, \boldsymbol{w}_h).$$

Hence, we may write

$$a^h(\boldsymbol{u}_I, \boldsymbol{w}_h) = \sum_{K\in\mathcal{T}_h} \left[a_K^h(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{w}_h) + a_K(\boldsymbol{u}_\pi - \boldsymbol{u}, \boldsymbol{w}_h)\right] + a(\boldsymbol{u}, \boldsymbol{w}_h).$$

Thus, from (3.20),

$$\alpha_\star |\boldsymbol{w}_h|_V^2 \le - \sum_{K\in\mathcal{T}_h} \left[a_K^h(\boldsymbol{u}_I - \boldsymbol{u}_\pi, \boldsymbol{w}_h) + a_K(\boldsymbol{u}_\pi - \boldsymbol{u}, \boldsymbol{w}_h)\right].$$

Then,

$$|\boldsymbol{w}_h|_V \lesssim \left[\sum_{K\in\mathcal{T}_h} \left(|\boldsymbol{u} - \boldsymbol{u}_\pi|_{V_K}^2 + |\boldsymbol{u} - \boldsymbol{u}_I|_{V_K}^2\right)\right]^{1/2}.$$

Applying (3.17) and (3.12), we deduce that

$$|\boldsymbol{w}_h|_V \lesssim h\, |\boldsymbol{u}|_{(H^2(\Omega))^2}. \tag{3.21}$$

By the triangle inequality, (3.12) and (3.21), we have

$$|\mathcal{P}^h \boldsymbol{u} - \boldsymbol{u}|_V \lesssim h\, |\boldsymbol{u}|_{(H^2(\Omega))^2}. \tag{3.22}$$

Combining (3.19) and (3.22), we get the desired result (3.18). □

Theorem 3.8. *Let $\boldsymbol{u} \in W^{1,\infty}(0,T;\boldsymbol{V})$ and $\boldsymbol{u}^h \in W^{1,\infty}(0,T;\boldsymbol{V}^h)$ be the solutions of Problems 3.2 and 3.5, respectively. Assume $\boldsymbol{u} \in L^\infty(0,T;(H^2(\Omega))^2)$, $\dot{\boldsymbol{u}} \in L^1(0,T;(H^2(\Omega))^2)$, $\dot{\boldsymbol{u}}|_{\Gamma_{C,i}} \in L^1(0,T;(H^2(\Gamma_{C,i}))^2)$ for $1 \le i \le i_0$. Then we have*

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_{L^\infty(0,T;\boldsymbol{V})} \lesssim h.$$

Proof. Let $\boldsymbol{I}^h\dot{\boldsymbol{u}}$ be the global nodal interpolant of $\dot{\boldsymbol{u}}$. We apply Theorem 2.3 and choose $\boldsymbol{v}^h = \boldsymbol{I}^h\dot{\boldsymbol{u}}$:

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}^h\|_{L^\infty(0,T;\boldsymbol{V})} &\lesssim \|\boldsymbol{u} - \mathcal{P}^h\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{V})} + \|\mathcal{P}^h\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}\|_{L^1(0,T;\boldsymbol{V})} + \|\dot{\boldsymbol{u}} - \boldsymbol{I}^h\dot{\boldsymbol{u}}\|_{L^1(0,T;V)}\\ &\quad + \|R(\boldsymbol{I}^h\dot{\boldsymbol{u}})\|_{L^1(0,T)}^{1/2} + \|\boldsymbol{\ell} - \boldsymbol{\ell}^h\|_{W^{1,1}(0,T;(\boldsymbol{V}^h)')},\end{aligned} \tag{3.23}$$

where

$$R(\boldsymbol{I}^h\dot{\boldsymbol{u}}(t);t) = a(\boldsymbol{u}(t), \boldsymbol{I}^h\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}(t)) + j(\boldsymbol{I}^h\dot{\boldsymbol{u}}(t)) - j(\dot{\boldsymbol{u}}(t)) - \langle \boldsymbol{\ell}(t), \boldsymbol{I}^h\dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}(t)\rangle \tag{3.24}$$

From (3.12),

$$\|\dot{\boldsymbol{u}} - \boldsymbol{I}^h\dot{\boldsymbol{u}}\|_{L^1(0,T;\boldsymbol{V})} \lesssim h\|\dot{\boldsymbol{u}}\|_{L^1(0,T;(H^2(\Omega))^2)}.$$

By (3.18),

$$\|\boldsymbol{u} - \mathcal{P}^h\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{V})} \lesssim h\|\boldsymbol{u}\|_{L^\infty(0,T;(H^2(\Omega))^2)},$$

$$\|\mathcal{P}^h\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}\|_{L^1(0,T;\boldsymbol{V})} \lesssim h\|\dot{\boldsymbol{u}}\|_{L^1(0,T;(H^2(\Omega))^2)}.$$

In view of (3.14) and the embedding theorem, we get

$$\|\boldsymbol{\ell} - \boldsymbol{\ell}^h\|_{W^{1,1}(0,T;(\boldsymbol{V}^h)')} \lesssim h\|\boldsymbol{f}_1\|_{W^{1,\infty}(0,T;(L^2(\Omega))^2)}.$$

It only remains to bound the term $\|R(\boldsymbol{I}^h\dot{\boldsymbol{u}})\|_{L^1(0,T)}^{1/2}$.

As in Han and Sofonea [1, Section 8.1], we can show that under the stated regularity conditions, the weak solution $\boldsymbol{u}$ satisfies the relations (3.1) and (3.2) a.e. in $\Omega \times (0,T)$ and the relation (3.4) a.e. on $\Gamma_N \times (0,T)$. By integration by parts, for all $\boldsymbol{v} \in \boldsymbol{V}$ and $t \in [0,T]$, we have

$$\begin{aligned}a(\boldsymbol{u}(t), \boldsymbol{v}) &= \int_\Gamma (\boldsymbol{\sigma}(t)\boldsymbol{\nu})\cdot\boldsymbol{v}\, ds - \int_\Omega \operatorname{div}\boldsymbol{\sigma}(t)\cdot\boldsymbol{v}\, dx\\ &= \int_{\Gamma_C} \boldsymbol{\sigma}_{\boldsymbol{\tau}}(t)\cdot\boldsymbol{v}_{\boldsymbol{\tau}}\, ds + \int_{\Gamma_N} \boldsymbol{f}_2(t)\cdot\boldsymbol{v}\, ds + \int_\Omega \boldsymbol{f}_1(t)\cdot\boldsymbol{v}\, dx.\end{aligned}$$

Then from (3.24),

$$R(\boldsymbol{I}^h\dot{\boldsymbol{u}}(t);t) = \int_{\Gamma_C} \left[\boldsymbol{\sigma}_{\boldsymbol{\tau}}(t)\cdot(\boldsymbol{I}^h\dot{\boldsymbol{u}}_{\boldsymbol{\tau}}(t) - \dot{\boldsymbol{u}}_{\boldsymbol{\tau}}(t)) + g\big(|\boldsymbol{I}^h\dot{\boldsymbol{u}}_{\boldsymbol{\tau}}(t)| - |\dot{\boldsymbol{u}}_{\boldsymbol{\tau}}(t)|\big)\right] ds.$$

Note that on Γ_C, $\boldsymbol{I}^h\dot{\boldsymbol{u}}$ is equal to the usual continuous $(\mathbb{P}_1)^2$ nodal interpolant of $\dot{\boldsymbol{u}}$. Then,

$$\|R(\boldsymbol{I}^h\dot{\boldsymbol{u}})\|_{L^1(0,T)} \lesssim \|\boldsymbol{I}^h\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}\|_{L^1(0,T;(L^2(\Gamma_C))^2)} \lesssim h^2\Big(\sum_{i=1}^{i_0} |\dot{\boldsymbol{u}}|_{L^1(0,T;(H^2(\Gamma_{C,i}))^2)}^2\Big)^{1/2}. \tag{3.25}$$

Using Theorem 2.3, we obtain the desired result. □

Then we consider the fully discrete approximation of Problem 3.2.

Problem 3.9. Find $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset \boldsymbol{V}^h$ such that

$$\boldsymbol{u}_0^{hk} = \mathcal{P}^h \boldsymbol{u}_0 \tag{3.26}$$

and for $n = 1, \dots, N$,

$$a^h(\boldsymbol{u}_n^{hk}, \boldsymbol{v}^h - \delta_n \boldsymbol{u}_n^{kh}) + j(\boldsymbol{v}^h) - j(\delta_n \boldsymbol{u}_n^{hk}) \ge \langle \boldsymbol{\ell}_n^h, \boldsymbol{v}^h - \delta_n \boldsymbol{u}_n^{hk}\rangle \quad \forall \boldsymbol{v}^h \in \boldsymbol{V}^h. \tag{3.27}$$

Theorem 3.10. *Let $\boldsymbol{u} \in W^{1,\infty}(0,T;\boldsymbol{V})$ and $\boldsymbol{u}^{hk} \subset \boldsymbol{V}^h$ be the solutions of Problems 3.2 and 3.9, respectively. Assume $\boldsymbol{u} \in C^1([0,T];(H^2(\Omega))^2)$, $\ddot{\boldsymbol{u}} \in L^1(0,T;\boldsymbol{V})$, $\dot{\boldsymbol{u}}|_{\Gamma_{C,i}} \in C([0,T];(H^2(\Gamma_{C,i}))^2)$ for $1 \le i \le i_0$. Then, we have the optimal order error estimate*

$$\max_{1\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}} \lesssim h + k. \tag{3.28}$$

Proof. We apply Theorem 2.6 with $\boldsymbol{v}_n^h = \boldsymbol{I}^h \dot{\boldsymbol{u}}_n$, the global interpolant of $\dot{\boldsymbol{u}}_n$, $1 \le n \le N$:

$$\begin{aligned}
\max_{1\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}} \lesssim{}& \max_{1\le n\le N} \|\boldsymbol{u}_n - \mathcal{P}^h \boldsymbol{u}_n\|_{\boldsymbol{V}} + \sum_{n=1}^N k_n \|\delta_n \mathcal{P}^h \boldsymbol{u}_n - \dot{\boldsymbol{u}}_n\|_{\boldsymbol{V}} + \sum_{n=1}^N k_n \|\dot{\boldsymbol{u}}_n - \boldsymbol{I}^h \dot{\boldsymbol{u}}_n\|_{\boldsymbol{V}} \\
&+ \max_{1\le n\le N} \|\boldsymbol{\ell}_n - \boldsymbol{\ell}_n^h\|_{(V^h)'} + \sum_{n=1}^{N-1} \|(\boldsymbol{\ell}_{n+1} - \boldsymbol{\ell}_n) - (\boldsymbol{\ell}_{n+1}^h - \boldsymbol{\ell}_n^h)\|_{(V^h)'} \\
&+ \Big(\sum_{n=1}^N k_n R_n(\boldsymbol{I}^h \dot{\boldsymbol{u}}_n) \Big)^{1/2},
\end{aligned} \tag{3.29}$$

where

$$R_n(\boldsymbol{I}^h \dot{\boldsymbol{u}}_n) := a(\boldsymbol{u}_n, \boldsymbol{I}^h \dot{\boldsymbol{u}}_n - \dot{\boldsymbol{u}}_n) + j(\boldsymbol{I}^h \dot{\boldsymbol{u}}_n) - j(\dot{\boldsymbol{u}}_n) - \langle \boldsymbol{\ell}_n, \boldsymbol{I}^h \dot{\boldsymbol{u}}_n - \dot{\boldsymbol{u}}_n\rangle.$$

By (3.18),

$$\|\boldsymbol{u}_n - \mathcal{P}^h \boldsymbol{u}_n\|_{\boldsymbol{V}} \lesssim h \|\boldsymbol{u}_n\|_{H^2(\Omega)^2} \lesssim h \|\boldsymbol{u}\|_{C(0,T;H^2(\Omega)^2)}.$$

Note that

$$\sum_{n=1}^N k_n \|\delta_n \mathcal{P}^h \boldsymbol{u}_n - \dot{\boldsymbol{u}}_n\|_{\boldsymbol{V}} \le \sum_{n=1}^N k_n \|\delta_n \mathcal{P}^h \boldsymbol{u}_n - \mathcal{P}^h \dot{\boldsymbol{u}}_n\|_{\boldsymbol{V}} + \sum_{n=1}^N k_n \|\mathcal{P}^h \dot{\boldsymbol{u}}_n - \dot{\boldsymbol{u}}_n\|_{\boldsymbol{V}}.$$

By (3.18), we have

$$\|\dot{\boldsymbol{u}}_n - \mathcal{P}^h \dot{\boldsymbol{u}}_n\|_{\boldsymbol{V}} \lesssim h \|\dot{\boldsymbol{u}}_n\|_{H^2(\Omega)^2} \lesssim h \|\dot{\boldsymbol{u}}\|_{C(0,T;H^2(\Omega)^2)},$$

$$\sum_{n=1}^N k_n \|\mathcal{P}^h \dot{\boldsymbol{u}}_n - \dot{\boldsymbol{u}}_n\|_{\boldsymbol{V}} \lesssim h \|\dot{\boldsymbol{u}}\|_{C(0,T;H^2(\Omega)^2)}.$$

We take Taylor expansion for $\boldsymbol{u}_{n-1}$ at t_n:

$$\boldsymbol{u}_{n-1} = \boldsymbol{u}_n - k_n \dot{\boldsymbol{u}}_n + \int_{t_n}^{t_{n-1}} (t_{n-1} - s) \ddot{\boldsymbol{u}}(s)\, ds.$$

Then,

$$\delta_n \mathcal{P}^h \boldsymbol{u}_n - \mathcal{P}^h \dot{\boldsymbol{u}}_n = k_n^{-1} (\mathcal{P}^h \boldsymbol{u}_n - \mathcal{P}^h \boldsymbol{u}_{n-1}) - \mathcal{P}^h \dot{\boldsymbol{u}}_n = k_n^{-1} \int_{t_{n-1}}^{t_n} (t_{n-1} - s) \mathcal{P}^h \ddot{\boldsymbol{u}}(s)\, ds.$$

Hence,

$$\begin{aligned}
\sum_{n=1}^N k_n \|\delta_n \mathcal{P}^h \boldsymbol{u}_n - \mathcal{P}^h \dot{\boldsymbol{u}}_n\|_{\boldsymbol{V}} &\lesssim k \sum_{n=1}^N \|k_n^{-1} \int_{t_{n-1}}^{t_n} (t_{n-1} - s) \mathcal{P}^h \ddot{\boldsymbol{u}}(s)\, ds\|_{\boldsymbol{V}} \\
&\lesssim k \int_0^T \|\mathcal{P}^h \ddot{\boldsymbol{u}}(s)\|_{\boldsymbol{V}}\, ds.
\end{aligned}$$

We then apply the stability inequality (2.12) to get

$$\sum_{n=1}^N k_n \|\delta_n \mathcal{P}^h \boldsymbol{u}_n - \mathcal{P}^h \dot{\boldsymbol{u}}_n\|_{\boldsymbol{V}} \lesssim k \|\ddot{\boldsymbol{u}}\|_{L^1(0,T;\boldsymbol{V})}.$$

By (3.12), we have

$$\sum_{n=1}^N k_n \|\dot{\boldsymbol{u}}_n - \boldsymbol{I}^h \dot{\boldsymbol{u}}_n\|_{\boldsymbol{V}} \lesssim h \|\dot{\boldsymbol{u}}\|_{C([0,T];(H^2(\Omega))^2)}.$$

By (3.14), we have

$$\|\boldsymbol{\ell}_n - \boldsymbol{\ell}_n^h\|_{V_h'} \lesssim h\|\boldsymbol{f}_1\|_{C([0,T];(L^2(\Omega))^2)}.$$

Similar to (3.14),

$$\|(\boldsymbol{\ell}_{n+1} - \boldsymbol{\ell}_n) - (\boldsymbol{\ell}_{n+1}^h - \boldsymbol{\ell}_n^h)\|_{V_h'} \lesssim h\|\boldsymbol{f}_{1,n+1} - \boldsymbol{f}_{1,n}\|_{(L^2(\Omega))^2}.$$

Combing with the embedding theorem,

$$\begin{aligned}\sum_{n=1}^{N-1} \|(\boldsymbol{\ell}_{n+1} - \boldsymbol{\ell}_n) - (\boldsymbol{\ell}_{n+1}^h - \boldsymbol{\ell}_n^h)\|_{V_h'} &\lesssim h \sum_{n=1}^{N-1} \|\boldsymbol{f}_{1,n+1} - \boldsymbol{f}_{1,n}\|_{(L^2(\Omega))^2} \\ &\lesssim h\|\dot{\boldsymbol{f}}_1\|_{L^\infty(0,T;(L^2(\Omega))^2)}.\end{aligned}$$

Similar to the derivation of (3.25), by integration by parts,

$$\sum_{n=1}^{N} k_n R_n(\boldsymbol{I}^h \dot{\boldsymbol{u}}_n) \lesssim \sum_{n=1}^{N} k_n \|\boldsymbol{I}^h \dot{\boldsymbol{u}}_n - \dot{\boldsymbol{u}}_n\|_{(L^2(\Gamma_C))^2}$$

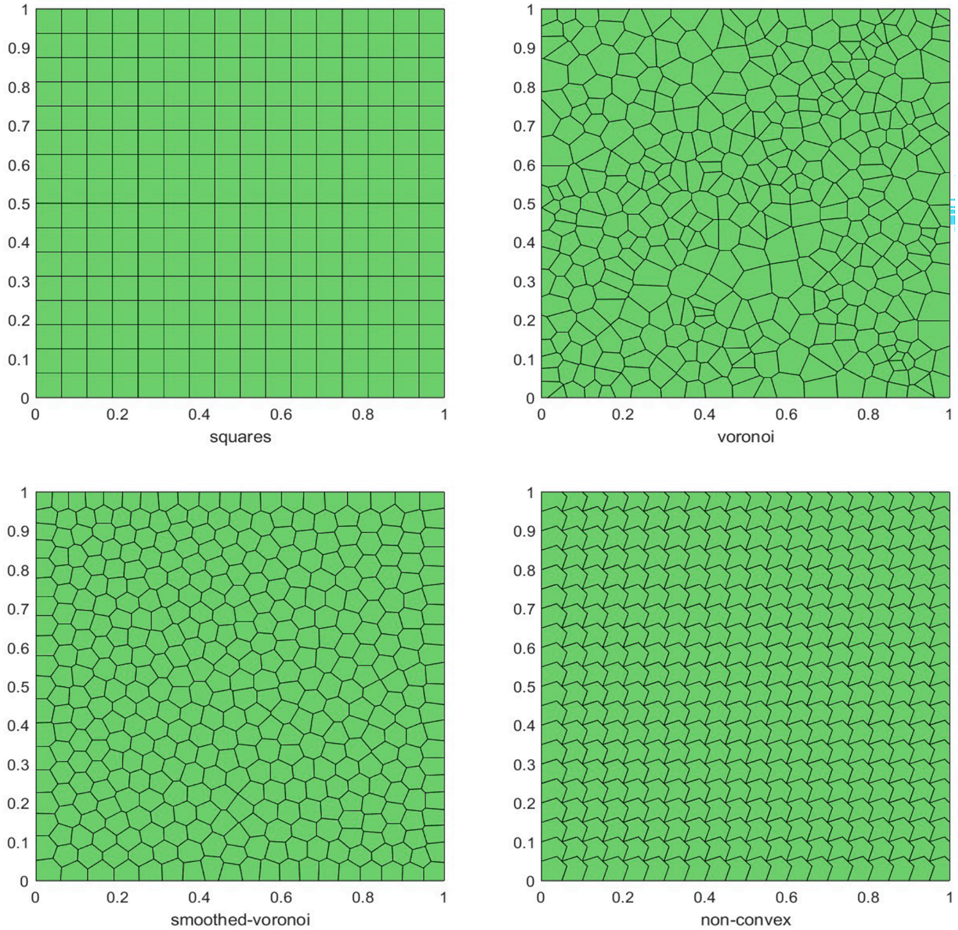

Fig. 1. Four different types of polygonal meshes.

Table 1
Numerical errors at $t = 1$ on square meshes for lowest-order VEM.

(k,h)	$(\frac{1}{4},\frac{1}{4})$	$(\frac{1}{8},\frac{1}{8})$	$(\frac{1}{16},\frac{1}{16})$	$(\frac{1}{32},\frac{1}{32})$	$(\frac{1}{64},\frac{1}{64})$
error	0.1745	0.0911	0.0458	0.0221	0.0099
order	\	0.9377	0.9921	1.0461	1.1681

$$\lesssim \sum_{n=1}^{N} k_n h^2 \sum_{i=1}^{i_0} \|\dot{\boldsymbol{u}}_n\|_{(H^2(\Gamma_{C,i}))^2} \lesssim h^2 \|\dot{\boldsymbol{u}}\|_{C([0,T];(H^2(\Gamma_{C,i}))^2)}.$$

Combining the above inequalities, we derive (3.28) from (3.29). □

4. Numerical solution of the discrete problem

In this section, we report numerical results on a two dimensional test problem. We use the VEM to discretize the spatial variable and uniform backward difference scheme in time. For the discrete problem, we solve it by an adaptive semi-smooth Newton method [24]; for details, cf. Feng et al. [15]. In the numerical simulation, the polygonal meshes are produced by an algorithm discussed in Talischi et al. [31] and the codes are written based on the program described in Sutton [32]. Four different types of meshes used in the following numerical example are presented in Fig. 1.

Let the domain $\Omega = (0,1) \times (0,1)$ represent the cross section of a three-dimensional linearly elastic body. The boundary $\partial\Omega$ is decomposed into three parts: $\Gamma_D, \Gamma_C, \Gamma_F$. On the boundary $\Gamma_D = \{1\} \times (0,1)$, the body is clamped and therefore the displacement field vanishes there. On the boundary $\Gamma_C = (0,1) \times \{0\}$, the body is in bilateral frictional contact with a rigid obstacle, and the friction is modeled with Tresca's law. The remaining part Γ_F is a traction boundary condition, the force $\boldsymbol{f}_2$ acts on the boundary whereas the part of the boundary represented by $(0,1) \times \{1\}$ is traction free. No volume force is assumed to act on the body Ω. For computation, we use the following data

$$\begin{aligned}
&E = 200\,\mathrm{daN/mm^2}, \quad \kappa = 0.3, \quad g = 4\ \mathrm{daN/mm^2},\\
&\boldsymbol{f}_1 = (0,0)^T\ \mathrm{daN/mm^2}, \quad \boldsymbol{f}_2(x_1,x_2,t) = (8(1.25 - x_2)t, -0.01t)\mathrm{daN/mm^2},\\
&\boldsymbol{u}_0 = \boldsymbol{0}, \quad T = 1s.
\end{aligned}$$

where E is the Young's modulus and κ is the Poisson's ratio of the material.

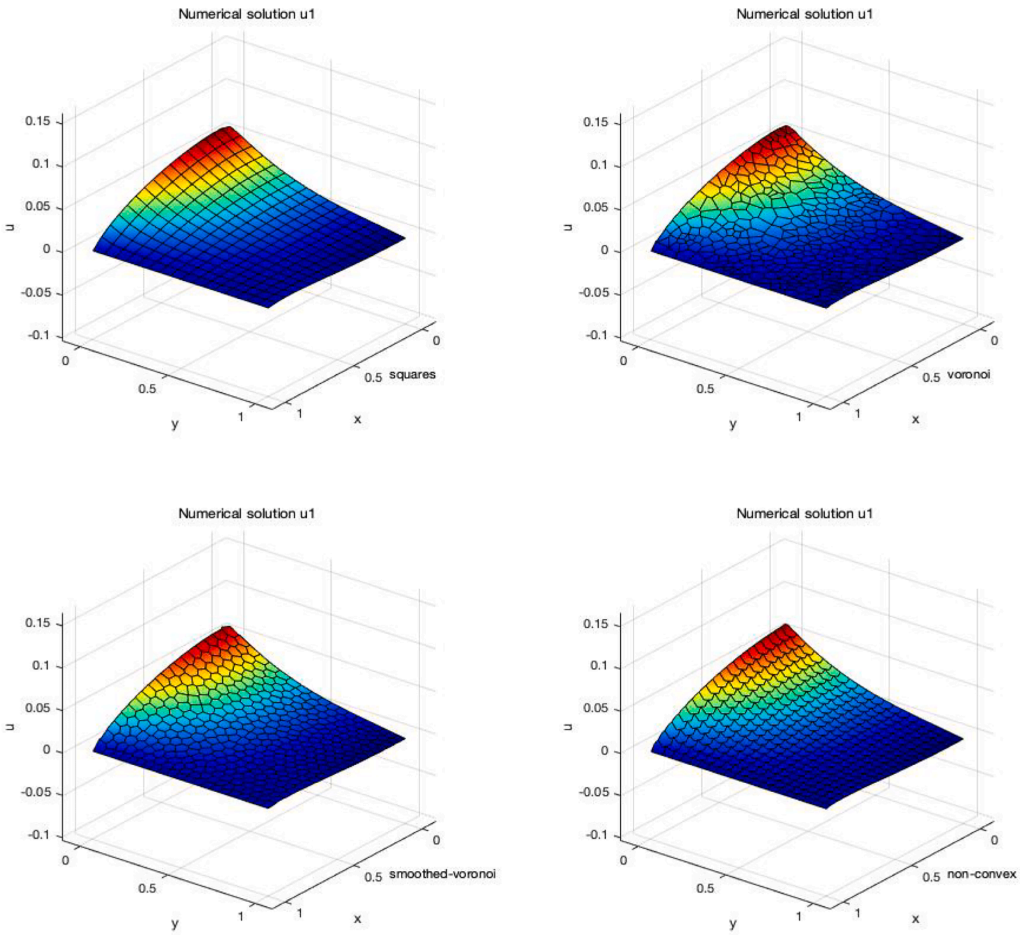

Fig. 2. The numerical solution related to different meshes.

Then the Lame coefficients are

$$\lambda = \frac{E\kappa}{(1+\kappa)(1-2\kappa)}, \quad \mu = \frac{E}{2(1+\kappa)}.$$

The numerical solutions in tangential direction, corresponding to different meshes are displayed in Fig. 2, respectively.

In Table 1 and Fig. 3, we report relative errors of the numerical solutions in the energy norm on square meshes, $\|\boldsymbol{u}_{\rm ref} - \boldsymbol{u}_h\|_E / \|\boldsymbol{u}_{\rm ref}\|_E$, where the energy norm is defined by the formula

$$\|\boldsymbol{v}\|_E := \frac{1}{\sqrt{2}}(C\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q^{1/2}.$$

The aim of this part is to illustrate the convergence of the discrete scheme and to provide numerical evidence of the optimal error estimate obtained in Section 3. To this end, we compute a sequence of numerical solutions by using uniform discretization with the spatial discretization parameter h and the time step k. Since the true solution $\boldsymbol{u}$ is not available, we use the numerical solution corresponding to a fine discretization of Ω and a small time step as the "reference" solution $\boldsymbol{u}_{\rm ref}$ in computing the solution errors. The numerical solution with $h = 1/256$ and $k = 1/256$ is taken to be the "reference" solution $\boldsymbol{u}_{\rm ref}$. Note that the theoretical error bound predicts an optimal first order convergence of the numerical solutions measured in the energy norm, under certain solution regularity assumptions.

The relative errors in the energy norm are shown in Fig. 3.

Remark 4.1. Many people are concerned about comparing the performance of the virtual element method with that of other numerical methods such as the finite element method. However, it is rather involved to address the issue. We refer the reader to the references [33–35] for discussions along this line. Note that if the mesh is triangular, the virtual element method related to W_K^h is the

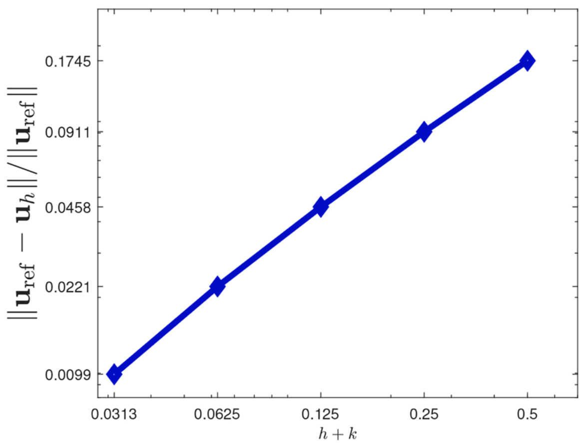

Fig. 3. Relative errors in energy norm.

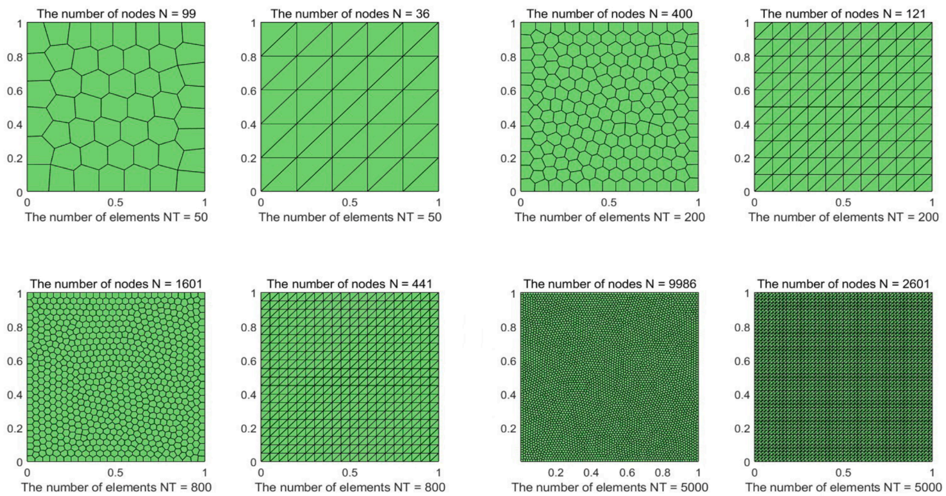

Fig. 4. Comparison of the number of grid nodes for different numbers of elements.

linear finite element method. Thus, it makes sense to compare the number of degrees of freedom of the virtual element method with that of the traditional linear finite element method on grids of same scales. It appears difficult to establish a rigorous and complete theoretical result for the comparison, and so let us present some relevant numerical values. Over a unit square, consider numbers of nodes for polygonal meshes and triangular meshes with the equal number of mesh elements n_e, as shown in Fig. 4. The node counts for the polygonal meshes and the triangular meshes are 99 vs. 36 for $n_e = 50$, 400 vs. 121 for $n_e = 200$, 1601 vs. 441 for $n_e = 800$, and 9986 vs. 2601 for $n_e = 5000$, respectively. When $n_e = 5000$, for the polygonal mesh, the longest element side is 0.0219, and for the triangular mesh, the element side parallel to the axes is 0.02. These numerical results suggest that the polygonal mesh tends to have more vertices than the triangular mesh with the same number of elements, and hence corresponds to a relatively larger number of degrees of freedom.

On the other hand, the virtual element method exhibits certain advantages on robustness against mesh distortion and handling problems with high regularity solutions (cf. Brezzi and Luisa [33], Mengolini et al. [34], Zhang et al. [35]). Moreover, the virtual element method works well even on more complex geometries with unstructured or non-convex meshes.

Evidently, further investigations are needed to determine relative strengths and weaknesses of the virtual element method as compared to the finite element method and other numerical methods.

CRediT authorship contribution statement

Fang Feng: Writing – review & editing, Writing – original draft, Visualization, Software, Methodology, Formal analysis; **Weimin Han:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Conceptualization; **Jianguo Huang:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Formal analysis, Conceptualization.

Data availability

Data will be made available on request.

Authors' contributions

All authors contributed equally to this manuscript.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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