CS:4420 Artificial Intelligence

Spring 2017

Constraint Satisfaction Problems

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Constraint Satisfaction Problems (CSPs)

Standard search problem:

> state is a “black box”—any old data structure
> that supports goal test, eval, successor

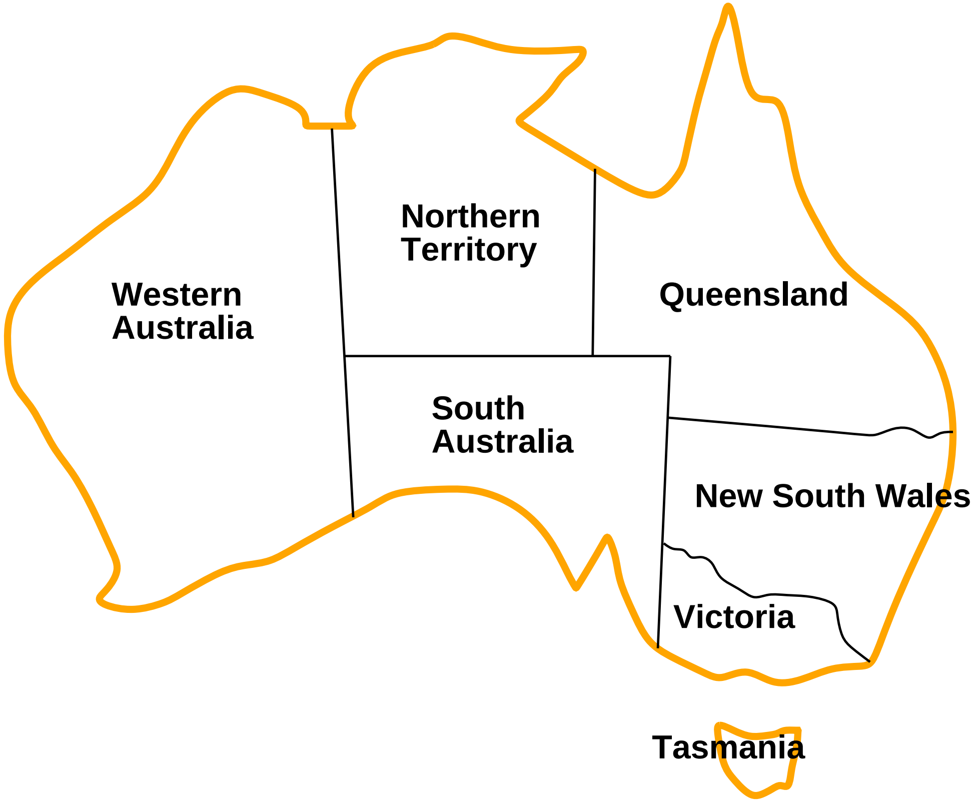
CSP:

> state is defined by variables X_i with values from domain D_i
>
> goal test is a set of constraints specifying
> allowable combinations of values for subsets of variables

Simple example of a formal representation language

Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map coloring



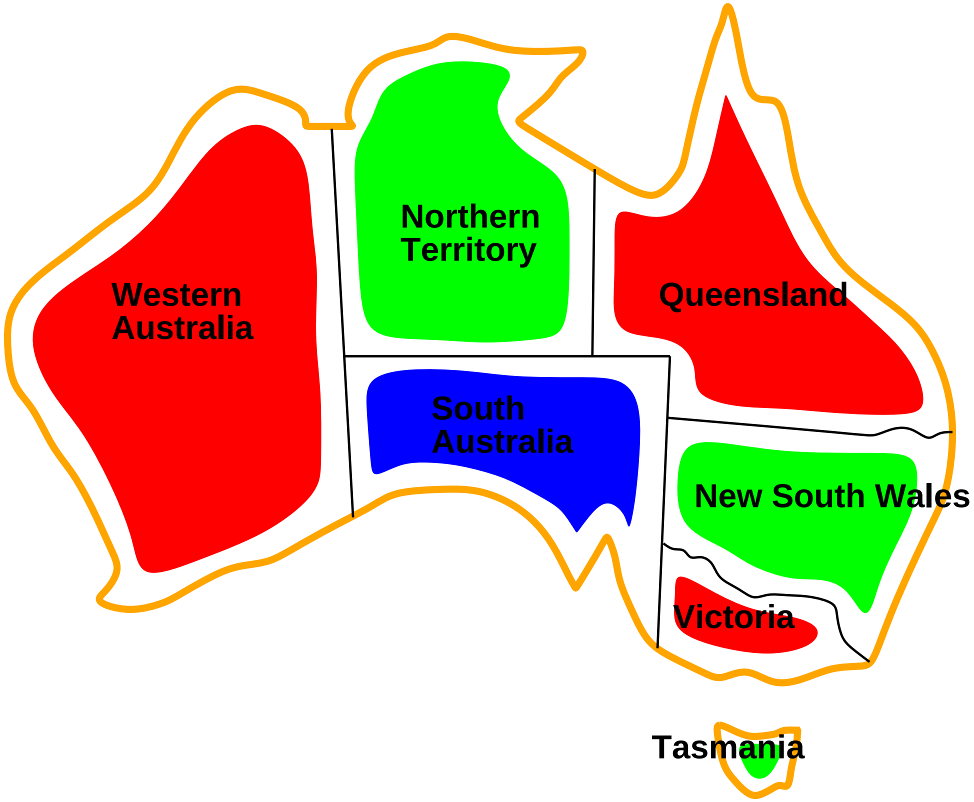
Variables: MA, NT, Q, NSW, V, SA, T

Domains: $D_i = \{r(ed), g(reen), b(lue)\}$

Constraints: adjacent regions must have different colors

e.g., $WA \neq NT$ (if the language allows this), or
$(WA, NT) \in \{(r, g), (r, b), (g, r), (g, b), \ldots\}$

Example: Map coloring contd.



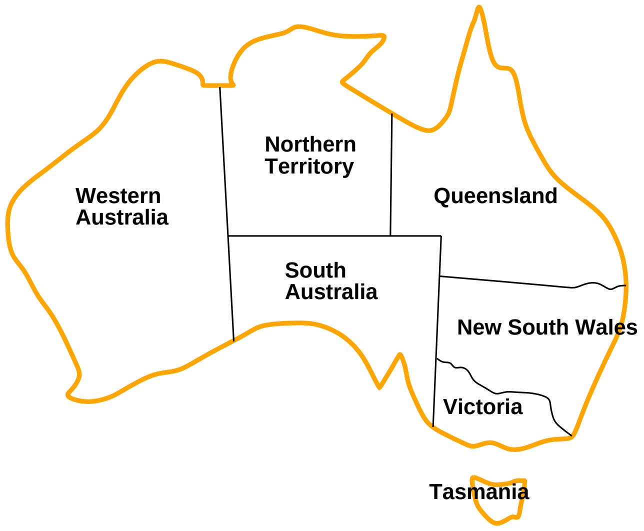
Solutions are assignments satisfying all constraints,

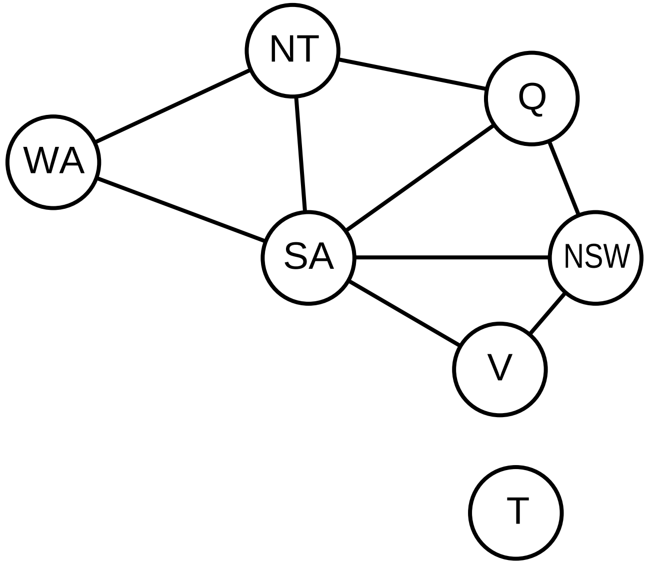
e.g., $\{WA = r, NT = g, Q = r, NSW = g, V = r, SA = b, T = g\}$

Constraint graph

Binary CSP: each constraint relates at most two variables

Constraint graph: nodes are variables, arcs show constraints





General-purpose CSP methods use the graph structure to speed up search

e.g., Tasmania is an independent subproblem!

Varieties of CSPs

Discrete variables

finite domains (size d)

- e.g., Boolean CSPs, incl. Boolean SAT (NP-complete)
- $O(d^n)$ complete assignments

infinite domains (integers, strings, etc.)

- e.g., job scheduling, variables are start/end days for each job
- need a constraint language,
  e.g., $startJob_1 + 5 \leq startJob_3$
- linear constraints solvable, nonlinear undecidable

Continuous variables

- e.g., start/end times for Hubble Telescope observations
- linear constraints solvable in polynolmial time by linear programming methods

Varieties of constraints

Unary constraints involve a single variable

e.g., $SA \neq g$

Binary constraints involve pairs of variables

e.g., $SA \neq WA$

Higher-order constraints involve 3 or more variables

e.g., cryptarithmetic column constraints

Preferences are soft constraints

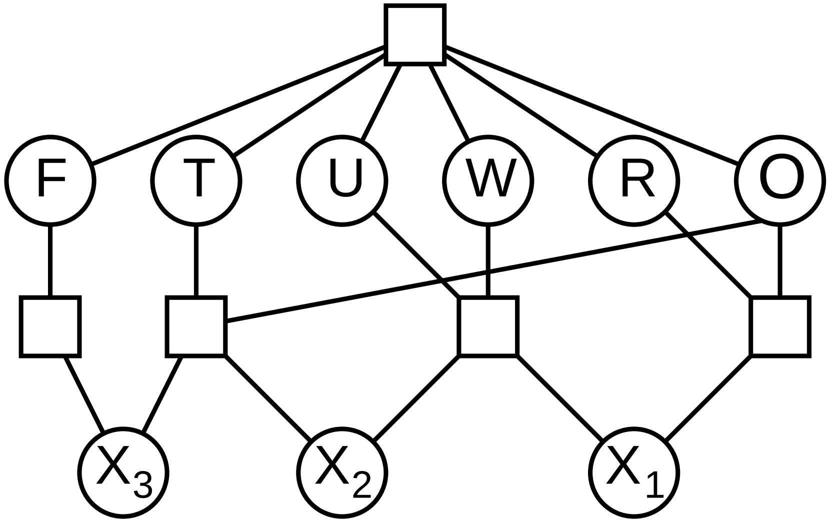
e.g., red is better than $green$

often representable by a cost for each variable assignment

→ constrained optimization problems

Example: Cryptarithmetic

$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$



Variables: $F,\ T,\ U,\ W,\ R,\ O,\ X_1,\ X_2,\ X_3$

Domain: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints: $\mathit{alldiff}\,(F, T, U, W, R, O)$

$O + O = R + 10 \cdot X_1$

$\ldots$

Real-world CSPs

Assignment problems
    e.g., who teaches what class

Timetabling problems
    e.g., which class is offered when and where?

Hardware configuration

Transportation scheduling

Factory scheduling

Floorplanning

Notice that many real-world problems involve real-valued variables

Standard search formulation (incremental)

Let's start with a basic, naive approach and then improve it

States are defined by the values assigned so far

> Initial state: the empty assignment, $\{\ \}$
>
> Successor function: assign a value to an unassigned variable that does not conflict with current assignment.
> fail if no legal assignments (not fixable!)
>
> Goal test: the current assignment is complete

Note:

1. This is the same for all CSPs!
2. Every solution appears at depth n with n variables $\Longrightarrow$ use depth-first search
3. Path is irrelevant, so can also use complete-state formulation
4. However, with domain of size d, branching factor $b = (n - \ell)d$ at depth ℓ, hence $n!d^n$ leaves!

Backtracking search

Variable assignments are commutative
i.e., $[WA = r$ then $NT = g]$ same as $[NT = g$ then $WA = r]$

Only need to consider assignments to a single variable at each node
$\Longrightarrow$ $b = d$ and there are d^n leaves

Depth-first search for CSPs with single-variable assignments is called backtracking search

Backtracking search is the basic uninformed algorithm for CSPs

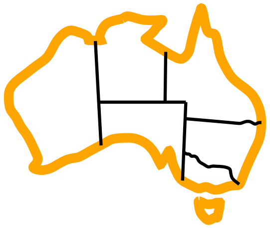
Can solve n-queens for $n \approx 25$

Backtracking search

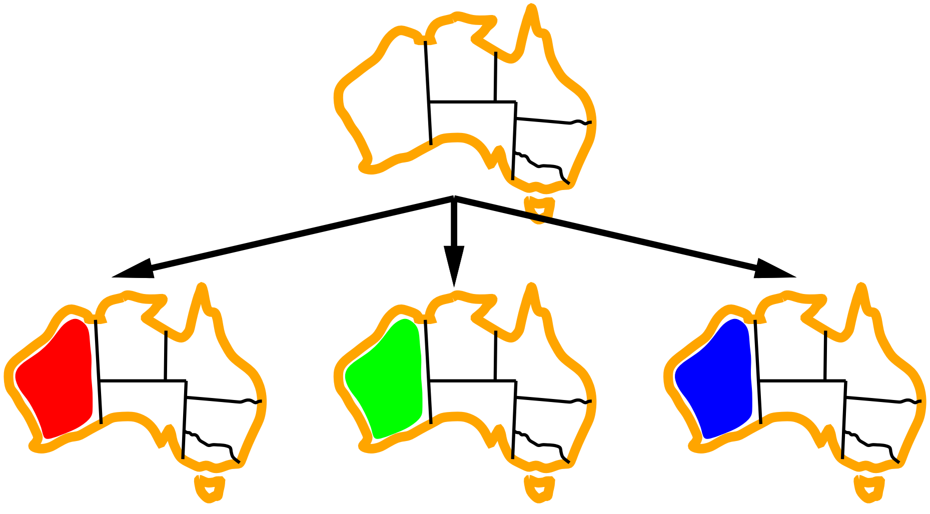
```
function BACKTRACKING-SEARCH($csp$) returns solution/failure
    return RECURSIVE-BACKTRACKING($[\,]$, $csp$)

function RECURSIVE-BACKTRACKING($assigned$, $csp$) returns solution/failure
    if $assigned$ is complete then return $assigned$
    $var \leftarrow$ SELECT-UNASSIGNED-VARIABLE(VARIABLES[$csp$], $assigned$, $csp$)
    for each $value$ in ORDER-DOMAIN-VALUES($var$, $assigned$, $csp$) do
        if $value$ is consistent with $assigned$ according to CONSTRAINTS[$csp$] then
            $result \leftarrow$ RECURSIVE-BACKTRACKING($[var = value|assigned]$, $csp$)
            if $result \neq failure$ then return $result$
    end
    return $failure$
```

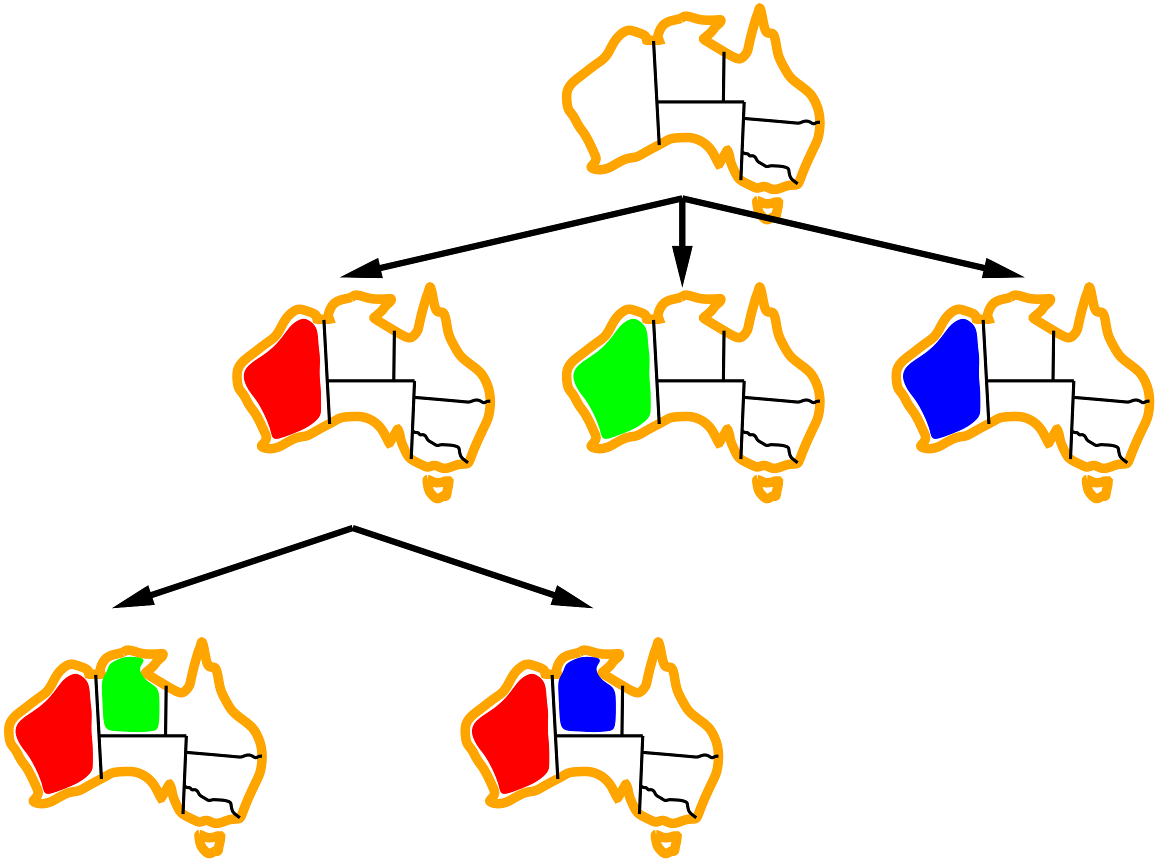
Backtracking example



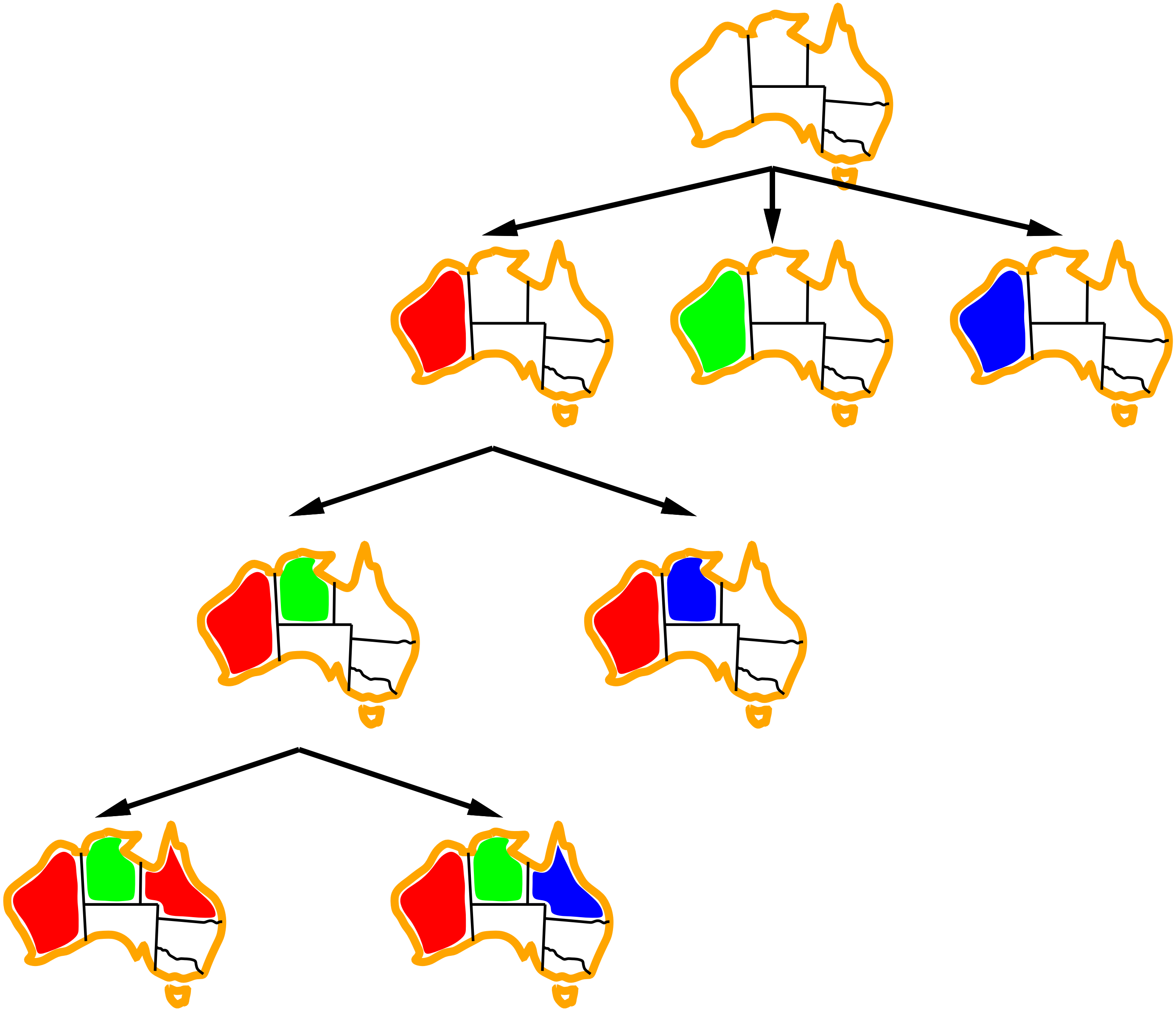
Backtracking example



Backtracking example



Backtracking example



Improving backtracking efficiency

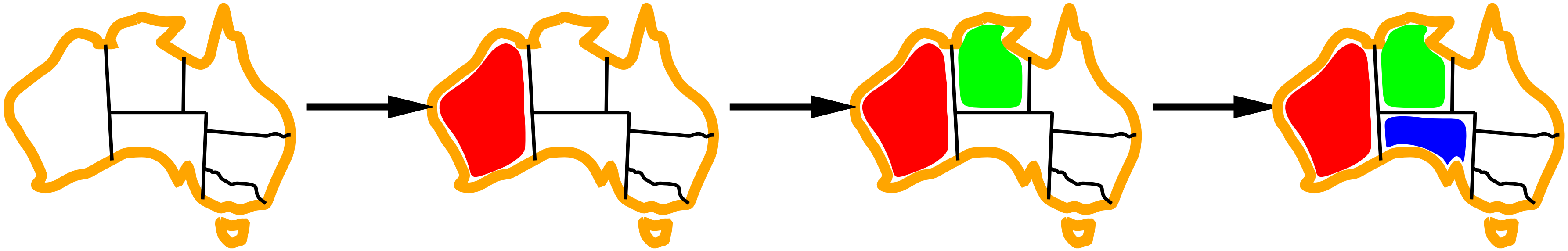
General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?

Variable choice heuristics

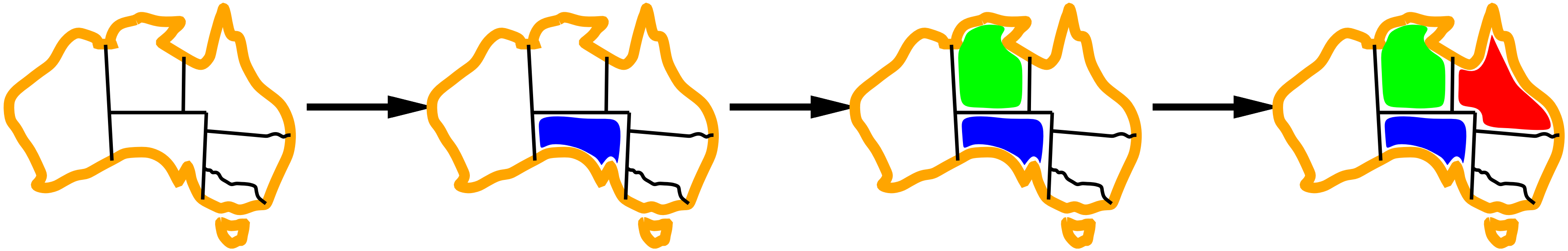
Minimum remaining values (MRV):

choose the variable with the fewest legal values



Degree heuristic:

choose the variable with the most constraints on remaining vars

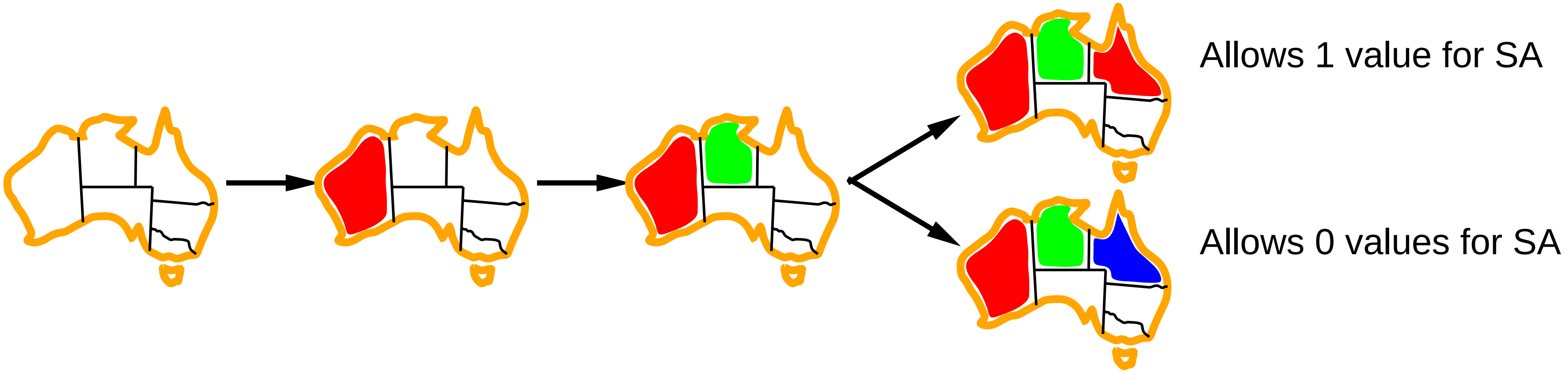


Latter ofter used as a tie-breaker for former

Value choice heuristics

Least constraining value:

- for a given a variable, choose the least constraining value: the one that rules out the fewest values in the remaining variables

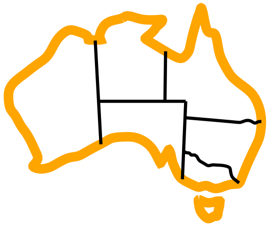


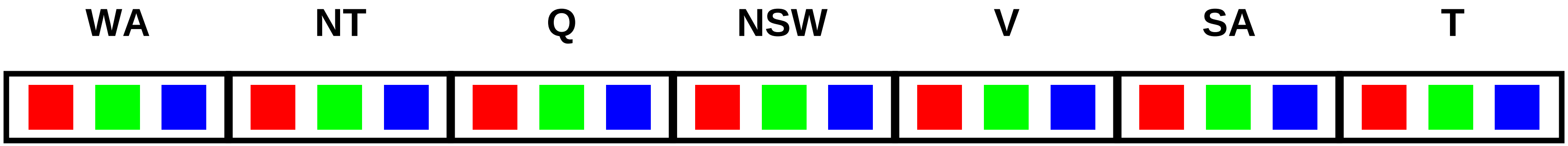
Combining these heuristics makes 1000-queens feasible

Forward checking

Idea: Keep track of remaining legal values for unassigned variables

Terminate search when any variable has no legal values





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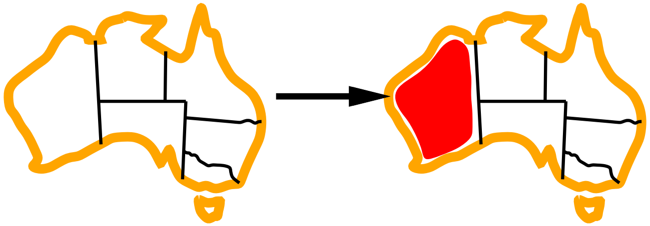

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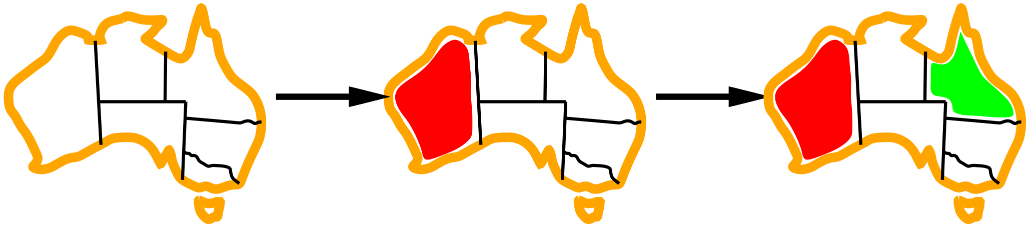

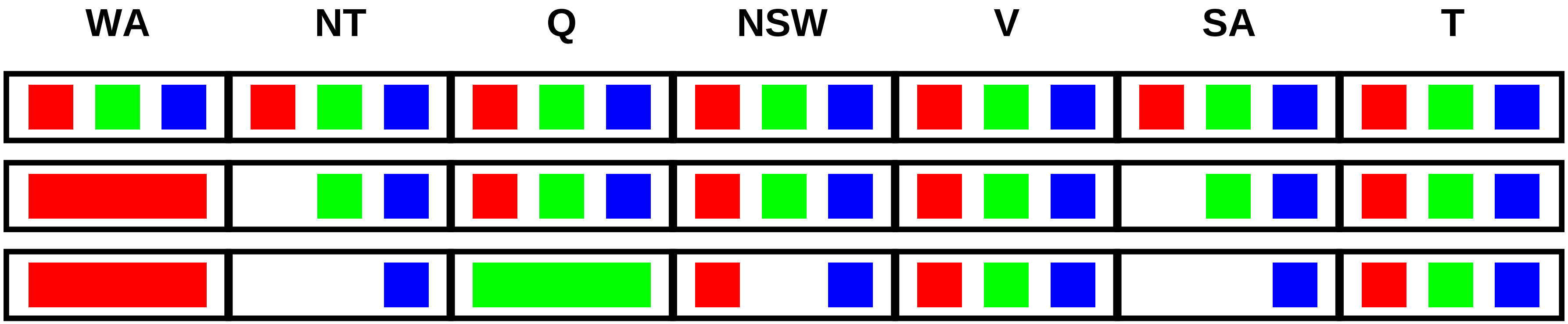

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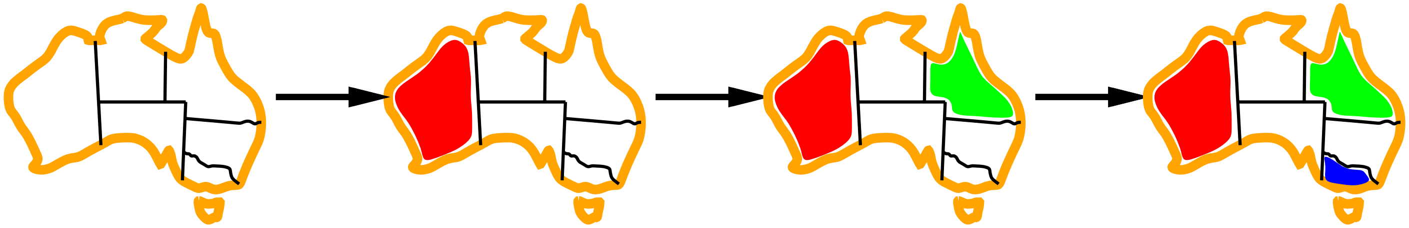

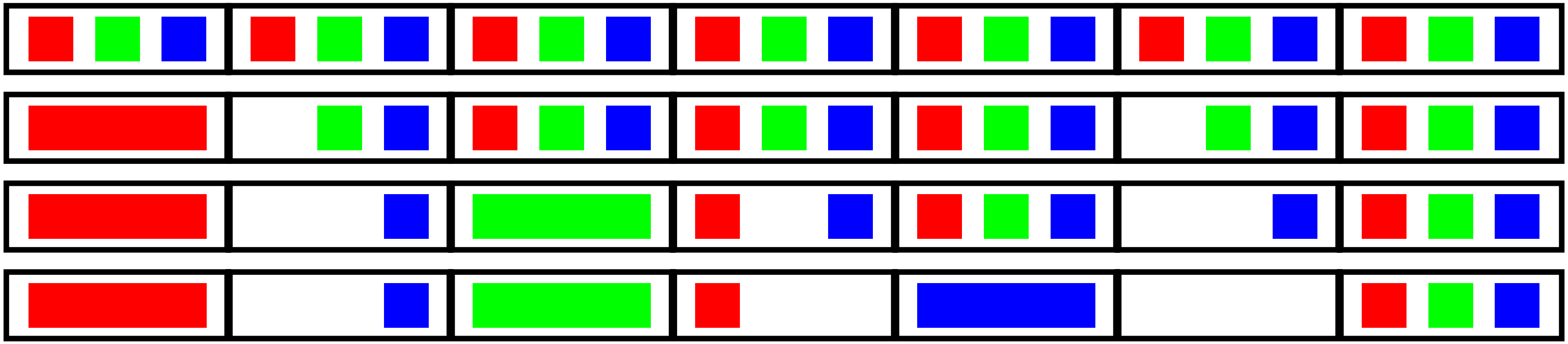

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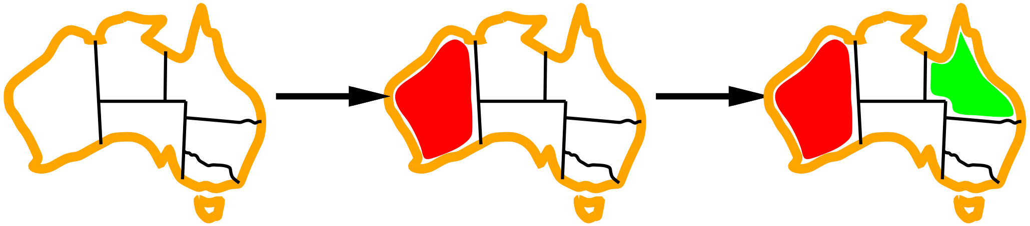
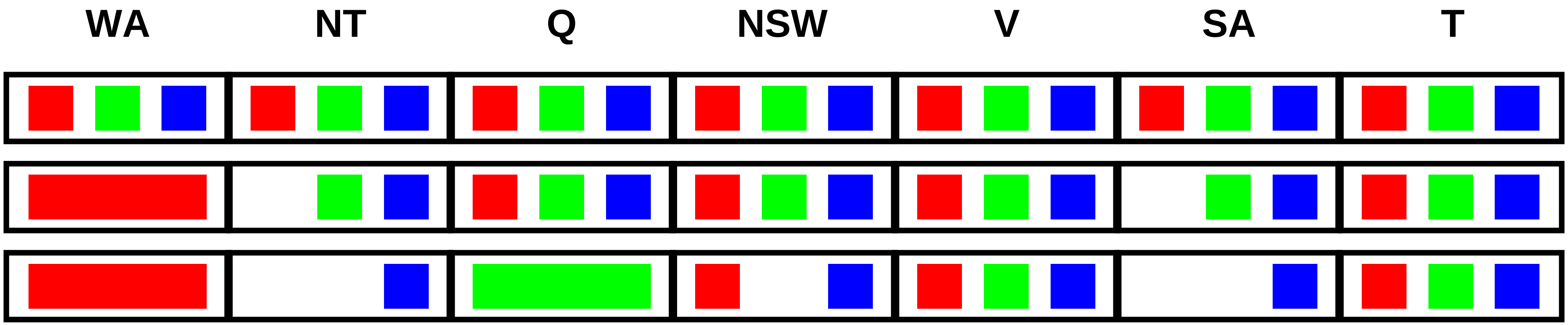
Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:





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NT and SA cannot both be blue!

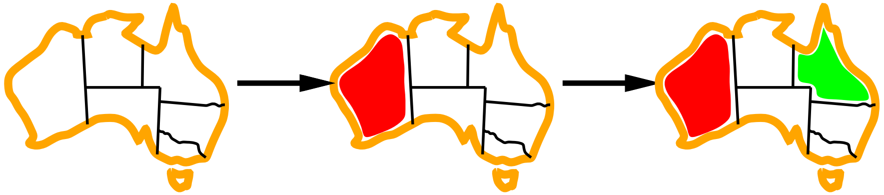
Constraint propagation repeatedly enforces constraints locally

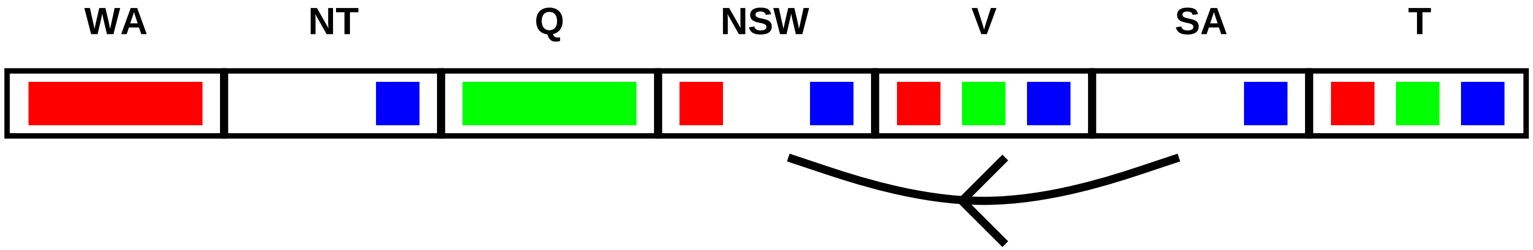
Arc consistency

Simplest form of propagation, makes each arc consistent

Arc $X \rightarrow Y$ is consistent iff

for every value x of X there is some allowed value y for Y





WA	NT	Q	NSW	V	SA	T
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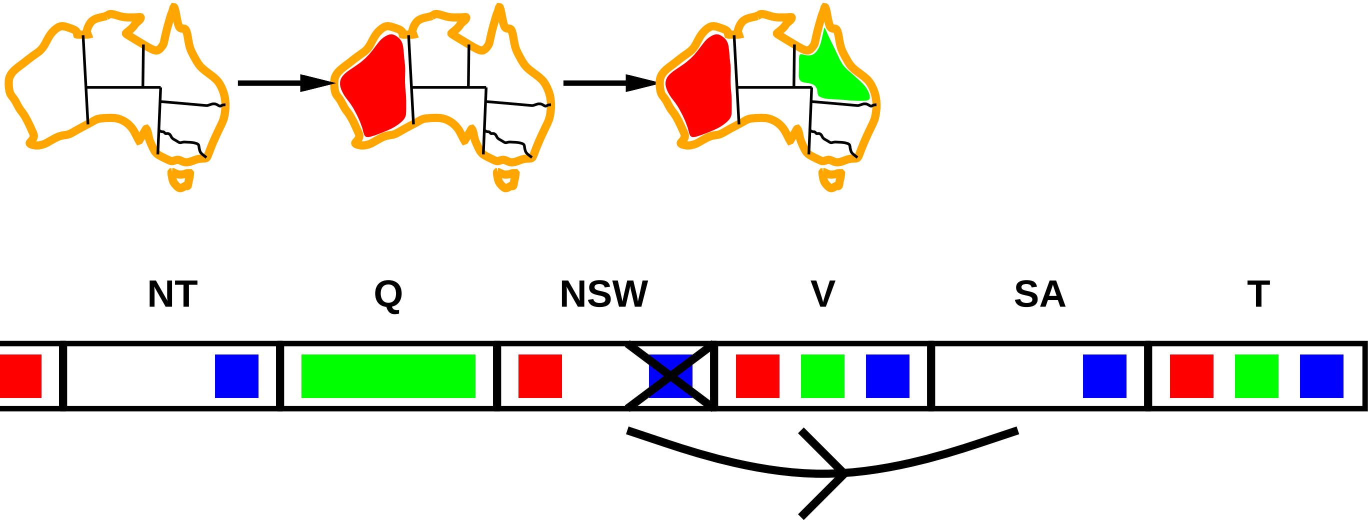

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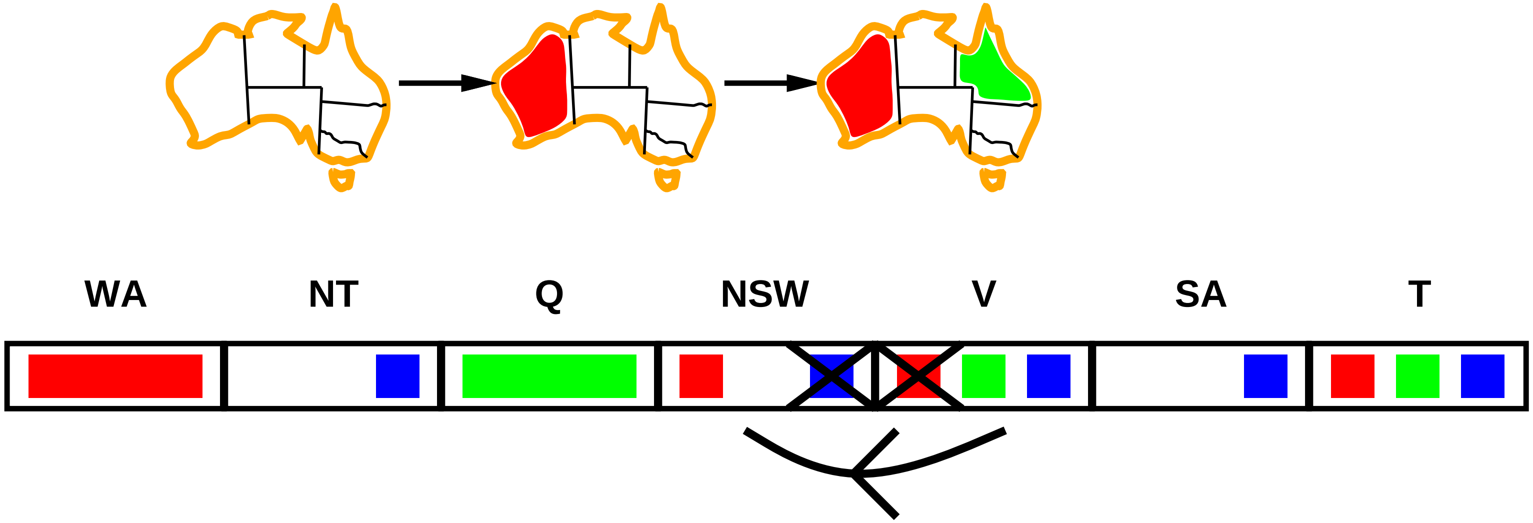

If X loses a value, neighbors of X need to be rechecked

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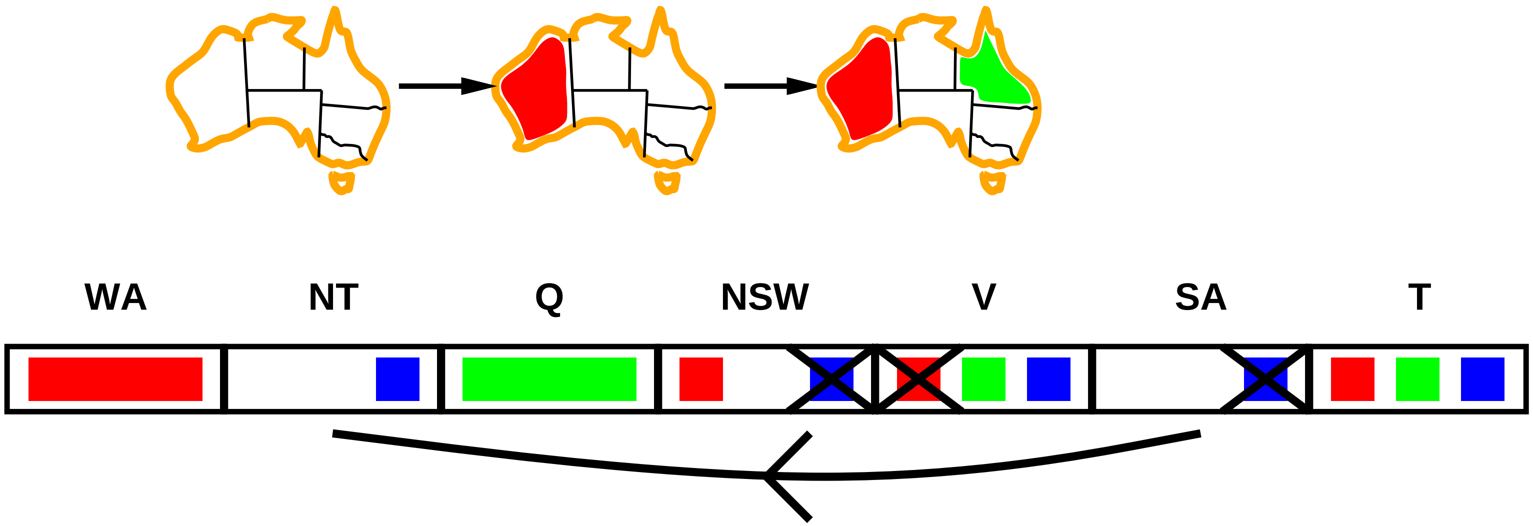


If X loses a value, neighbors of X need to be rechecked

Arc consistency detects failure earlier than forward checking

Can be run as a preprocessor and/or after each assignment

Arc consistency algorithm

```
function AC-3( $csp$) returns the CSP, possibly with reduced domains
    inputs: $csp$, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
    local variables: $queue$, a queue of arcs, initially all the arcs in $csp$

    while $queue$ is not empty do
        $(X_i, X_j) \leftarrow$ REMOVE-FIRST($queue$)
        if REMOVE-INCONSISTENT-VALUES($X_i, X_j$) then
            for each $X_k$ in NEIGHBORS[$X_i$] do
                add $(X_k, X_i)$ to $queue$
```

```
function REMOVE-INCONSISTENT-VALUES( $X_i, X_j$) returns true iff we remove a value
    $removed \leftarrow false$
    for each $x$ in DOMAIN[$X_i$] do
        if no value $y$ in DOMAIN[$X_j$] allows $(x,y)$ to satisfy the constraint between $X_i$ and $X_j$
            then delete $x$ from DOMAIN[$X_i$];  $removed \leftarrow true$
    return $removed$
```

$O(n^2d^3)$, can be reduced to $O(n^2d^2)$ (but detecting all is NP-hard)

Further notions of consistency

Node consistency: A single variable X is node-consistent if all the values in X's domain $D(X)$ satisfy the unary constraints on X

Ex.

$D(X) = \{1,2,3\}$ $C_1 = (X > 0)$ X node-consist. with C_1

$D(X) = \{1,2,3\}$ $C_2 = (X > 5)$ X not node-consist. with C_2

Further notions of consistency

Arc-consistency for n-constraints

Generalized arc consistency: A variable X_i is generalized arc-consistent wrt an n-ary constraint $C(X_1, \ldots, X_i, \ldots, X_n)$ if,
for every $v \in D(X_i)$,
there is a $(v_1, \ldots, v, \ldots, v_n) \in D(X_1) \times \cdots \times D(X_i) \times \cdots \times D(X_n)$
that satisfies C

Ex.

$D(X) = D(Y) = D(Z) = \{1, 2, 3\}$

$C_1 = (X + Y > Z)$ Y generalized arc-consist. with C_1

$C_2 = (X + Y < Z)$ Z not generalized arc-consist. with C_2

Further notions of consistency

Chained arc-consistency

Path consistency: A two-variable set $\{X, Z\}$ is path-consistent wrt a third variable Y if,
for every assignment satisfying the constraints on $\{X, Z\}$,
there is an assignment to Y that satisfies the constraints on $\{X, Y\}$ and $\{Y, Z\}$

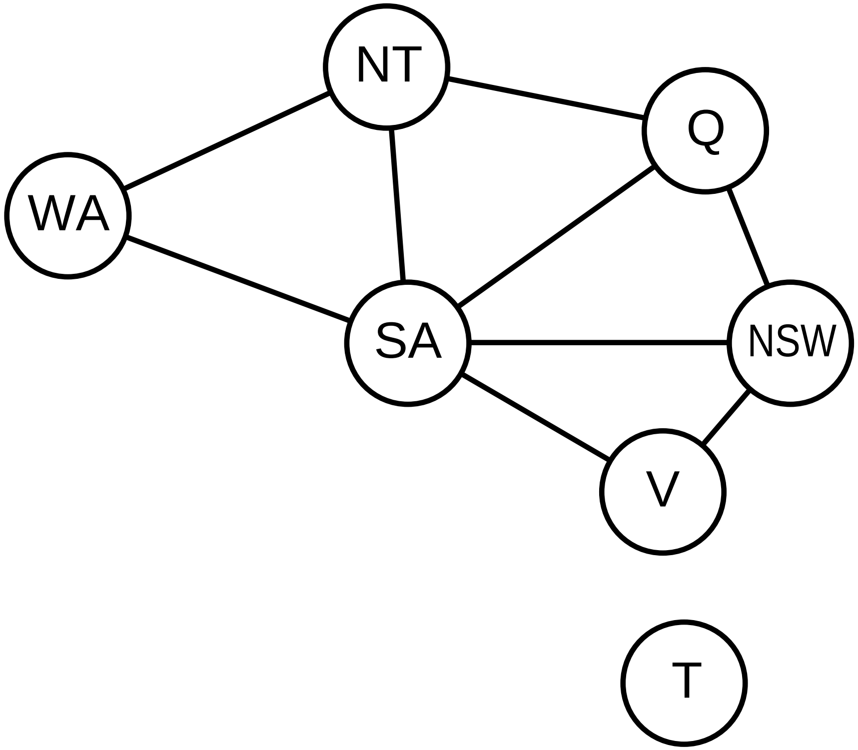
Ex.

$D(X) = D(Y) = D(Z) = \{1, 2, 3, 4\}$

$\{X > 2 \cdot Z,\ X > Y,\ Y = Z + 1\}$ $\{X, Z\}$ path-consistent wrt Y

$\{X > 2 \cdot Z,\ X < Y,\ Y = Z + 1\}$ not $\{X, Z\}$ path-consistent wrt Y

Problem structure



Tasmania and mainland are independent subproblems

Identifiable as connected components of constraint graph

Problem structure

Suppose each subproblem has c variables out of n total

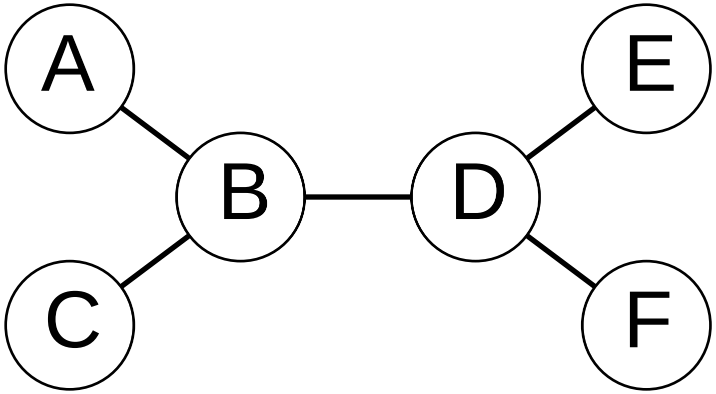
Worst-case solution cost is $n/c \cdot d^c$, linear in n

E.g., $n = 80$, $d = 2$, $c = 20$

2^{80} = 4 billion years at 10 million nodes/sec

$4 \cdot 2^{20}$ = 0.4 seconds at 10 million nodes/sec

Tree-structured CSPs



Theorem: If the constraint graph has no loops, the CSP can be solved in $O(n\,d^2)$ time

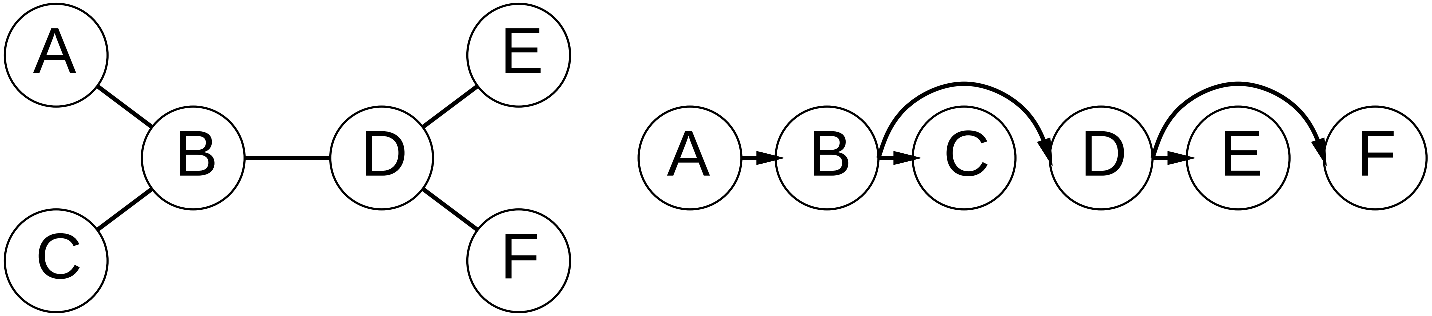
Compare to general CSPs, where worst-case time is $O(d^n)$

This property also applies to logical and probabilistic reasoning:
an important example of the relation between

- syntactic restrictions and
- the complexity of reasoning

Algorithm for tree-structured CSPs

1. Choose a variable as root, order variables from root to leaves so that every node's parent precedes it in the ordering

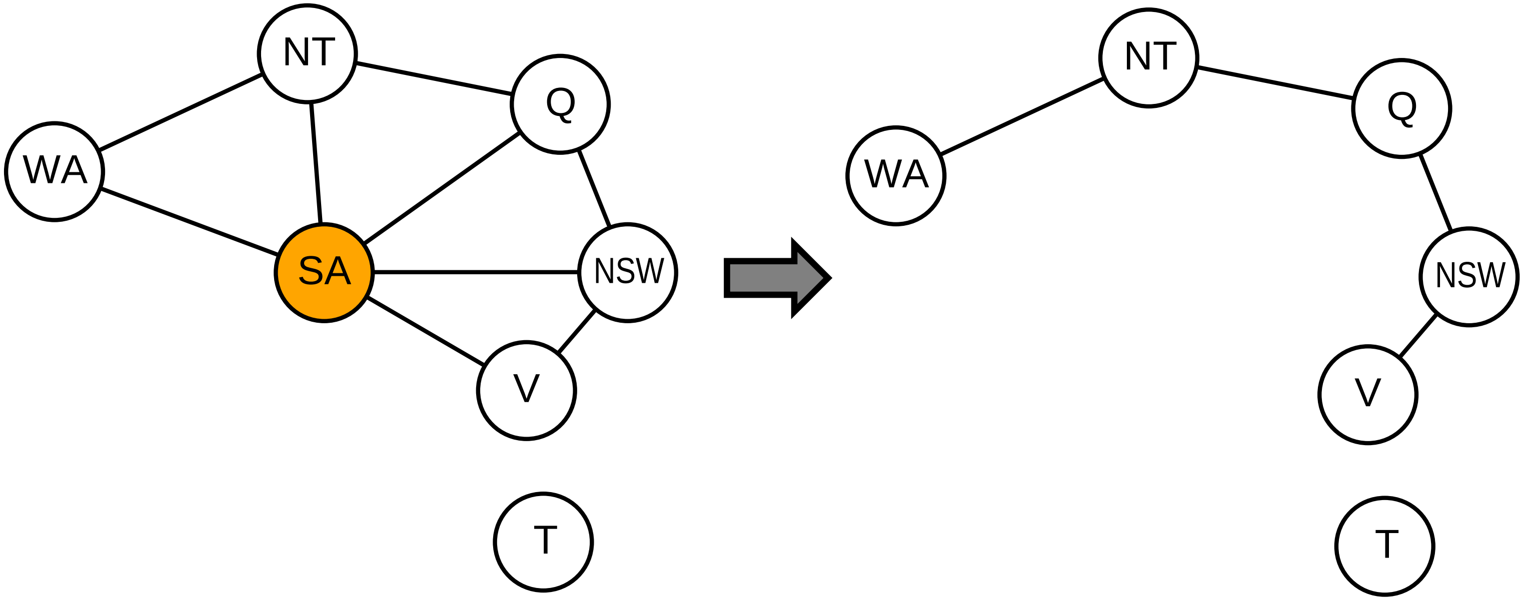


2. For j from n down to 2, apply $\textsc{RemoveInconsistentValues}(Parent(X_j), X_j)$

3. For j from 1 to n, assign X_j consistently with $Parent(X_j)$

Nearly tree-structured CSPs

Conditioning: instantiate a variable, prune its neighbors' domains



Cutset conditioning: instantiate (in all ways) a set of variables so that the remaining constraint graph is a tree

Cutset size $c \implies$ runtime $O(d^c \cdot (n-c)d^2)$, very fast for small c

Further Optimizations

- Tree decomposition
- Symmetry breaking

Iterative algorithms for CSPs

Hill-climbing, simulated annealing typically work with
"complete" states, i.e., all variables assigned

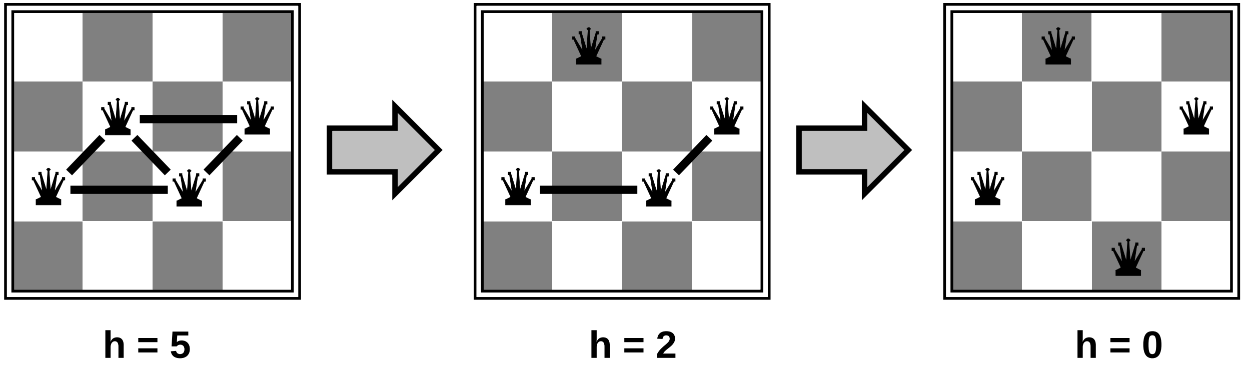
To apply to CSPs:
    allow states with unsatisfied constraints
    operators reassign variable values

Variable selection: randomly select any conflicted variable

Value selection by min-conflicts heuristic:
    choose value that violates the fewest constraints
    i.e., hillclimb with $h(n) =$ total number of violated constraints

Example: 4-Queens

States:	4 queens in 4 columns ($4^4 = 256$ states)
Operators:	move queen in column
Goal test:	no attacks
Evaluation:	$h(n) =$ number of attacks

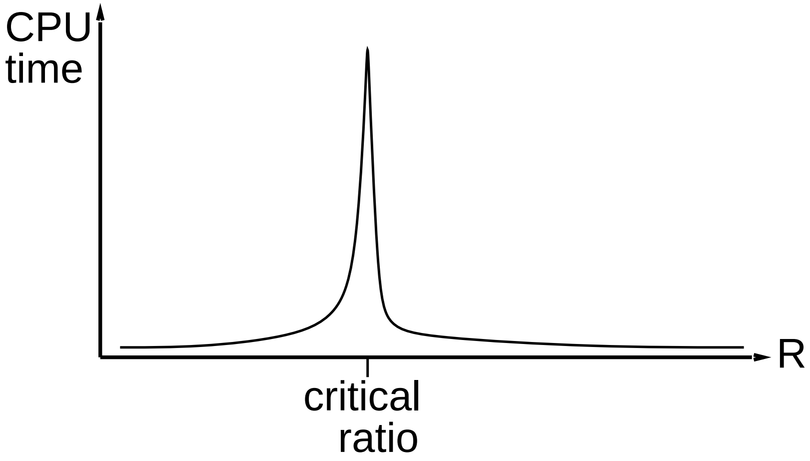


Performance of min-conflicts

Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)

The same appears to be true for any randomly-generated CSP except in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



The critical ration corresponds to a phase transition for the problems, from satisfiable to unsatisfiable