CS:4350 Logic in Computer Science

First-Order Logic

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Credits

Part of these slides are based on Chap. 2 of *Logic in Computer Science* by M. Huth and M. Ryan, Cambridge University Press, 2nd edition, 2004, and on some slides by S. Russel and P. Norvig

Outline

First-order Logic

Propositional logic talks about facts, statements that can be either true or false

However, unlike natural language, it cannot directly talk about

- *Objects*: people, houses, numbers, theories, colors, baseball games, wars, centuries, …
- *Relations*: red, round, bogus, prime, brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, …
- *Functions*: father of, best friend, successor of, one more than, end of, …

First-order logic (FOL) extends PL to do all of the above

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Syntax of FOL: Basic Elements

Variables	x y z $\dots$
Constant symbols	a b kingJohn potus 0 1 2 $\dots$
Function symbols	sqrt(_) leftLeg(_) _ $+$ _ $\dots$
Predicate symbols	Married(_, _) Likes(_, _) _ $>$ _ Even(_) $\dots$
Equality	_ $=$ _
Connectives	$\neg$ _ _ $\wedge$ _ _ $\vee$ _ _ $\rightarrow$ _ _ $\leftrightarrow$ _
Quantifiers	$\forall x$ _ $\exists x$ _ $\dots$

Terms

- Every variable is a term
- Ever constant symbol is a term
- If $t_1, t_2, \ldots, t_n$ are terms and f is a function symbol of arity $n > 0$, then $f(t_1, t_2, \ldots, t_n)$ is a term

Examples

x y a kim potus 0 1 $x + 2$ (infix syntax for $+(x, 2)$)

$x + (2 - y)$ $\mathrm{father}(\mathrm{spouse}(\mathrm{kim}))$ $\mathrm{avg}(2, x, 10)$

Atomic Formulas

- $\top$ and $\bot$ are atomic formulas
- Every nullary predicate symbol is an atomic formula
- If t_1, t_2 are terms then $t_1 = t_2$ is an atomic formula
- If $t_1, t_2, \ldots, t_n$ are terms and p is a predicate symbol of arity $n > 0$,
  then $p(t_1, t_2, \ldots, t_n)$ is an atomic formula

Examples

$$x = y \quad \mathrm{Even}(x + 2) \quad \mathrm{Likes}(\mathrm{father}(\mathrm{kim}), \mathrm{potus})$$

$$x + (2 - y) > 0 \quad \mathrm{father}(\mathrm{spouse}(\mathrm{kim})) = \mathrm{joe} \quad \mathrm{avg}(2, x, 10) > x$$

Formulas

Formulas are constructed from atomic formulas similarly to QBFs

- Every atomic formula is a formula
- If F and G are formulas,
  then $\neg F$, $F \rightarrow G$ and $F \leftrightarrow G$ are formulas
- If $F_1, \ldots, F_n$ are formulas, where $n \geq 2$,
  then $F_1 \wedge \cdots \wedge F_n$ and $F_1 \vee \cdots \vee F_n$ are formulas
- If x is a variable and F is a formula,
  then $\forall x\, F$ and $\exists x\, F$ are formulas

Precedence and associativity rules are as with QBFs

Example $\forall x\, \forall y\, (\mathrm{Married}(x, y) \rightarrow \mathrm{Married}(y, x))$

$x > 2 \vee 1 < x \qquad \exists y\, (y > 1 \wedge \neg(y > 2))$

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$x > 2 \vee 1 < x$ $\qquad$ $\exists y\, (y > 1 \wedge \neg(y > 2))$

Truth in FOL

Formulas are true or false with respect to

- an *interpretation* $\mathcal{I}$ of the constant, function and predicate symbols
- a *universe* $\mathcal{U}$ of concrete values, or *elements*

$\mathcal{U}$ is a set containing ≥ 1 elements

$\mathcal{I}$ maps

$$\begin{array}{rcl} \text{variables} & \mapsto & \mathcal{U} \\ \text{constant symbols} & \mapsto & \mathcal{U} \\ \text{predicate symbols} & \mapsto & \text{relations over } \mathcal{U} \\ \text{function symbols} & \mapsto & \text{functional relations over } \mathcal{U} \end{array}$$

Truth in FOL

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- an *interpretation* $\mathcal{I}$ of the constant, function and predicate symbols
- a *universe* $\mathcal{U}$ of concrete values, or *elements*

An atomic formula $p(t_1, \ldots, t_n)$ is true in an interpretation

iff

the elements denoted to by $t_1, \ldots, t_n$ are in the relation denoted by p

Truth example

Consider the interpretation in which

potus ↦ Joe Biden
fistLady ↦ Jill Biden
Married ↦ the set consisting of all pairs of married people

In this interpretation,

- *Married*(*potus*, *firstLady*) is true
- *Married*(*potus*, *potus*) is false

Truth example

Consider the interpretation in which

$\mathit{potus} \mapsto$ Joe Biden
$\mathit{fistLady} \mapsto$ Jill Biden
$\mathit{Married} \mapsto$ the set consisting of all pairs of married people

In this interpretation,

- $\mathit{Married}(\mathit{potus}, \mathit{firstLady})$ is true
- $\mathit{Married}(\mathit{potus}, \mathit{potus})$ is false

Semantics of First-Order Logic

Formally:

An *interpretation* $\mathcal{I}$ is a triple $(\mathcal{U}, (\_)^{\mathcal{I}}, \sigma)$ where

- $\mathcal{U}$ is a non-empty set of objects, the *universe or domain*
- σ is a mapping from variables to $\mathcal{U}$, a *valuation* or *environment*
- $c^{\mathcal{I}}$ is an element in $\mathcal{U}$ for every constant symbol c
- $f^{\mathcal{I}}$ is a function from $\mathcal{U}^n$ to $\mathcal{U}$ (a subset of $\mathcal{U}^n \times \mathcal{U}$) for every function symbol f of arity n
- $r^{\mathcal{I}}$ is a relation over $\mathcal{U}^n$ (a subset of $\mathcal{U}^n$) for every predicate symbol r of arity n

Note

- An interpretation gives meaning to the non-logical symbols in formulas (constant, function, predicate symbols, and variables)
- The meaning of $=$, connectives and quantifiers is fixed for all interpretations

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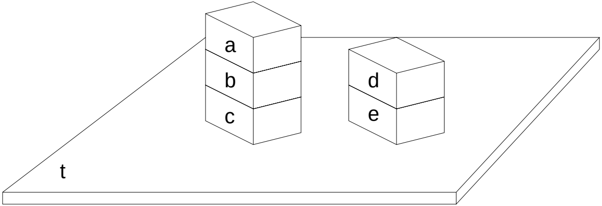
An Interpretation $\mathcal{I}$ in the Blocks World

constant symbols: $b_1, b_2, b_3, b_4, b_5, b_6$
function symbols: support
predicate symbols: On, Above, Clear



$$b_1{}^{\mathcal{I}} = a,\ b_2{}^{\mathcal{I}} = b,\ b_3{}^{\mathcal{I}} = c,\ b_4{}^{\mathcal{I}} = d,\ b_5{}^{\mathcal{I}} = e,\ b_6{}^{\mathcal{I}} = t$$

$$\begin{aligned}
\mathrm{support}^{\mathcal{I}} &= \{(a,b),(b,c),(c,t),(d,e),(e,t),(t,t)\} \\
\mathrm{On}^{\mathcal{I}} &= \{(a,b),(b,c),(c,t),(d,e),(e,t)\} \\
\mathrm{Above}^{\mathcal{I}} &= \{(a,b),(a,c),(a,t),(b,c),(b,t),(c,t),(d,e),(d,t),(e,t)\} \\
\mathrm{Clear}^{\mathcal{I}} &= \{(a),(d)\}
\end{aligned}$$

Semantics of FOL Terms

$\mathcal{I}$ interpretation with universe $\mathcal{U}$ and valuation σ

If e is an FOL expression, $[\![e]\!]^{\mathcal{I}}$ denotes the *meaning of e in $\mathcal{I}$*

For terms t, $[\![t]\!]^{\mathcal{I}}$ is an element of $\mathcal{U}$:

$$\begin{array}{rcll} [\![x]\!]^{\mathcal{I}} & \stackrel{\text{def}}{=} & \sigma(x) & \text{for all variables } x \\ [\![c]\!]^{\mathcal{I}} & \stackrel{\text{def}}{=} & c^{\mathcal{I}} & \text{for all constant symbols } c \\ [\![f(t_1, \ldots, t_n)]\!]^{\mathcal{I}} & \stackrel{\text{def}}{=} & f^{\mathcal{I}}([\![t_1]\!]^{\mathcal{I}}, \ldots, [\![t_n]\!]^{\mathcal{I}}) & \text{for all } n\text{-ary function symbols } f \end{array}$$

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Example

Consider the symbols mother, spouse and the interpretation $\mathcal{I}$ with valuation σ where

$\text{mother}^{\mathcal{I}}$	is a unary function mapping people to their mother
$\text{spouse}^{\mathcal{I}}$	is a unary function mapping people to their spouse
σ	is $\{x \mapsto \text{Bart Simpson},\ y \mapsto \text{Homer Simpson}, \ldots\}$

What is the meaning of spouse(mother(x)) in $\mathcal{I}$?

$$\begin{aligned}
[\![\text{spouse}(\text{mother}(x))]\!]^{\mathcal{I}} &= \\
&= \\
&= \\
&= \\
&= \\
&=
\end{aligned}$$

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&= \text{Homer}
\end{aligned}$$

Semantics of FOL Formulas

$\mathcal{I}$ interpretation with valuation σ

The meaning $[\![F]\!]^{\mathcal{I}}$ of a formula F is either 1 (true) or 0 (false):

$$\begin{array}{rcccl}
[\![t_1 = t_2]\!]^{\mathcal{I}} & \stackrel{\text{def}}{=} & 1 & \text{iff} & [\![t_1]\!]^{\mathcal{I}} \text{ is the same as } [\![t_2]\!]^{\mathcal{I}} \\
[\![r(t_1, \ldots, t_n)]\!]^{\mathcal{I}} & \stackrel{\text{def}}{=} & 1 & \text{iff} & ([\![t_1]\!]^{\mathcal{I}}, \ldots, [\![t_n]\!]^{\mathcal{I}}) \in r^{\mathcal{I}} \\
[\![\neg F]\!]^{\mathcal{I}} & \stackrel{\text{def}}{=} & 1 & \text{iff} & [\![F]\!]^{\mathcal{I}} = 0 \\
[\![F_1 \wedge \cdots \wedge F_n]\!]^{\mathcal{I}} & \stackrel{\text{def}}{=} & 1 & \text{iff} & [\![F_i]\!]^{\mathcal{I}} = 1 \text{ for all } i = 1, \ldots, n \\
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 & & & & \text{at most on } x \\
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Models, Validity, etc. as usual

An interpretation $\mathcal{I}$ *satisfies* a formula F, or is a *model of* F, written $\mathcal{I} \models F$, if $[\![F]\!]^{\mathcal{I}} = 1$

A formula is *satisfiable* if it has at least one model

Ex: $\forall x\, x \geq y$ $\neg \forall x\, x \geq y$ $P(x)$ $\neg P(x)$

A formula is *unsatisfiable* if it has no models

Ex: $P(x) \land \neg P(x)$ $\neg(x = x)$ $\forall x\, \forall y\, Q(x, y) \land \neg Q(a, b)$

A formula is *valid* if it is satisfied by every interpretation

Ex: $P(x) \to P(x)$ $x = x$ $\forall x\, P(x) \to \exists x\, P(x)$

Note: As in PL, F is valid/unsatisfiable iff $\neg F$ is unsatisfiable/valid

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A formula is *unsatisfiable* if it has no models

Ex: $P(x) \wedge \neg P(x) \quad \neg(x = x) \quad \forall x\, \forall y\, Q(x, y) \wedge \neg Q(a, b)$

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Note: As in PL, F is valid/unsatisfiable iff $\neg F$ is unsatisfiable/valid

Models, Validity, etc. for Sets of Formulas

An interpretation *satisfies* a set S of formulas, or is a *model of* S, written $\mathcal{I} \models S$, if it is a model for *every* formula in S

A set S of formulas is *satisfiable* if it has at least one model

Ex: $\{\forall x\, x \geq 0,\ \forall x\, x + 1 > x\}$

S is *unsatisfiable*, or *inconsistent*, if it has no models

Ex: $\{P(x),\ \neg P(x)\}$

S *entails* a formula F, written $S \models F$, if every model for S is also a model for F

Ex: $\{\forall x\, (P(x) \to Q(x)),\ P(a)\} \models Q(a)$

Note: As in PL, $S \models F$ iff $S \cup \{\neg F\}$ is unsatisfiable

Models, Validity, etc. for Sets of Formulas

An interpretation *satisfies* a set S of formulas, or is a *model of* S, written $\mathcal{I} \models S$, if it is a model for *every* formula in S

A set S of formulas is *satisfiable* if it has at least one model

Ex: $\{\forall x\, x \geq 0,\ \forall x\, x + 1 > x\}$

S is *unsatisfiable*, or *inconsistent*, if it has no models

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S *entails* a formula F, written $S \models F$, if every model for S is also a model for F

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Free and bound variables

The notions of

- quantifier scope,
- free/bound occurrence of a variable in a formula, and
- closed formula

are defined exactly as with QBFs

Theorem 1
Let F be a closed formula and let $\mathcal{I}$ and $\mathcal{I}'$ be two interpretations that differ only for their variable valuations. Then,

$$\mathcal{I} \models F \text{ iff } \mathcal{I}' \models F .$$

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$\mathcal{I}$ an interpretation

The satisfiability of a closed formula in $\mathcal{I}$ does not depend on how $\mathcal{I}$ interprets the variables

However, it does depend on how $\mathcal{I}$ interprets the non-logical symbols

Example

$$\exists x\,(2 < x \wedge x < 3)$$

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is true over the reals and false over the integers

Lots of Models

An FOL formula F can have either no models at all or infinitely many

Levels of freedom in constructing a model:

- Cardinality of universe: finite $1, 2, \ldots, n, \ldots$ or infinite
  - Interpretation of each predicate symbol
    - Interpretation of each function symbol
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Symbol	Interpretation choices in a universe U of cardinality n	
a	n	(# of elements of U)
$P(\_)$	2^n	(# of subsets of U)
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Equality

Recall that $t_1 = t_2$ is true in an interpretation iff t_1 and t_2 denote the same element of the universe

Examples

- $a = b$
- $t = t$
- $a \neq a$
- $1 = 25$
- $x * x = x$
- $a = b \rightarrow b = a$
- $a = b \wedge b = c \rightarrow a = c$
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Qualifying Arguments and Quantifiers

FOL is an untyped logic:

- We assume a single set, the universe $\mathcal{U}$, containing everything we want to talk about
- All variables range over the entire $\mathcal{U}$
- Function and predicate symbols apply to any elements of $\mathcal{U}$

As in dynamically-typed programming languages (Javascript, Python, ...), this makes it possible to write practically non-sensical expressions

This issue can be addressed through the use of *qualification*

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Qualifying Universal Quantification

How do we interpret this formula?

$$\forall x \text{ Smart(x)}$$

We typically want to qualify the quantification

Which set of elements are we saying are all smart?

- People? Dogs? Students at Iowa? Students at Iowa taking this course? …
- $\forall x\ (\text{Person(x)} \rightarrow \text{Smart(x)})$
- $\forall x\ (\text{Dog(x)} \rightarrow \text{Smart(x)})$
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General Quantification Schemas

Universal quantification

$$\forall \boldsymbol{x}\ (\text{Qualifier for } \boldsymbol{x} \rightarrow \text{Statement involving } \boldsymbol{x})$$

Existential quantification

$$\exists \boldsymbol{x}\ (\text{Qualifier for } \boldsymbol{x} \wedge \text{Statement involving } \boldsymbol{x})$$

Incorrect Qualifications

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This states that everything is a dog and is smart!

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$$\exists x\,(\mathrm{Dog(x)} \to \mathrm{Smart(x)})$$

Incorrect Qualifications

$$\forall x\,(\mathrm{Dog}(x) \land \mathrm{Smart}(x))$$

This states that everything is a dog and is smart!

$$\exists x\,(\mathrm{Dog}(x) \rightarrow \mathrm{Smart}(x))$$

This is satisfied by any interpretation where $\mathrm{Dog}(x)$ is always false!

Useful Quantifier Equivalences

Exactly as with QBFs:

$$\forall x\, \forall y\, F \;\equiv\; \forall y\, \forall x\, F \qquad\qquad \exists x\, \exists y\, F \;\equiv\; \exists y\, \exists x\, F$$

$$\neg \forall x\, F \;\equiv\; \exists x\, \neg F \qquad\qquad \neg \exists x\, F \;\equiv\; \forall x\, \neg F$$

$$\forall x\, (F \wedge G) \;\equiv\; \forall x\, F \wedge \forall x\, G \qquad\qquad \exists x\, (F \vee G) \;\equiv\; \exists x\, F \vee \exists x\, G$$

Conditional Quantifier Equivalences

Exactly as with QBFs:

$$\begin{aligned} \forall x\, G &\equiv G & \exists x\, G &\equiv G \\ \forall x\, (F \vee G) &\equiv \forall x\, F \vee G & \exists x\, (F \wedge G) &\equiv \exists x\, F \wedge G \\ \forall x\, (F \rightarrow G) &\equiv \exists x\, F \rightarrow G & \exists x\, (F \rightarrow G) &\equiv \forall x\, F \rightarrow G \\ \forall x\, (G \rightarrow F) &\equiv G \rightarrow \forall x\, F & \exists x\, (G \rightarrow F) &\equiv G \rightarrow \exists x\, F \end{aligned}$$

if x is not free in G

From English to FOL

First step

Choose a set of constant, function and predicate symbols to represent specific individuals, functions, and relations, respectively

Example

Constant	Intended meaning	Function	Intended meaning
ann	some person named Ann	mother(x)	x's mother
jane	some person named Jane	father(x)	x's father

Predicate	Intended meaning	Predicate	Intended meaning
Person(x)	x is a person	Brothers(x, y)	x and y are brothers
Married(x)	x is married	Sisters(x, y)	x and y are sisters
Dog(x)	x is a dog	Siblings(x, y)	x and y are siblings
Male(x)	x is a male	Cousin(x, y)	x and y are first cousins
Female(x)	x is a female	Spouse(x, y)	y is x's spouse
Mammal(x)	x is a mammal	Parent(x, y)	y is a parent of x

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From English to FOL, Examples

Dogs are mammals

Brothers are siblings

"Siblings" is a symmetric relation

Jane is Ann’s mother

Ann’s mother and father are married

Jane is married to some man

Ann is Jane’s only daughter

One’s mother is one’s female parent

Everybody is somebody’s child

Some people have no children

First cousins are people whose parents are siblings

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One’s mother is one’s female parent

Everybody is somebody’s child

Some people have no children

First cousins are people whose parents are siblings

From English to FOL, Examples

Dogs are mammals	$\forall x\,(\mathrm{Dog}(x) \rightarrow \mathrm{Mammal}(x))$
Brothers are siblings	$\forall x\,\forall y\,(\mathrm{Brothers}(x, y) \rightarrow \mathrm{Siblings}(x, y))$
"Siblings" is a symmetric relation	$\forall x\,\forall y\,(\mathrm{Siblings}(x, y) \rightarrow \mathrm{Siblings}(y, x))$
Jane is Ann's mother	$\mathrm{jane} = \mathrm{mother}(\mathrm{ann})$
Ann's mother and father are married	$\mathrm{Spouse}(\mathrm{mother}(\mathrm{ann}), \mathrm{father}(\mathrm{ann}))$
Jane is married to some man	
Ann is Jane's only daughter	
One's mother is one's female parent	
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Jane is Ann's mother	$\mathrm{jane} = \mathrm{mother}(\mathrm{ann})$
Ann's mother and father are married	$\mathrm{Spouse}(\mathrm{mother}(\mathrm{ann}), \mathrm{father}(\mathrm{ann}))$
Jane is married to some man	$\exists x\,(\mathit{Person}(x) \land \mathit{Male}(x) \land \mathrm{Spouse}(\mathrm{jane}, x))$
Ann is Jane's only daughter	

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Jane is Ann’s mother	$\mathrm{jane} = \mathrm{mother}(\mathrm{ann})$
Ann’s mother and father are married	$\mathrm{Spouse}(\mathrm{mother}(\mathrm{ann}), \mathrm{father}(\mathrm{ann}))$
Jane is married to some man	$\exists x\,(\mathit{Person}(x) \land \mathit{Male}(x) \land \mathrm{Spouse}(\mathrm{jane}, x))$
Ann is Jane’s only daughter	$\mathrm{jane} = \mathrm{mother}(\mathrm{ann}) \land$ $\forall x\,(\mathrm{Female}(x) \land \mathrm{mother}(x) = \mathrm{jane} \rightarrow x = \mathrm{ann})$
One’s mother is one’s female parent	
Everybody is somebody’s child	
Some people have no children	
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Jane is Ann’s mother	$\mathrm{jane} = \mathrm{mother}(\mathrm{ann})$
Ann’s mother and father are married	$\mathrm{Spouse}(\mathrm{mother}(\mathrm{ann}), \mathrm{father}(\mathrm{ann}))$
Jane is married to some man	$\exists x\,(\mathit{Person}(x) \wedge \mathit{Male}(x) \wedge \mathrm{Spouse}(\mathrm{jane}, x))$
Ann is Jane’s only daughter	$\mathrm{jane} = \mathrm{mother}(\mathrm{ann}) \wedge$ $\forall x\,(\mathrm{Female}(x) \wedge \mathrm{mother}(x) = \mathrm{jane} \rightarrow x = \mathrm{ann})$
One’s mother is one’s female parent	$\forall x\,\forall y\,(y = \mathrm{mother}(x) \leftrightarrow \mathrm{Female}(y) \wedge \mathrm{Parent}(x,y))$
Everybody is somebody’s child	
Some people have no children	
First cousins are people whose parents are siblings	

From English to FOL, Examples

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Jane is Ann's mother	$\mathrm{jane} = \mathrm{mother}(\mathrm{ann})$
Ann's mother and father are married	$\mathrm{Spouse}(\mathrm{mother}(\mathrm{ann}), \mathrm{father}(\mathrm{ann}))$
Jane is married to some man	$\exists x\,(\mathit{Person}(x) \land \mathit{Male}(x) \land \mathrm{Spouse}(\mathrm{jane}, x))$
Ann is Jane's only daughter	$\mathrm{jane} = \mathrm{mother}(\mathrm{ann}) \land \forall x\,(\mathrm{Female}(x) \land \mathrm{mother}(x) = \mathrm{jane} \rightarrow x = \mathrm{ann})$
One's mother is one's female parent	$\forall x\,\forall y\,(y = \mathrm{mother}(x) \leftrightarrow \mathrm{Female}(y) \land \mathrm{Parent}(x, y))$
Everybody is somebody's child	$\forall x\,(\mathrm{Person}(x) \rightarrow \exists y\,(\mathrm{Person}(y) \land \mathrm{Parent}(x, y)))$
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Jane is Ann's mother	$\mathrm{jane} = \mathrm{mother}(\mathrm{ann})$
Ann's mother and father are married	$\mathrm{Spouse}(\mathrm{mother}(\mathrm{ann}), \mathrm{father}(\mathrm{ann}))$
Jane is married to some man	$\exists x\, (\mathit{Person}(x) \wedge \mathit{Male}(x) \wedge \mathrm{Spouse}(\mathrm{jane}, x))$
Ann is Jane's only daughter	$\mathrm{jane} = \mathrm{mother}(\mathrm{ann}) \wedge \forall x\, (\mathrm{Female}(x) \wedge \mathrm{mother}(x) = \mathrm{jane} \rightarrow x = \mathrm{ann})$
One's mother is one's female parent	$\forall x\, \forall y\, (y = \mathrm{mother}(x) \leftrightarrow \mathrm{Female}(y) \wedge \mathrm{Parent}(x, y))$
Everybody is somebody's child	$\forall x\, (\mathrm{Person}(x) \rightarrow \exists y\, (\mathrm{Person}(y) \wedge \mathrm{Parent}(x, y)))$
Some people have no children	$\exists x\, (\mathrm{Person}(x) \wedge \forall y\, \neg \mathrm{Parent}(y, x))$
First cousins are people whose parents are siblings	

From English to FOL, Examples

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Ann is Jane’s only daughter	$\mathrm{jane} = \mathrm{mother}(\mathrm{ann}) \land \forall x\,(\mathrm{Female}(x) \land \mathrm{mother}(x) = \mathrm{jane} \rightarrow x = \mathrm{ann})$
One’s mother is one’s female parent	$\forall x\,\forall y\,(y = \mathrm{mother}(x) \leftrightarrow \mathrm{Female}(y) \land \mathrm{Parent}(x, y))$
Everybody is somebody’s child	$\forall x\,(\mathrm{Person}(x) \rightarrow \exists y\,(\mathrm{Person}(y) \land \mathrm{Parent}(x, y)))$
Some people have no children	$\exists x\,(\mathrm{Person}(x) \land \forall y\,\neg \mathrm{Parent}(y, x))$
First cousins are people whose parents are siblings	$\forall x_1\,\forall x_2\,(\mathrm{Cousins}(x_1, x_2) \leftrightarrow \mathrm{Person}(x) \land \mathrm{Person}(y) \land \exists p_1\,\exists p_2\,(\mathrm{Siblings}(p_1, p_2) \land \mathrm{Parent}(x_1, p_1) \land \mathrm{Parent}(x_2, p_2)))$

From FOL to English, Examples

$\forall x\, \neg(\mathrm{Person}t(x) \wedge \mathrm{Siblings}(x, x))$

$\forall x\, \forall y\, (\mathrm{Brothers}(x, y) \rightarrow \mathrm{Male}(x) \wedge \mathrm{Male}(y))$

$\forall x\, (\mathrm{Person}(x) \rightarrow (\mathrm{Male}(x) \vee \mathrm{Female}(x)) \wedge \neg(\mathrm{Male}(x) \wedge \mathrm{Female}(x)))$

$\forall x\, (\mathrm{Person}(x) \wedge \mathrm{Married}(x) \rightarrow \exists y\, \mathrm{Spouse}(x, y))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \wedge \mathrm{Spouse}(x, y) \rightarrow \mathrm{Married}(x))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \wedge \mathrm{Spouse}(x, y) \rightarrow \neg\mathrm{Siblings}(x, y))$

$\neg\forall x\, (\mathrm{Person}(x) \wedge \exists y\, \mathrm{Parent}(y, x) \rightarrow \mathrm{Married}(x))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \wedge \mathrm{Parent}(x, y) \rightarrow \mathrm{Person}(y))$

$\forall x\, \exists y\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$

$\exists y\, \forall x\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$

From FOL to English, Examples

$\forall x\, \neg(\mathrm{Person}t(x) \land \mathrm{Siblings}(x, x))$ No one is his or her own sibling

$\forall x\, \forall y\, (\mathrm{Brothers}(x, y) \rightarrow \mathrm{Male}(x) \land \mathrm{Male}(y))$

$\forall x\, (\mathrm{Person}(x) \rightarrow (\mathrm{Male}(x) \lor \mathrm{Female}(x)) \land \neg(\mathrm{Male}(x) \land \mathrm{Female}(x)))$

$\forall x\, (\mathrm{Person}(x) \land \mathrm{Married}(x) \rightarrow \exists y\, \mathrm{Spouse}(x, y))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \mathrm{Married}(x))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \neg\mathrm{Siblings}(x, y))$

$\neg\forall x\, (\mathrm{Person}(x) \land \exists y\, \mathrm{Parent}(y, x) \rightarrow \mathrm{Married}(x))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Parent}(x, y) \rightarrow \mathrm{Person}(y))$

$\forall x\, \exists y\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$

$\exists y\, \forall x\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$

From FOL to English, Examples

$\forall x\, \neg(\mathrm{Person}t(x) \land \mathrm{Siblings}(x, x))$ No one is his or her own sibling

$\forall x\, \forall y\, (\mathrm{Brothers}(x, y) \rightarrow \mathrm{Male}(x) \land \mathrm{Male}(y))$ Brothers are male

$\forall x\, (\mathrm{Person}(x) \rightarrow (\mathrm{Male}(x) \lor \mathrm{Female}(x)) \land \neg(\mathrm{Male}(x) \land \mathrm{Female}(x)))$

$\forall x\, (\mathrm{Person}(x) \land \mathrm{Married}(x) \rightarrow \exists y\, \mathrm{Spouse}(x, y))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \mathrm{Married}(x))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \neg\mathrm{Siblings}(x, y))$

$\neg\forall x\, (\mathrm{Person}(x) \land \exists y\, \mathrm{Parent}(y, x) \rightarrow \mathrm{Married}(x))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Parent}(x, y) \rightarrow \mathrm{Person}(y))$

$\forall x\, \exists y\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$

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From FOL to English, Examples

$\forall x\, \neg(\text{Person}t(x) \wedge \text{Siblings}(x, x))$ No one is his or her own sibling

$\forall x\, \forall y\, (\text{Brothers}(x, y) \rightarrow \text{Male}(x) \wedge \text{Male}(y))$ Brothers are male

$\forall x\, (\text{Person}(x) \rightarrow (\text{Male}(x) \vee \text{Female}(x)) \wedge \neg(\text{Male}(x) \wedge \text{Female}(x)))$

Every person is either male or female but not both

$\forall x\, (\text{Person}(x) \wedge \text{Married}(x) \rightarrow \exists y\, \text{Spouse}(x, y))$

$\forall x\, \forall y\, (\text{Person}(x) \wedge \text{Spouse}(x, y) \rightarrow \text{Married}(x))$

$\forall x\, \forall y\, (\text{Person}(x) \wedge \text{Spouse}(x, y) \rightarrow \neg\text{Siblings}(x, y))$

$\neg\forall x\, (\text{Person}(x) \wedge \exists y\, \text{Parent}(y, x) \rightarrow \text{Married}(x))$

$\forall x\, \forall y\, (\text{Person}(x) \wedge \text{Parent}(x, y) \rightarrow \text{Person}(y))$

$\forall x\, \exists y\, (\text{Person}(x) \rightarrow y = \text{mother}(x))$

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From FOL to English, Examples

$\forall x\, \neg(\mathrm{Person}t(x) \land \mathrm{Siblings}(x, x))$ No one is his or her own sibling

$\forall x\, \forall y\, (\mathrm{Brothers}(x, y) \rightarrow \mathrm{Male}(x) \land \mathrm{Male}(y))$ Brothers are male

$\forall x\, (\mathrm{Person}(x) \rightarrow (\mathrm{Male}(x) \lor \mathrm{Female}(x)) \land \neg(\mathrm{Male}(x) \land \mathrm{Female}(x)))$

Every person is either male or female but not both

$\forall x\, (\mathrm{Person}(x) \land \mathrm{Married}(x) \rightarrow \exists y\, \mathrm{Spouse}(x, y))$ Married people have spouses

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \mathrm{Married}(x))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \neg\mathrm{Siblings}(x, y))$

$\neg\forall x\, (\mathrm{Person}(x) \land \exists y\, \mathrm{Parent}(y, x) \rightarrow \mathrm{Married}(x))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Parent}(x, y) \rightarrow \mathrm{Person}(y))$

$\forall x\, \exists y\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$

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From FOL to English, Examples

$\forall x\, \neg(\text{Person}t(x) \land \text{Siblings}(x, x))$	No one is his or her own sibling
$\forall x\, \forall y\, (\text{Brothers}(x, y) \rightarrow \text{Male}(x) \land \text{Male}(y))$	Brothers are male
$\forall x\, (\text{Person}(x) \rightarrow (\text{Male}(x) \lor \text{Female}(x)) \land \neg(\text{Male}(x) \land \text{Female}(x)))$	Every person is either male or female but not both
$\forall x\, (\text{Person}(x) \land \text{Married}(x) \rightarrow \exists y\, \text{Spouse}(x, y))$	Married people have spouses
$\forall x\, \forall y\, (\text{Person}(x) \land \text{Spouse}(x, y) \rightarrow \text{Married}(x))$	Only married people have spouses
$\forall x\, \forall y\, (\text{Person}(x) \land \text{Spouse}(x, y) \rightarrow \neg\text{Siblings}(x, y))$	
$\neg\forall x\, (\text{Person}(x) \land \exists y\, \text{Parent}(y, x) \rightarrow \text{Married}(x))$	
$\forall x\, \forall y\, (\text{Person}(x) \land \text{Parent}(x, y) \rightarrow \text{Person}(y))$	
$\forall x\, \exists y\, (\text{Person}(x) \rightarrow y = \text{mother}(x))$	
$\exists y\, \forall x\, (\text{Person}(x) \rightarrow y = \text{mother}(x))$	

From FOL to English, Examples

$\forall x\, \neg(\mathrm{Person}t(x) \land \mathrm{Siblings}(x, x))$ No one is his or her own sibling

$\forall x\, \forall y\, (\mathrm{Brothers}(x, y) \rightarrow \mathrm{Male}(x) \land \mathrm{Male}(y))$ Brothers are male

$\forall x\, (\mathrm{Person}(x) \rightarrow (\mathrm{Male}(x) \lor \mathrm{Female}(x)) \land \neg(\mathrm{Male}(x) \land \mathrm{Female}(x)))$
Every person is either male or female but not both

$\forall x\, (\mathrm{Person}(x) \land \mathrm{Married}(x) \rightarrow \exists y\, \mathrm{Spouse}(x, y))$ Married people have spouses

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \mathrm{Married}(x))$ Only married people have spouses

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \neg\mathrm{Siblings}(x, y))$
People cannot be married to their own siblings

$\neg\forall x\, (\mathrm{Person}(x) \land \exists y\, \mathrm{Parent}(y, x) \rightarrow \mathrm{Married}(x))$

$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Parent}(x, y) \rightarrow \mathrm{Person}(y))$

$\forall x\, \exists y\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$

$\exists y\, \forall x\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$

From FOL to English, Examples

$\forall x\, \neg(\mathrm{Person}t(x) \land \mathrm{Siblings}(x, x))$	No one is his or her own sibling
$\forall x\, \forall y\, (\mathrm{Brothers}(x, y) \rightarrow \mathrm{Male}(x) \land \mathrm{Male}(y))$	Brothers are male
$\forall x\, (\mathrm{Person}(x) \rightarrow (\mathrm{Male}(x) \lor \mathrm{Female}(x)) \land \neg(\mathrm{Male}(x) \land \mathrm{Female}(x)))$	Every person is either male or female but not both
$\forall x\, (\mathrm{Person}(x) \land \mathrm{Married}(x) \rightarrow \exists y\, \mathrm{Spouse}(x, y))$	Married people have spouses
$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \mathrm{Married}(x))$	Only married people have spouses
$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \neg\mathrm{Siblings}(x, y))$	People cannot be married to their own siblings
$\neg\forall x\, (\mathrm{Person}(x) \land \exists y\, \mathrm{Parent}(y, x) \rightarrow \mathrm{Married}(x))$	Not everybody who has children is married
$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Parent}(x, y) \rightarrow \mathrm{Person}(y))$	
$\forall x\, \exists y\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$	
$\exists y\, \forall x\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$	

From FOL to English, Examples

$\forall x\, \neg(\mathrm{Person}t(x) \wedge \mathrm{Siblings}(x, x))$	No one is his or her own sibling
$\forall x\, \forall y\, (\mathrm{Brothers}(x, y) \rightarrow \mathrm{Male}(x) \wedge \mathrm{Male}(y))$	Brothers are male
$\forall x\, (\mathrm{Person}(x) \rightarrow (\mathrm{Male}(x) \vee \mathrm{Female}(x)) \wedge \neg(\mathrm{Male}(x) \wedge \mathrm{Female}(x)))$	Every person is either male or female but not both
$\forall x\, (\mathrm{Person}(x) \wedge \mathrm{Married}(x) \rightarrow \exists y\, \mathrm{Spouse}(x, y))$	Married people have spouses
$\forall x\, \forall y\, (\mathrm{Person}(x) \wedge \mathrm{Spouse}(x, y) \rightarrow \mathrm{Married}(x))$	Only married people have spouses
$\forall x\, \forall y\, (\mathrm{Person}(x) \wedge \mathrm{Spouse}(x, y) \rightarrow \neg \mathrm{Siblings}(x, y))$	People cannot be married to their own siblings
$\neg \forall x\, (\mathrm{Person}(x) \wedge \exists y\, \mathrm{Parent}(y, x) \rightarrow \mathrm{Married}(x))$	Not everybody who has children is married
$\forall x\, \forall y\, (\mathrm{Person}(x) \wedge \mathrm{Parent}(x, y) \rightarrow \mathrm{Person}(y))$	People's parents are people too
$\forall x\, \exists y\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$	
$\exists y\, \forall x\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$	

From FOL to English, Examples

$\forall x\, \neg(\mathrm{Person}t(x) \land \mathrm{Siblings}(x, x))$	No one is his or her own sibling
$\forall x\, \forall y\, (\mathrm{Brothers}(x, y) \rightarrow \mathrm{Male}(x) \land \mathrm{Male}(y))$	Brothers are male
$\forall x\, (\mathrm{Person}(x) \rightarrow (\mathrm{Male}(x) \lor \mathrm{Female}(x)) \land \neg(\mathrm{Male}(x) \land \mathrm{Female}(x)))$	Every person is either male or female but not both
$\forall x\, (\mathrm{Person}(x) \land \mathrm{Married}(x) \rightarrow \exists y\, \mathrm{Spouse}(x, y))$	Married people have spouses
$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \mathrm{Married}(x))$	Only married people have spouses
$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \neg \mathrm{Siblings}(x, y))$	People cannot be married to their own siblings
$\neg \forall x\, (\mathrm{Person}(x) \land \exists y\, \mathrm{Parent}(y, x) \rightarrow \mathrm{Married}(x))$	Not everybody who has children is married
$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Parent}(x, y) \rightarrow \mathrm{Person}(y))$	People’s parents are people too
$\forall x\, \exists y\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$	Everyone has a mother
$\exists y\, \forall x\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$	

From FOL to English, Examples

$\forall x\, \neg(\mathrm{Person}t(x) \land \mathrm{Siblings}(x, x))$	No one is his or her own sibling
$\forall x\, \forall y\, (\mathrm{Brothers}(x, y) \rightarrow \mathrm{Male}(x) \land \mathrm{Male}(y))$	Brothers are male
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$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Spouse}(x, y) \rightarrow \neg\mathrm{Siblings}(x, y))$	People cannot be married to their own siblings
$\neg\forall x\, (\mathrm{Person}(x) \land \exists y\, \mathrm{Parent}(y, x) \rightarrow \mathrm{Married}(x))$	Not everybody who has children is married
$\forall x\, \forall y\, (\mathrm{Person}(x) \land \mathrm{Parent}(x, y) \rightarrow \mathrm{Person}(y))$	People's parents are people too
$\forall x\, \exists y\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$	Everyone has a mother
$\exists y\, \forall x\, (\mathrm{Person}(x) \rightarrow y = \mathrm{mother}(x))$	Everyone has the *same* mother

Natural Deduction for FOL

The natural deduction inference system for PL extends to FOL

We need additional of rules for

- equality
- quantifiers

Freeness

Let x be a variable, t a term, and F a formula of FOL

Recall F_x^t denotes the result of replacing every free occurrence of x in F by t

t is *free for* x *in* F if no free occurrence of x in F occurs in the scope of $\exists\forall y$
for any variable y of t
iff every variable of t remains free in F_x^t

Example $F\colon\ S(x) \wedge \forall y\,(P(z) \to Q(y))$

$F_x^{f(y)}\colon\ S(f(y)) \wedge \forall y\,(P(z) \to Q(y))$ $\qquad F_z^{f(y)}\colon\ S(x) \wedge \forall y\,(P(f(y)) \to Q(y))$

Term $f(y)$ is free for x in F but not for z

Freeness

Let x be a variable, t a term, and F a formula of FOL

Recall F_x^t denotes the result of replacing every free occurrence of x in F by t

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Example $F\colon\ S(x) \land \forall y\,(P(z) \to Q(y))$

$F_x^{f(y)}\colon\ S(f(y)) \land \forall y\,(P(z) \to Q(y))$ $\qquad$ $F_z^{f(y)}\colon\ S(x) \land \forall y\,(P(f(y)) \to Q(y))$

Term $f(y)$ is free for x in F but not for z

Freeness

Let x be a variable, t a term, and F a formula of FOL

Recall F_x^t denotes the result of replacing every free occurrence of x in F by t

t is *free for x in F* if no free occurrence of x in F occurs in the scope of $\exists\!\!\forall y$ for any variable y of t
iff every variable of t remains free in F_x^t

Example $F\colon\ S(x) \land \forall y\,(P(z) \to Q(y))$

$F_x^{f(y)}\colon\ S(f(y)) \land \forall y\,(P(z) \to Q(y)) \qquad F_z^{f(y)}\colon\ S(x) \land \forall y\,(P(f(y)) \to Q(y))$

Term $f(y)$ is free for x in F but not for z

$=$ introduction and elimination

$$\frac{}{t = t}\ {=}\mathrm{i} \qquad\qquad \frac{s = t \quad A^s_x \quad s, t \text{ free for } x \text{ in } A}{A^t_x}\ {=}\mathrm{e}$$

There rules are sufficient to derive all main properties of equality:

$\vdash\ a = a$

$a = b\ \vdash\ b = a$

$a = b, b = c \vdash\ a = c$

$a = b\ \vdash\ f(a) = f(b)$

$a = b\ \vdash\ P(a) \leftrightarrow P(b)$

$=$ introduction and elimination

$$\frac{}{t = t}\,{=}\mathrm{i} \qquad\qquad \frac{s = t \quad A^s_x \quad s, t \text{ free for } x \text{ in } A}{A^t_x}\,{=}\mathrm{e}$$

There rules are sufficient to derive all main properties of equality:

$\vdash\ a = a$

$a = b \vdash\ b = a$

$a = b, b = c \vdash\ a = c$

$a = b \vdash\ f(a) = f(b)$

$a = b \vdash\ P(a) \leftrightarrow P(b)$

Example derivation

$$\frac{}{t = t} =\mathrm{i} \qquad \frac{s = t \quad A^s_x}{A^t_x} =\mathrm{e}$$

$$a = b \vdash b = a$$

Example derivation

$$\frac{}{t = t} =\text{i} \qquad \frac{s = t \quad A^s_x}{A^t_x} =\text{e}$$

$$a = b \vdash b = a$$

Proof

1 $a = b$ premise

Example derivation

$$\frac{}{t = t} =\mathrm{i} \qquad \frac{s = t \quad A_x^s}{A_x^t} =\mathrm{e}$$

$$a = b \vdash b = a$$

Proof

1 $a = b$ premise

2 $a = a$ $=$i

Example derivation

$$\frac{}{t = t}\,{=}\mathrm{i} \qquad \frac{s = t \quad A^s_x}{A^t_x}\,{=}\mathrm{e}$$

$$a = b \vdash b = a$$

Proof

1 $a = b$ premise

2 $a = a$ $=\mathrm{i}$

3 $b = a$ $=\mathrm{e}$ 1 applied to left-hand side of 2

Example derivation

$$\frac{}{t = t}\;{=}\mathrm{i} \qquad \frac{s = t \quad A^s_x}{A^t_x}\;{=}\mathrm{e}$$

$$a = b \vdash b = a$$

Proof

1 $a = b$ premise

2 $a = a$ $=\mathrm{i}$

3 $b = a$ $=\mathrm{e}$ 1 applied to left-hand side of 2

How could be apply equality 1 to equality 2?

Example derivation

$$\frac{}{t = t} =\mathrm{i} \qquad \frac{s = t \quad A_x^s}{A_x^t} =\mathrm{e}$$

$$a = b \vdash b = a$$

Proof

1 $a = b$ premise

2 $a = a$ $=\mathrm{i}$

3 $b = a$ $=\mathrm{e}$ 1 applied to left-hand side of 2

How could be apply equality 1 to equality 2? By seeing 2 as $(x = a)_x^a$:

$$\frac{a = b \quad (x = a)_x^a}{(x = a)_x^b} =\mathrm{e}$$

Example derivation

$$\frac{}{t = t} = \mathrm{i} \qquad \frac{s = t \quad A_x^s}{A_x^t} = \mathrm{e}$$

$$a = b,\ b = c \vdash\ a = c$$

Example derivation

$$\frac{}{t = t} =\mathrm{i} \qquad \frac{s = t \quad A_x^s}{A_x^t} =\mathrm{e}$$

$$a = b,\ b = c \vdash a = c$$

Proof 1 $a = b$ premise

Example derivation

$$\frac{}{t = t}\,{=}\mathrm{i} \qquad \frac{s = t \quad A^s_x}{A^t_x}\,{=}\mathrm{e}$$

$$a = b,\ b = c \vdash a = c$$

Proof 1 $a = b$ premise

2 $b = c$ premise

Example derivation

$$\frac{}{t = t}\ {=}\mathrm{i} \qquad \frac{s = t \quad A_x^s}{A_x^t}\ {=}\mathrm{e}$$

$$a = b,\ b = c \vdash a = c$$

Proof

1	$a = b$	premise	
2	$b = c$	premise	
3	$a = c$	$=\mathrm{e}$	2 applied to right-hand side of 1

Example derivation

$$\frac{A \to B \quad B \to A}{A \leftrightarrow B}\leftrightarrow\mathrm{i} \qquad \frac{}{t = t}=\mathrm{i} \qquad \frac{s = t \quad A_x^s}{A_x^t}=\mathrm{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Example derivation

$$\frac{A \rightarrow B \quad B \rightarrow A}{A \leftrightarrow B}\leftrightarrow\text{i} \qquad \frac{}{t = t}=\text{i} \qquad \frac{s = t \quad A^s_x}{A^t_x}=\text{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof 1 $a = b$ premise

Example derivation

$$\frac{A \to B \quad B \to A}{A \leftrightarrow B}\leftrightarrow\mathrm{i} \qquad \frac{}{t = t}=\mathrm{i} \qquad \frac{s = t \quad A_x^s}{A_x^t}=\mathrm{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof

1	$a = b$	premise	
2	$P(a)$	assumption	
3	$P(b)$	$=\mathrm{e}$	1 applied to 2

Example derivation

$$\frac{A \to B \quad B \to A}{A \leftrightarrow B}\leftrightarrow\mathrm{i} \qquad \frac{}{t = t}=\mathrm{i} \qquad \frac{s = t \quad A^s_x}{A^t_x}=\mathrm{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof

1	$a = b$	premise
2	$P(a)$	assumption
3	$P(b)$	$=\mathrm{e}$ 1 applied to 2
4	$P(a) \to P(b)$	$\to\mathrm{i}$ 2–3

Example derivation

$$\frac{A \to B \quad B \to A}{A \leftrightarrow B}\leftrightarrow\mathrm{i} \qquad \frac{}{t = t}=\mathrm{i} \qquad \frac{s = t \quad A^{s}_{x}}{A^{t}_{x}}=\mathrm{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof

$$\begin{array}{llll}
1 & a = b & \text{premise} & \\
\hline
\multicolumn{1}{|l}{2} & P(a) & \text{assumption} & \multicolumn{1}{l|}{} \\
\multicolumn{1}{|l}{3} & P(b) & =\mathrm{e} & \multicolumn{1}{l|}{\text{1 applied to 2}} \\
\hline
4 & P(a) \to P(b) & \to\mathrm{i} & \text{2–3} \\
5 & a = a & =\mathrm{i} &
\end{array}$$

Example derivation

$$\frac{A \to B \quad B \to A}{A \leftrightarrow B}\leftrightarrow\mathrm{i} \qquad \frac{}{t = t}=\mathrm{i} \qquad \frac{s = t \quad A^{s}_{x}}{A^{t}_{x}}=\mathrm{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof

1	$a = b$	premise	
2	$P(a)$	assumption	
3	$P(b)$	$=\mathrm{e}$	1 applied to 2
4	$P(a) \to P(b)$	$\to\mathrm{i}$	2–3
5	$a = a$	$=\mathrm{i}$	
6	$b = a$	$=\mathrm{e}$	1 applied to 5

Example derivation

$$\frac{A \to B \quad B \to A}{A \leftrightarrow B}\leftrightarrow\mathrm{i} \qquad \frac{}{t = t}=\mathrm{i} \qquad \frac{s = t \quad A_x^s}{A_x^t}=\mathrm{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof

1	$a = b$	premise	
2	$P(a)$	assumption	
3	$P(b)$	$=\mathrm{e}$	1 applied to 2
4	$P(a) \to P(b)$	$\to\mathrm{i}$	2–3
5	$a = a$	$=\mathrm{i}$	
6	$b = a$	$=\mathrm{e}$	1 applied to 5
7	$P(b) \to P(b)$	$=\mathrm{e}$	1 applied to 4

Example derivation

$$\frac{A \to B \quad B \to A}{A \leftrightarrow B} \leftrightarrow\mathrm{i} \qquad \frac{}{t = t} =\mathrm{i} \qquad \frac{s = t \quad A_x^s}{A_x^t} =\mathrm{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof

1	$a = b$	premise	
2	$P(a)$	assumption	
3	$P(b)$	$=\mathrm{e}$	1 applied to 2
4	$P(a) \to P(b)$	$\to\mathrm{i}$	2–3
5	$a = a$	$=\mathrm{i}$	
6	$b = a$	$=\mathrm{e}$	1 applied to 5
7	$P(b) \to P(b)$	$=\mathrm{e}$	1 applied to 4
8	$P(b) \to P(a)$	$=\mathrm{e}$	6 applied to 7

Example derivation

$$\frac{A \to B \quad B \to A}{A \leftrightarrow B} \leftrightarrow\mathrm{i} \qquad \frac{}{t = t} =\mathrm{i} \qquad \frac{s = t \quad A^{s}_{x}}{A^{t}_{x}} =\mathrm{e}$$

$$a = b \vdash P(a) \leftrightarrow P(b)$$

Proof

1	$a = b$	premise	
2	$P(a)$	assumption	
3	$P(b)$	$=\mathrm{e}$	1 applied to 2
4	$P(a) \to P(b)$	$\to\mathrm{i}$	2–3
5	$a = a$	$=\mathrm{i}$	
6	$b = a$	$=\mathrm{e}$	1 applied to 5
7	$P(b) \to P(b)$	$=\mathrm{e}$	1 applied to 4
8	$P(b) \to P(a)$	$=\mathrm{e}$	6 applied to 7
9	$P(a) \leftrightarrow P(b)$	$\leftrightarrow\mathrm{i}$	1, 2

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

x_0 fresh variable

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

$\forall$ introduction and elimination

$$\frac{\boxed{\begin{array}{l} x_0 \\ \qquad \vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

x_0 fresh variable

$$\frac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 1 Prove $\forall z\, P(z) \vdash P(a)$

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad \vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

x_0 fresh variable

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 1 Prove $\forall z\, P(z) \vdash P(a)$

1 $\forall z\, P(z)$ premise

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad x_0 \text{ fresh variable} \qquad\qquad \dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 1 Prove $\forall z\, P(z) \vdash P(a)$

1	$\forall z\, P(z)$	premise	
2	$P(a)$	$\forall$e	1

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

x_0 fresh variable

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 2 Prove $\forall z\,(P(z) \wedge Q(z)) \vdash \forall y\, Q(y)$

∀ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

x_0 fresh variable

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 2 Prove $\forall z\,(P(z) \wedge Q(z)) \vdash \forall y\, Q(y)$

1 $\forall z\,(P(z) \wedge Q(z))$ premise

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

x_0 fresh variable

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 2 Prove $\forall z\,(P(z) \wedge Q(z)) \vdash \forall y\, Q(y)$

1 $\forall z\,(P(z) \wedge Q(z))$ premise

x_0 2

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

x_0 fresh variable

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 2 Prove $\forall z\,(P(z) \wedge Q(z)) \vdash \forall y\, Q(y)$

	1	$\forall z\,(P(z) \wedge Q(z))$	premise
x_0	2		
	3	$P(x_0) \wedge Q(x_0)$	$\forall$e 1

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad x_0 \text{ fresh variable}$$

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 2 Prove $\forall z\,(P(z) \land Q(z)) \vdash \forall y\, Q(y)$

	1	$\forall z\,(P(z) \land Q(z))$	premise
x_0	2		
	3	$P(x_0) \land Q(x_0)$	$\forall\mathrm{e}$ 1
	4	$Q(x_0)$	$\land\mathrm{e}_2$ 2

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

x_0 fresh variable

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 2 Prove $\forall z\,(P(z) \land Q(z)) \vdash \forall y\, Q(y)$

	1	$\forall z\,(P(z) \land Q(z))$	premise	
x_0	2			
	3	$P(x_0) \land Q(x_0)$	$\forall$e	1
	4	$Q(x_0)$	$\land\mathrm{e}_2$	2
	5	$\forall y\, Q(y)$	$\forall$i	2–5

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

x_0 fresh variable

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 3 Prove $\vdash \forall x\, x = x$

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad\vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

x_0 fresh variable

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 3 Prove $\vdash \forall x\, x = x$

x_0 1

$\forall$ introduction and elimination

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad \vdots \\ \qquad A_x^{x_0} \end{array}}}{\forall x\, A}\;\forall\mathrm{i}$$

x_0 fresh variable

$$\dfrac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\;\forall\mathrm{e}$$

Example 3 Prove $\vdash \forall x\, x = x$

$$\begin{array}{llll} x_0 & 1 & & \\ & 2 & x_0 = x_0 & =\mathrm{i} \end{array}$$

$\forall$ introduction and elimination

$$\frac{\boxed{\begin{array}{l} x_0 \\ \qquad\quad \vdots \\ \qquad\quad A_x^{x_0} \end{array}}}{\forall x\, A}\ \forall\mathrm{i}$$

x_0 fresh variable

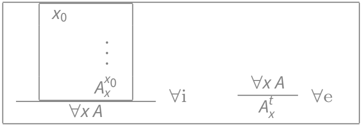
$$\frac{\forall x\, A \qquad t \text{ free for } x \text{ in } A}{A_x^t}\ \forall\mathrm{e}$$

Example 3 Prove $\vdash \forall x\, x = x$

$$\boxed{\begin{array}{llll} x_0 & 1 & & \\ & 2 & x_0 = x_0 & =\mathrm{i} \end{array}}$$

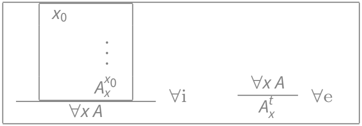
$$\begin{array}{llll} 3 & \forall x\, x = x & \forall\mathrm{i} & 1\text{–}2 \end{array}$$

Example derivation



$$\vdash\ \forall x\,\forall y\,(x = y \rightarrow f(x) = f(y))$$

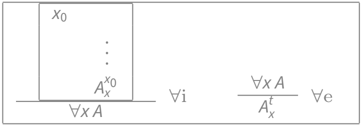
Example derivation



$$\vdash\ \forall x\, \forall y\, (x = y \rightarrow f(x) = f(y))$$

x_0 1

Example derivation

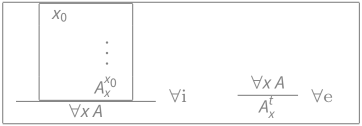


$$\vdash \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

x_0 1

y_0 2

Example derivation



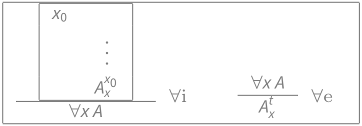
$$\vdash \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

x_0 1

y_0 2

3 $x_0 = y_0$ assumption

Example derivation



x_0

$\vdots$

$A_x^{x_0}$

$\forall x\, A$ ∀i

$$\frac{\forall x\, A}{A_x^t}\ \forall\mathrm{e}$$

$$\vdash \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

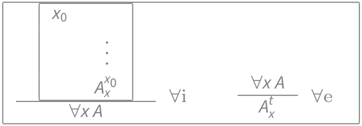
x_0 1

y_0 2

3 $x_0 = y_0$ assumption

4 $f(x_0) = f(x_0)$ =i

Example derivation



$$\vdash\ \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

x_0	1			
y_0	2			
	3	$x_0 = y_0$	assumption	
	4	$f(x_0) = f(x_0)$	$=$i	
	5	$f(x_0) = f(y_0)$	$=$e	3 applied to 4

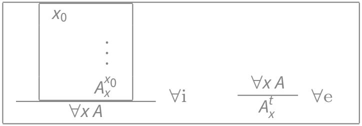
Example derivation

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A^{x_0}_x \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad \dfrac{\forall x\, A}{A^t_x}\ \forall\mathrm{e}$$

$$\vdash\ \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

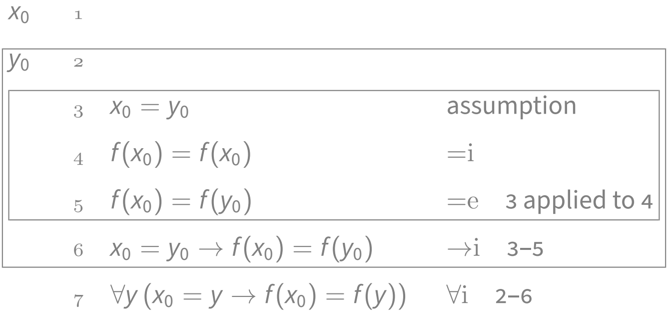
x_0	1			
y_0	2			
	3	$x_0 = y_0$	assumption	
	4	$f(x_0) = f(x_0)$	$=$i	
	5	$f(x_0) = f(y_0)$	$=$e	3 applied to 4
	6	$x_0 = y_0 \to f(x_0) = f(y_0)$	$\to$i	3–5

Example derivation



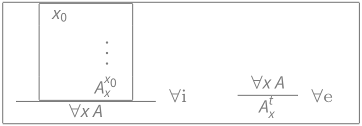
$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad A^{x_0}_x \end{array}}}{\forall x\, A}\ \forall\mathrm{i} \qquad \dfrac{\forall x\, A}{A^t_x}\ \forall\mathrm{e}$$

$$\vdash\ \forall x\, \forall y\, (x = y \to f(x) = f(y))$$

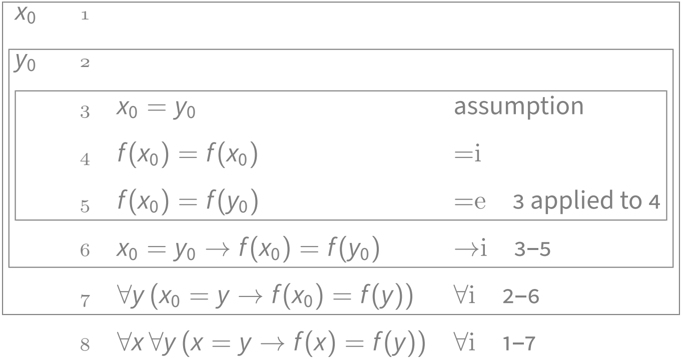


x_0	1			
y_0	2			
	3	$x_0 = y_0$	assumption	
	4	$f(x_0) = f(x_0)$	$=$i	
	5	$f(x_0) = f(y_0)$	$=$e	3 applied to 4
	6	$x_0 = y_0 \to f(x_0) = f(y_0)$	$\to$i	3–5
	7	$\forall y\, (x_0 = y \to f(x_0) = f(y))$	$\forall$i	2–6

Example derivation



$$\vdash \ \forall x\, \forall y\, (x = y \to f(x) = f(y))$$



x_0	1			
y_0	2			
	3	$x_0 = y_0$	assumption	
	4	$f(x_0) = f(x_0)$	$=$i	
	5	$f(x_0) = f(y_0)$	$=$e	3 applied to 4
	6	$x_0 = y_0 \to f(x_0) = f(y_0)$	$\to$i	3–5
	7	$\forall y\, (x_0 = y \to f(x_0) = f(y))$	$\forall$i	2–6
	8	$\forall x\, \forall y\, (x = y \to f(x) = f(y))$	$\forall$i	1–7

$\exists$ introduction and elimination

$$\dfrac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i} \qquad\qquad \dfrac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

x_0 fresh var

$\exists$ introduction and elimination

$$\frac{A^t_x \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i} \qquad\qquad \frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e} \qquad x_0 \text{ fresh var}$$

Example 1 Prove $P(a) \vdash \exists z\, P(z)$

$\exists$ introduction and elimination

$$\frac{A^t_x \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i} \qquad\qquad \frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

x_0 fresh var

Example 1 Prove $P(a) \vdash \exists z\, P(z)$

1 $P(a)$ premise

$\exists$ introduction and elimination

$$\frac{A^t_x \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\;\exists\mathrm{i}$$

$$\frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\;\exists\mathrm{e}$$

x_0 fresh var

Example 1 Prove $P(a) \vdash \exists z\, P(z)$

1 $P(a)$ premise

2 $\exists z\, P(z)$ $\exists$i 1

$\exists$ introduction and elimination

$$\frac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i} \qquad\qquad \frac{\exists x\, A \quad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

x_0 fresh var

Example 2 Prove $\exists x\, P(x),\ \forall x\, \neg P(x)\ \vdash\ \bot$

$\exists$ introduction and elimination

$$\frac{A^t_x \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i} \qquad\qquad \frac{\exists x\, A \qquad \boxed{\begin{array}{cc} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

x_0 fresh var

Example 2 Prove $\exists x\, P(x), \forall x\, \neg P(x) \vdash \bot$

1 $\exists x\, P(x)$ premise

$\exists$ introduction and elimination

$$\frac{A^{t}_{x} \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i} \qquad\qquad \frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A^{x_0}_{x} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

x_0 fresh var

Example 2 Prove $\exists x\, P(x),\ \forall x\, \neg P(x) \vdash \bot$

1 $\exists x\, P(x)$ premise

2 $\forall x\, \neg P(x)$ premise

$\exists$ introduction and elimination

$$\frac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i} \qquad\qquad \frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

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	1	$\exists x\, P(x)$	premise
	2	$\forall x\, \neg P(x)$	premise
x_0	3	$P(x_0)$	assumption

∃ introduction and elimination

$$\dfrac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i} \qquad\qquad \dfrac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

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	2	$\forall x\, \neg P(x)$	premise
x_0	3	$P(x_0)$	assumption
	4	$\neg P(x_0)$	$\forall$e 3

$\exists$ introduction and elimination

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	4	$\neg P(x_0)$	$\forall$e 3
	5	$\bot$	$\neg$e 3, 4

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$$\frac{A_x^t \qquad t \text{ free for } x \text{ in } A}{\exists x\, A}\ \exists\mathrm{i} \qquad\qquad \frac{\exists x\, A \quad \boxed{\begin{array}{ll} x_0 & A_x^{x_0} \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

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	1	$\exists x\, P(x)$	premise	
	2	$\forall x\, \neg P(x)$	premise	
x_0	3	$P(x_0)$	assumption	
	4	$\neg P(x_0)$	$\forall$e	3
	5	$\bot$	$\neg$e	3, 4
	6	$\bot$	$\exists$e	1, 3–5

Example derivation

$$\frac{A^t_x}{\exists x\, A}\ \exists\mathrm{i} \qquad \frac{\exists x\, A \quad \boxed{\begin{array}{ll} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

$$\forall z\, (P(z) \rightarrow Q(z)),\ \exists y\, P(y)\ \vdash\ \exists x\, Q(x)$$

Example derivation

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Example derivation

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$$\forall z\,(P(z) \to Q(z)),\ \exists y\, P(y) \ \vdash\ \exists x\, Q(x)$$

1	$\forall z\,(P(z) \to Q(z))$	premise
2	$\exists y\, P(y)$	premise

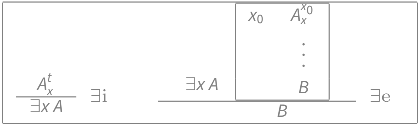
Example derivation

$$\frac{A^t_x}{\exists x\, A}\ \exists\mathrm{i} \qquad \frac{\exists x\, A \quad \boxed{\begin{matrix} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{matrix}}}{B}\ \exists\mathrm{e}$$

$$\forall z\,(P(z) \to Q(z)),\ \exists y\, P(y) \vdash \exists x\, Q(x)$$

	1	$\forall z\,(P(z) \to Q(z))$	premise
	2	$\exists y\, P(y)$	premise
x_0	3	$P(x_0)$	assumption

Example derivation



$$\forall z\,(P(z) \to Q(z)),\ \exists y\, P(y)\ \vdash\ \exists x\, Q(x)$$

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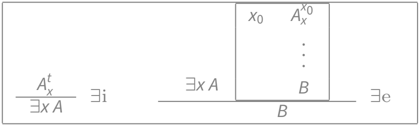
	2	$\exists y\, P(y)$	premise
x_0	3	$P(x_0)$	assumption
	4	$P(x_0) \to Q(x_0)$	∀e 1

Example derivation



$$\frac{A^t_x}{\exists x\, A}\ \exists\mathrm{i} \qquad \frac{\exists x\, A \qquad \boxed{\begin{array}{ll} x_0 & A^{x_0}_x \\ & \vdots \\ & B \end{array}}}{B}\ \exists\mathrm{e}$$

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	2	$\exists y\, P(y)$	premise
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	4	$P(x_0) \rightarrow Q(x_0)$	$\forall$e 1
	5	$Q(x_0)$	$\rightarrow$e 3, 4

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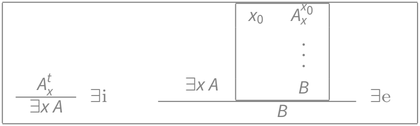
	4	$P(x_0) \to Q(x_0)$	$\forall$e	1
	5	$Q(x_0)$	$\to$e	3, 4
	6	$\exists x\, Q(x)$	$\exists$i	5

Example derivation



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	4	$P(x_0) \to Q(x_0)$	$\forall$e 1
	5	$Q(x_0)$	$\to$e 3, 4
	6	$\exists x\, Q(x)$	$\exists$i 5
	7	$\exists x\, Q(x)$	$\exists$e 2, 3–6

Soundness of Natural Deduction

Let $F, F_1, \dots, F_n$ be FOL formulas

Theorem 2 (Soundness)
If $F_1, \dots, F_n \vdash F$ *then* $F_1, \dots, F_n \models F$.

As in Propositional Logic, the proof of reduces to proving that

- formulas derivable from no premises are valid
- new derivation rules preserve models

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The first proof, by Kurt Gödel, was a milestone result in mathematical logic

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Undecidability of FOL

The problem of determining the validity of FOL formulas is only semi-decidable:

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