CS:4350 Logic in Computer Science

Satisfiability and Randomization

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The University of Iowa

Credits

These slides are largely based on slides originally developed by **Andrei Voronkov** at the University of Manchester. Adapted by permission.

Outline

Random Clause Generation

How can one generate a random clause?

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How can one generate a random clause?

Let's first generate a *random literal*

Random Clause Generation

How can one generate a random clause?

Let's first generate a *random literal*

- Fix a number n of boolean variables

Random Clause Generation

How can one generate a random clause?

Let's first generate a *random literal*

- Fix a number n of boolean variables
- Select a literal among $p_1, \ldots, p_n, \neg p_1, \ldots, \neg p_n$ with an equal probability

Random Clause Generation

How can one generate a random clause?

Let's first generate a *random literal*

A *random clause* is a disjunction of random literals

- Fix a number n of boolean variables
- Select a literal among $p_1, \dots, p_n, \neg p_1, \dots, \neg p_n$ with an equal probability

Random Clause Generation

How can one generate a random clause?

Let's first generate a *random literal*

A *random clause* is a disjunction of random literals

- Fix a number n of boolean variables
- Select a literal among $p_1, \ldots, p_n, \neg p_1, \ldots, \neg p_n$ with an equal probability
- Fix the length k of the clause

Random Clause Generation

How can one generate a random clause?

Let's first generate a *random literal*

A *random clause* is a disjunction of random literals

- Fix a number n of boolean variables
- Select a literal among $p_1, \ldots, p_n, \neg p_1, \ldots, \neg p_n$ with an equal probability
- Fix the length k of the clause

Suppose we generate random clauses one by one

How does the set of models of this set change?

Example (obtained by a program) for $n = 5$ and $k = 2$

S

p_1	p_2	p_3	p_4	p_5
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	0	1	1
0	0	1	0	0
0	0	1	0	1
0	0	1	1	0
0	0	1	1	1
0	1	0	0	0
0	1	0	0	1
0	1	0	1	0
0	1	0	1	1
0	1	1	0	0
0	1	1	0	1
0	1	1	1	0
0	1	1	1	1

p_1	p_2	p_3	p_4	p_5
1	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1
1	1	1	0	0
1	1	1	0	1
1	1	1	1	0
1	1	1	1	1

Number of models for clause set ***S*** over 5 vars: 32

Example (obtained by a program) for $n = 5$ and $k = 2$

S

$\neg p_2 \vee \neg p_3$

p_1	p_2	p_3	p_4	p_5
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	0	1	1
0	0	1	0	0
0	0	1	0	1
0	0	1	1	0
0	0	1	1	1
0	1	0	0	0
0	1	0	0	1
0	1	0	1	0
0	1	0	1	1
0	1	1	0	0
0	1	1	0	1
0	1	1	1	0
0	1	1	1	1

p_1	p_2	p_3	p_4	p_5
1	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1
1	1	1	0	0
1	1	1	0	1
1	1	1	1	0
1	1	1	1	1

Number of models for clause set S over 5 vars: 32

Example (obtained by a program) for $n = 5$ and $k = 2$

$\boldsymbol{S}$

$\neg p_2 \vee \neg p_3$

p_1	p_2	p_3	p_4	p_5
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	0	1	1
0	0	1	0	0
0	0	1	0	1
0	0	1	1	0
0	0	1	1	1
0	1	0	0	0
0	1	0	0	1
0	1	0	1	0
0	1	0	1	1

p_1	p_2	p_3	p_4	p_5
1	0	0	0	0
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1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set $\boldsymbol{S}$ over 5 vars: 24

Example (obtained by a program) for $n = 5$ and $k = 2$

S

$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$

p_1	p_2	p_3	p_4	p_5
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	0	1	1
0	0	1	0	0
0	0	1	0	1
0	0	1	1	0
0	0	1	1	1
0	1	0	0	0
0	1	0	0	1
0	1	0	1	0
0	1	0	1	1

p_1	p_2	p_3	p_4	p_5
1	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set S over 5 vars: 24

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$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$

p_1	p_2	p_3	p_4	p_5
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	0	1	1
0	0	1	0	0
0	0	1	0	1
0	0	1	1	0
0	0	1	1	1

p_1	p_2	p_3	p_4	p_5
1	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set ***S*** over 5 vars: 20

Example (obtained by a program) for $n = 5$ and $k = 2$

S

$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$

p_1	p_2	p_3	p_4	p_5
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	0	1	1
0	0	1	0	0
0	0	1	0	1
0	0	1	1	0
0	0	1	1	1

p_1	p_2	p_3	p_4	p_5
1	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set S over 5 vars: 20

Example (obtained by a program) for $n = 5$ and $k = 2$

S

$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$

p_1	p_2	p_3	p_4	p_5
0	0	0	0	0
0	0	0	0	1
0	0	0	1	0
0	0	0	1	1
0	0	1	0	0
0	0	1	0	1
0	0	1	1	0
0	0	1	1	1

p_1	p_2	p_3	p_4	p_5
1	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set S over 5 vars: 20

Example (obtained by a program) for $n = 5$ and $k = 2$

S

$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set S over 5 vars: 12

Example (obtained by a program) for $n = 5$ and $k = 2$

$\boldsymbol{S}$

$\neg p_2 \lor \neg p_3$
$\neg p_2 \lor p_1$
$\neg p_2 \lor p_2$
$p_1 \lor p_1$
$\neg p_5 \lor p_5$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set $\boldsymbol{S}$ over 5 vars: 12

Example (obtained by a program) for $n = 5$ and $k = 2$

$\boldsymbol{S}$

$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$
$\neg p_5 \vee p_5$
$p_4 \vee p_5$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set $\boldsymbol{S}$ over 5 vars: 12

Example (obtained by a program) for $n = 5$ and $k = 2$

S
$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$
$\neg p_5 \vee p_5$
$p_4 \vee p_5$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set S over 5 vars: 9

Example (obtained by a program) for $n = 5$ and $k = 2$

S

$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$
$\neg p_5 \vee p_5$
$p_4 \vee p_5$
$\neg p_5 \vee \neg p_3$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set **S** over 5 vars: 9

Example (obtained by a program) for $n = 5$ and $k = 2$

S

$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$
$\neg p_5 \vee p_5$
$p_4 \vee p_5$
$\neg p_5 \vee \neg p_3$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set ***S*** over 5 vars: 7

Example (obtained by a program) for $n = 5$ and $k = 2$

$\boldsymbol{S}$

$\neg p_2 \vee \neg p_3$

$\neg p_2 \vee p_1$

$\neg p_2 \vee p_2$

$p_1 \vee p_1$

$\neg p_5 \vee p_5$

$p_4 \vee p_5$

$\neg p_5 \vee \neg p_3$

$p_2 \vee \neg p_4$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set $\boldsymbol{S}$ over 5 vars: 7

Example (obtained by a program) for $n = 5$ and $k = 2$

S

$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$
$\neg p_5 \vee p_5$
$p_4 \vee p_5$
$\neg p_5 \vee \neg p_3$
$p_2 \vee \neg p_4$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	1
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1

Number of models for clause set **S** over 5 vars: 4

Example (obtained by a program) for $n = 5$ and $k = 2$

S	p_1	p_2	p_3	p_4	p_5	p_1	p_2	p_3	p_4	p_5
						1	0	0	0	1
$\neg p_2 \vee \neg p_3$										
$\neg p_2 \vee p_1$										
$\neg p_2 \vee p_2$										
$p_1 \vee p_1$										
$\neg p_5 \vee p_5$										
$p_4 \vee p_5$										
$\neg p_5 \vee \neg p_3$										
$p_2 \vee \neg p_4$						1	1	0	0	1
$p_5 \vee \neg p_2$						1	1	0	1	0
						1	1	0	1	1

Number of models for clause set S over 5 vars: 4

Example (obtained by a program) for $n = 5$ and $k = 2$

S

$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$
$\neg p_5 \vee p_5$
$p_4 \vee p_5$
$\neg p_5 \vee \neg p_3$
$p_2 \vee \neg p_4$
$p_5 \vee \neg p_2$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	1
1	1	0	0	1
1	1	0	1	1

Number of models for clause set S over 5 vars: 3

Example (obtained by a program) for $n = 5$ and $k = 2$

S

$\neg p_2 \lor \neg p_3$
$\neg p_2 \lor p_1$
$\neg p_2 \lor p_2$
$p_1 \lor p_1$
$\neg p_5 \lor p_5$
$p_4 \lor p_5$
$\neg p_5 \lor \neg p_3$
$p_2 \lor \neg p_4$
$p_5 \lor \neg p_2$
$p_5 \lor p_2$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	1
1	1	0	0	1
1	1	0	1	1

Number of models for clause set S over 5 vars: 3

Example (obtained by a program) for $n = 5$ and $k = 2$

S		p_1	p_2	p_3	p_4	p_5
		1	0	0	0	1

$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$
$\neg p_5 \vee p_5$
$p_4 \vee p_5$
$\neg p_5 \vee \neg p_3$
$p_2 \vee \neg p_4$
$p_5 \vee \neg p_2$
$p_5 \vee p_2$

p_1	p_2	p_3	p_4	p_5

Number of models for clause set S over 5 vars: 1

Example (obtained by a program) for $n = 5$ and $k = 2$

S
$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$
$\neg p_5 \vee p_5$
$p_4 \vee p_5$
$\neg p_5 \vee \neg p_3$
$p_2 \vee \neg p_4$
$p_5 \vee \neg p_2$
$p_5 \vee p_2$
$\neg p_1 \vee \neg p_4$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	1

Number of models for clause set S over 5 vars: 1

Example (obtained by a program) for $n = 5$ and $k = 2$

S
$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$
$\neg p_5 \vee p_5$
$p_4 \vee p_5$
$\neg p_5 \vee \neg p_3$
$p_2 \vee \neg p_4$
$p_5 \vee \neg p_2$
$p_5 \vee p_2$
$\neg p_1 \vee \neg p_4$
$p_5 \vee p_2$

p_1	p_2	p_3	p_4	p_5

p_1	p_2	p_3	p_4	p_5
1	0	0	0	1

Number of models for clause set S over 5 vars: 1

Example (obtained by a program) for $n = 5$ and $k = 2$

S	p_1	p_2	p_3	p_4	p_5		p_1	p_2	p_3	p_4	p_5
							1	0	0	0	1
$\neg p_2 \lor \neg p_3$											
$\neg p_2 \lor p_1$											
$\neg p_2 \lor p_2$											
$p_1 \lor p_1$											
$\neg p_5 \lor p_5$											
$p_4 \lor p_5$											
$\neg p_5 \lor \neg p_3$											
$p_2 \lor \neg p_4$											
$p_5 \lor \neg p_2$											
$p_5 \lor p_2$											
$\neg p_1 \lor \neg p_4$											
$p_5 \lor p_2$											
$\neg p_1 \lor \neg p_5$											

Number of models for clause set S over 5 vars: 1

Example (obtained by a program) for $n = 5$ and $k = 2$

S		p_1	p_2	p_3	p_4	p_5		p_1	p_2	p_3	p_4	p_5

$\neg p_2 \vee \neg p_3$
$\neg p_2 \vee p_1$
$\neg p_2 \vee p_2$
$p_1 \vee p_1$
$\neg p_5 \vee p_5$
$p_4 \vee p_5$
$\neg p_5 \vee \neg p_3$
$p_2 \vee \neg p_4$
$p_5 \vee \neg p_2$
$p_5 \vee p_2$
$\neg p_1 \vee \neg p_4$
$p_5 \vee p_2$
$\neg p_1 \vee \neg p_5$

This set of 13 clauses is unsatisfiable

Number of models for clause set S over 5 vars: 0

SAT and k-SAT

SAT is satisfiability checking for sets of clauses

k-SAT is satisfiability checking for sets of clauses of length k or less

SAT and k-SAT

SAT is satisfiability checking for sets of clauses

k-SAT is satisfiability checking for sets of clauses of length k or less

- SAT is NP-complete
- 2-SAT is decidable in linear time
- 3-SAT is NP-complete!

SAT and k-SAT

SAT is satisfiability checking for sets of clauses

k-SAT is satisfiability checking for sets of clauses of length k or less

- SAT is NP-complete
- 2-SAT is decidable in linear time
- 3-SAT is NP-complete!

SAT and k-SAT

SAT is satisfiability checking for sets of clauses

k-SAT is satisfiability checking for sets of clauses of length k or less

- SAT is NP-complete
- 2-SAT is decidable in linear time
- 3-SAT is NP-complete!

All you need is 3-SAT, 3-SAT is all you need

There is a simple reduction of SAT to 3-SAT based, again on naming:

1. Take a clause having more than 3 literals:

$$L_1 \vee L_2 \vee L_3 \vee L_4 \vee \cdots \vee L_n$$

   and replace it by two clauses:

$$L_1 \vee L_2 \vee n$$
$$\neg n \vee L_3 \vee L_4 \vee \cdots \vee L_n$$

   where n is a fresh variable

2. Repeat until all clauses have at most 3 literals

The final clause set is equisatisfiable with the original clause

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$$L_1 \vee L_2 \vee n$$
$$\neg n \vee L_3 \vee L_4 \vee \cdots \vee L_n$$

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All you need is 3-SAT, 3-SAT is all you need

There is a simple reduction of SAT to 3-SAT based, again on naming:

1. Take a clause having more than 3 literals:

$$L_1 \vee L_2 \vee L_3 \vee L_4 \vee \cdots \vee L_n$$

   and replace it by two clauses:

$$\begin{array}{l} L_1 \vee L_2 \vee n \\ \neg n \vee L_3 \vee L_4 \vee \cdots \vee L_n \end{array}$$

   where n is a fresh variable

2. Repeat until all clauses have at most 3 literals

The final clause set is equisatisfiable with the original clause

Random Clause Generation

What is the probability that a set of clauses of a given size is unsatisfiable?

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Fix:

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Generate m clauses, each with k literals chosen randomly with an equal probability from $\{p_1, \ldots, p_n, \neg p_1, \ldots, \neg p_n\}$

Random Clause Generation

What is the probability that a set of clauses of a given size is unsatisfiable?

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- number k of literals per clause, so we will generate k-SAT instances
- number m of the clauses

Generate m clauses, each with k literals chosen randomly with an equal probability from $\{p_1, \ldots, p_n, \neg p_1, \ldots, \neg p_n\}$

Note: Probability of *unsat* is a monotonic function of m:
the larger the clause set, the higher the probability that it is unsatisfiable

Random Clause Generation

What is the probability that a set of clauses of a given size is unsatisfiable?

Fix:

- number n of propositional variables
- number k of literals per clause, so we will generate k-SAT instances
- real number r: ratio of clauses per variable

Generate $[r \cdot n]$ clauses, each with k literals chosen randomly with an equal probability from $\{p_1, \ldots, p_n, \neg p_1, \ldots, \neg p_n\}$

Note: Probability of *unsat* is a monotonic function of r:
the larger the clause set, the higher the probability that it is unsatisfiable

Roulette

We will generate random instances of 2-SAT with 5-variables

SAT Roulette

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You will bet on whether the resulting set of clauses is **satisfiable** or **unsatisfiable**

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- What will you bet on if we generate 5 clauses?
- 100 clauses?
- 15 clauses?

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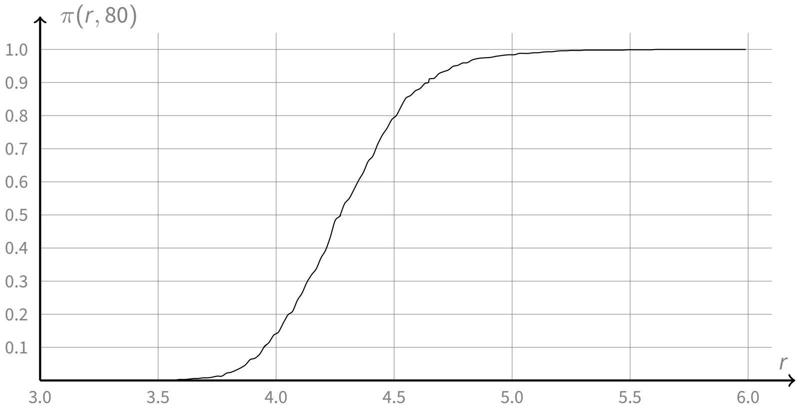
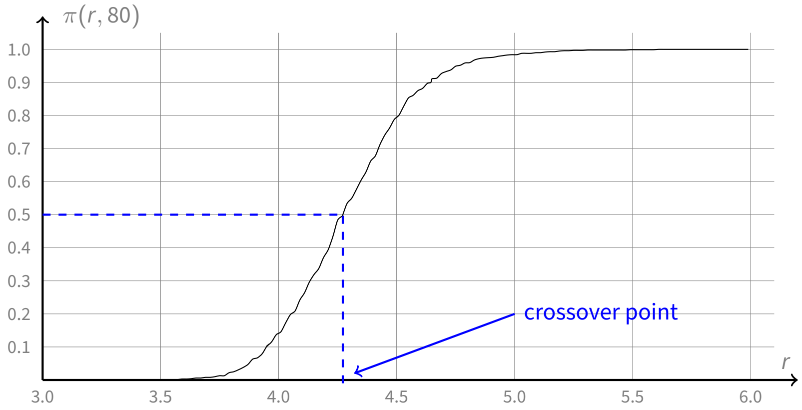
Probability of obtaining an unsatisfiable set



$\pi(r, n) =$ prob. that a randomly generated set of $\lceil r \cdot n \rceil$ 3-clauses over n variables is unsat

Probability of obtaining an unsatisfiable set

Crossover point: the value of r at which the probability crosses 0.5



$\pi(r, n) =$ prob. that a randomly generated set of $[r \cdot n]$ 3-clauses over n variables is unsat

ϵ-window

For any (small) number $\epsilon > 0$, the ϵ-*window* is the interval of values of r where

$$\epsilon \leq \pi(r, n) \leq 1 - \epsilon$$

Example $\epsilon = 0.1$

ϵ-window

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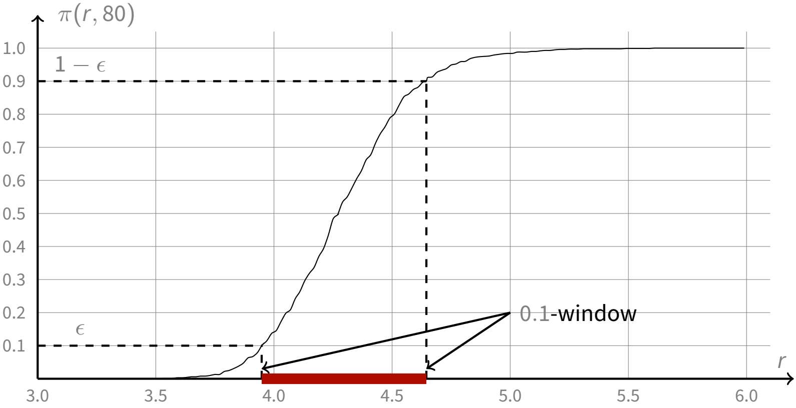
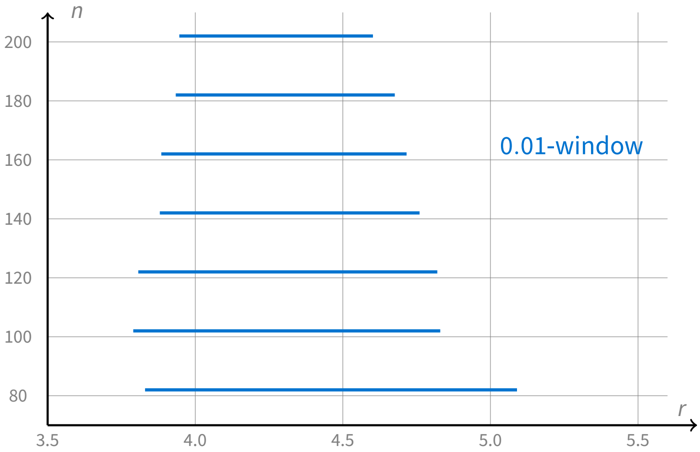
$$\epsilon \leq \pi(r, n) \leq 1 - \epsilon$$

Example $\epsilon = 0.1$



Scaling Window Effect



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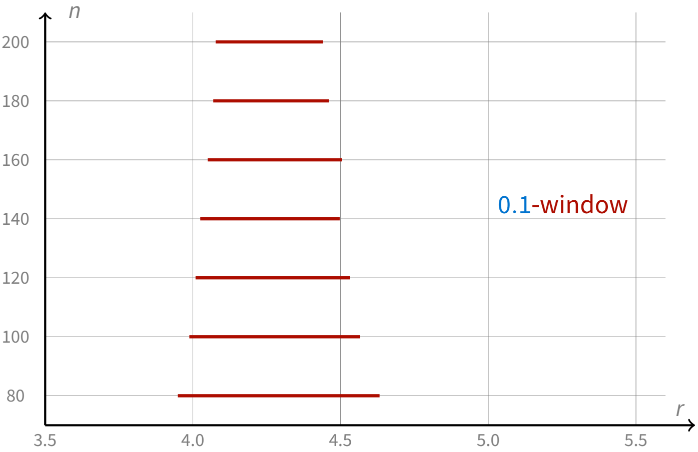
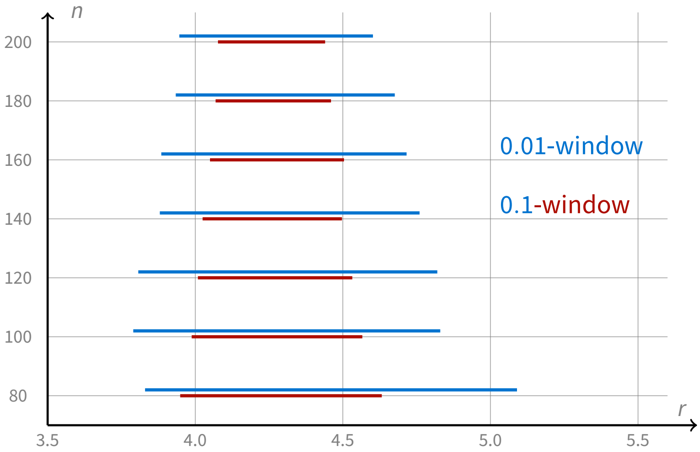
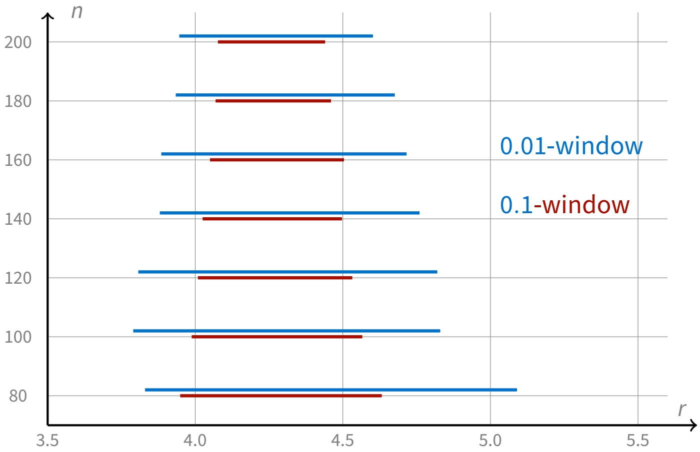


Scaling Window Effect

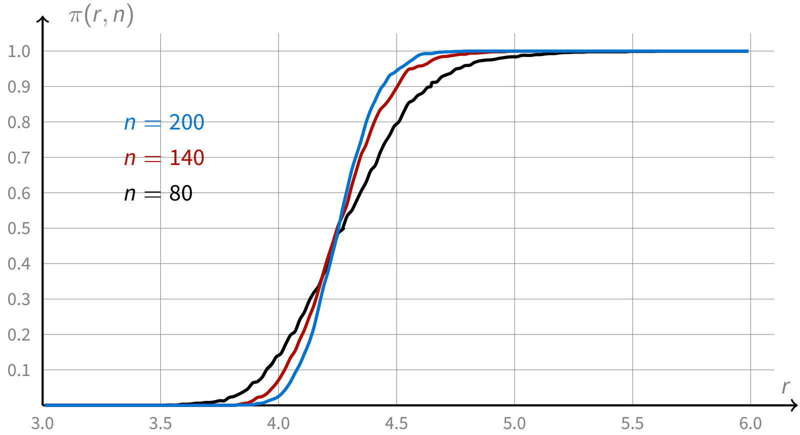


Scaling Window Effect



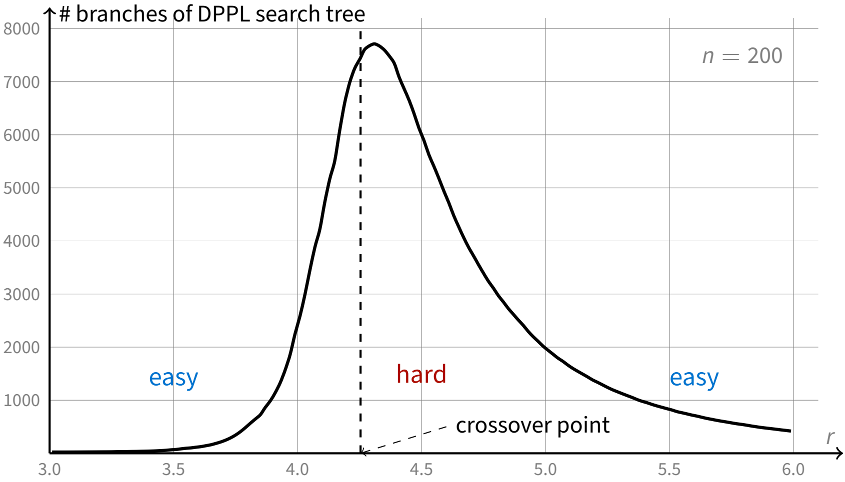
Conjecture: for $n \to \infty$ every ϵ-window degenerates into a point

Sharp Phase Transition



$\pi(r, n) =$ prob. that a randomly generated set of $[r \cdot n]$ 3-clauses over n variables is unsat

Easy-Hard-Easy Pattern



Satisfiability Algorithm that Cannot Establish Unsatisfiability

procedure *CHAOS*(S)
input: set of clauses S
output: interpretation $\mathcal{I}$ such that $\mathcal{I} \models S$ or *"don't know"*

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begin
  repeat *MAX-TRIES* times

end

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output: interpretation $\mathcal{I}$ such that $\mathcal{I} \models S$ or *"don't know"*
parameters: positive integer *MAX-TRIES*
begin
  repeat *MAX-TRIES* times
    $\mathcal{I} := $ random interpretation
    if $\mathcal{I} \models S$ **then return** $\mathcal{I}$
  return *"don't know"*
end

Satisfiability Algorithm that Cannot Establish Unsatisfiability

```
procedure CHAOS(S)
input: set of clauses S
output: interpretation I such that I ⊨ S or "don't know"
parameters: positive integer MAX-TRIES
begin
  repeat MAX-TRIES times
    I := random interpretation
    if I ⊨ S then return I
  return "don't know"
end
```

Note:

Satisfiability has short witnesses: interpretations, always checkable in poly-time

Satisfiability Algorithm that Cannot Establish Unsatisfiability

procedure *CHAOS*(S)
input: set of clauses S
output: interpretation $\mathcal{I}$ such that $\mathcal{I} \models S$ or *"don't know"*
parameters: positive integer *MAX-TRIES*
begin
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    $\mathcal{I} := $ random interpretation
    if $\mathcal{I} \models S$ **then return** $\mathcal{I}$
  return *"don't know"*
end

Note:

Satisfiability has short witnesses: interpretations, always checkable in poly-time

Unsatisfiability has long witnesses: proofs (e.g. tableaux), **not** always checkable in poly-time

Randomized Algorithms for SAT

1. Choose a random interpretation $\mathcal{I}$
2. Until $\mathcal{I}$ satisfies the clause set, choose a variable and *flip* it

$$flip(\mathcal{I}, p)(q) = \begin{cases} \mathcal{I}(q) & \text{if } p \neq q \\ 1 & \text{if } p = q \text{ and } \mathcal{I}(p) = 0 \\ 0 & \text{if } p = q \text{ and } \mathcal{I}(p) = 1 \end{cases}$$

The variables to flip are chosen using heuristics or randomly, or both

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procedure $GSAT(S)$
input: set of clauses S
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output: interpretation $\mathcal{I}$ such that $\mathcal{I} \models S$ or *"don't know"*
parameters: integers *MAX-TRIES*, *MAX-FLIPS*
begin
  repeat *MAX-TRIES* times
   $\mathcal{I}$:= random interpretation

end

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begin
 repeat *MAX-TRIES* times
  $\mathcal{I}$:= random interpretation
  repeat *MAX-FLIPS* times
   if $\mathcal{I} \models S$ **then return** $\mathcal{I}$
   p := a variable such that $\mathit{flip}(\mathcal{I}, p)$ satisfies
     the maximal number of clauses in S
   $\mathcal{I} := \mathit{flip}(\mathcal{I}, p)$

end

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procedure *GSAT*(S)
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output: interpretation $\mathcal{I}$ such that $\mathcal{I} \models S$ or *"don't know"*
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      p := a variable such that $\mathit{flip}(\mathcal{I}, p)$ satisfies
            the maximal number of clauses in S
      $\mathcal{I}$:= $\mathit{flip}(\mathcal{I}, p)$
  return *"don't know"*
end

GSAT example

0		0		1
p_1	$\vee$	$\neg p_2$	$\vee$	p_3
		$\neg p_2$	$\vee$	$\neg p_3$
$\neg p_1$			$\vee$	$\neg p_3$
$\neg p_1$	$\vee$	p_2		
p_1	$\vee$	p_2		

GSAT example

0		0		1	
p_1	$\vee$	$\neg p_2$	$\vee$	p_3	✓
		$\neg p_2$	$\vee$	$\neg p_3$	✓
$\neg p_1$			$\vee$	$\neg p_3$	✓
$\neg p_1$	$\vee$	p_2			✓
p_1	$\vee$	p_2			

flip no.	interpretation p_1	p_2	p_3	satisfied clauses	p_1	p_2	p_3	candidates for flipping	flipped variable
1	0	0	1	4					

GSAT example

0		0		1	
p_1	$\lor$	$\neg p_2$	$\lor$	p_3	✓
		$\neg p_2$	$\lor$	$\neg p_3$	✓
$\neg p_1$			$\lor$	$\neg p_3$	
$\neg p_1$	$\lor$	p_2			
p_1	$\lor$	p_2			✓

flip no.	interpretation			satisfied clauses				candidates for flipping	flipped variable
	p_1	p_2	p_3		p_1	p_2	p_3		
1	0	0	1	4	3				

GSAT example

0		1		1	
p_1	$\vee$	$\neg p_2$	$\vee$	p_3	✓
		$\neg p_2$	$\vee$	$\neg p_3$	
$\neg p_1$			$\vee$	$\neg p_3$	✓
$\neg p_1$	$\vee$	p_2			✓
p_1	$\vee$	p_2			✓

flip no.	interpretation			satisfied clauses				candidates for flipping	flipped variable
	p_1	p_2	p_3		p_1	p_2	p_3		
1	0	0	1	4	3	4			

GSAT example

0		1		1	
p_1	$\vee$	$\neg p_2$	$\vee$	p_3	✓
		$\neg p_2$	$\vee$	$\neg p_3$	✓
$\neg p_1$			$\vee$	$\neg p_3$	✓
$\neg p_1$	$\vee$	p_2			✓
p_1	$\vee$	p_2			

flip no.	interpretation			satisfied clauses				candidates for flipping	flipped variable
	p_1	p_2	p_3		p_1	p_2	p_3		
1	0	0	1	4	3	4	4		

GSAT example

0		1		1
p_1	$\vee$	$\neg p_2$	$\vee$	p_3
		$\neg p_2$	$\vee$	$\neg p_3$
$\neg p_1$			$\vee$	$\neg p_3$
$\neg p_1$	$\vee$	p_2		
p_1	$\vee$	p_2		

flip no.	interpretation			satisfied clauses				candidates for flipping	flipped variable
	p_1	p_2	p_3		p_1	p_2	p_3		
1	0	0	1	4	3	4	4	p_2, p_3	

GSAT example

0		1		1
p_1	$\vee$	$\neg p_2$	$\vee$	p_3
		$\neg p_2$	$\vee$	$\neg p_3$
$\neg p_1$			$\vee$	$\neg p_3$
$\neg p_1$	$\vee$	p_2		
p_1	$\vee$	p_2		

flip no.	interpretation			satisfied clauses				candidates for flipping	flipped variable
	p_1	p_2	p_3		p_1	p_2	p_3		
1	0	0	1	4	3	4	4	p_2, p_3	p_2

GSAT example

0		1		1	
p_1	$\vee$	$\neg p_2$	$\vee$	p_3	✓
		$\neg p_2$	$\vee$	$\neg p_3$	
$\neg p_1$			$\vee$	$\neg p_3$	✓
$\neg p_1$	$\vee$	p_2			✓
p_1	$\vee$	p_2			✓

flip no.	interpretation			satisfied clauses				candidates for flipping	flipped variable
	p_1	p_2	p_3		p_1	p_2	p_3		
1	0	0	1	4	3	4	4	p_2, p_3	p_2
2	0	1	1	4	3	4	4		

GSAT example

0		1		0	
p_1	$\lor$	$\neg p_2$	$\lor$	p_3	
		$\neg p_2$	$\lor$	$\neg p_3$	✓
$\neg p_1$			$\lor$	$\neg p_3$	✓
$\neg p_1$	$\lor$	p_2			✓
p_1	$\lor$	p_2			✓

flip no.	interpretation			satisfied clauses				candidates for flipping	flipped variable
	p_1	p_2	p_3		p_1	p_2	p_3		
1	0	0	1	4	3	4	4	p_2, p_3	p_2
2	0	1	1	4	3	4	4	p_2, p_3	p_3
3	0	1	0						

GSAT example

0		1		0	
p_1	$\vee$	$\neg p_2$	$\vee$	p_3	
		$\neg p_2$	$\vee$	$\neg p_3$	✓
$\neg p_1$			$\vee$	$\neg p_3$	✓
$\neg p_1$	$\vee$	p_2			✓
p_1	$\vee$	p_2			✓

flip no.	interpretation			satisfied clauses				candidates for flipping	flipped variable
	p_1	p_2	p_3		p_1	p_2	p_3		
1	0	0	1	4	3	4	4	p_2, p_3	p_2
2	0	1	1	4	3	4	4	p_2, p_3	p_3
3	0	1	0	4					

GSAT example

0		1		0
p_1	$\vee$	$\neg p_2$	$\vee$	p_3
		$\neg p_2$	$\vee$	$\neg p_3$
$\neg p_1$			$\vee$	$\neg p_3$
$\neg p_1$	$\vee$	p_2		
p_1	$\vee$	p_2		

flip no.	interpretation p_1	p_2	p_3	satisfied clauses	p_1	p_2	p_3	candidates for flipping	flipped variable
1	0	0	1	4	3	4	4	p_2, p_3	p_2
2	0	1	1	4	3	4	4	p_2, p_3	p_3
3	0	1	0	4	5	4	4		

GSAT example

1		1		0	
p_1	$\vee$	$\neg p_2$	$\vee$	p_3	✓
		$\neg p_2$	$\vee$	$\neg p_3$	✓
$\neg p_1$			$\vee$	$\neg p_3$	✓
$\neg p_1$	$\vee$	p_2			✓
p_1	$\vee$	p_2			✓

flip no.	interpretation			satisfied clauses				candidates for flipping	flipped variable
	p_1	p_2	p_3		p_1	p_2	p_3		
1	0	0	1	4	3	4	4	p_2, p_3	p_2
2	0	1	1	4	3	4	4	p_2, p_3	p_3
3	0	1	0	4	5	4	4	p_1	p_1
	1	1	0	5					

GSAT with random walks

procedure *GSATwithWalks*(S)
input: set of clauses S
output: interpretation $\mathcal{I}$ such that $\mathcal{I} \models S$ or *"don't know"*

GSAT with random walks

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input: set of clauses S
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parameters: integers *MAX-TRIES*, *MAX-FLIPS*
real number $0 \leq \pi \leq 1$ (probability of a sideways move)

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parameters: integers *MAX-TRIES*, *MAX-FLIPS*
real number $0 \leq \pi \leq 1$ (probability of a sideways move)
begin
repeat *MAX-TRIES* times
$\mathcal{I}$:= random interpretation

end

GSAT with random walks

procedure $\mathit{GSATwithWalks}(S)$
input: set of clauses S
output: interpretation $\mathcal{I}$ such that $\mathcal{I} \models S$ or *"don't know"*
parameters: integers *MAX-TRIES*, *MAX-FLIPS*
real number $0 \leq \pi \leq 1$ (probability of a sideways move)
begin
repeat *MAX-TRIES* times
$\mathcal{I}$:= random interpretation
repeat *MAX-FLIPS* times
if $\mathcal{I} \models S$ **then return** $\mathcal{I}$
with probability π
p := a variable such that $\mathit{flip}(\mathcal{I}, p)$ satisfies the maximal number of clauses in S
with probability $1 - \pi$
randomly select p among all variables occurring in clauses falsified by $\mathcal{I}$
$\mathcal{I} := \mathit{flip}(\mathcal{I}, p)$
return *"don't know"*
end

WSAT

procedure $\mathit{WSAT}(S)$
input: set of clauses S
output: interpretation $\mathcal{I}$ such that $\mathcal{I} \models S$ or *"don't know"*
parameters: integers *MAX-TRIES*, *MAX-FLIPS*

WSAT

procedure *WSAT*(S)
input: set of clauses S
output: interpretation $\mathcal{I}$ such that $\mathcal{I} \models S$ or *"don't know"*
parameters: integers *MAX-TRIES*, *MAX-FLIPS*
begin
 repeat *MAX-TRIES* times
  $\mathcal{I}$:= random interpretation

end

WSAT

procedure *WSAT*(S)
input: set of clauses S
output: interpretation $\mathcal{I}$ such that $\mathcal{I} \models S$ or *"don't know"*
parameters: integers *MAX-TRIES*, *MAX-FLIPS*
begin
 repeat *MAX-TRIES* times
  $\mathcal{I} :=$ random interpretation
  repeat *MAX-FLIPS* times
   if $\mathcal{I} \models S$ **then return** $\mathcal{I}$
   randomly select a clause $C \in S$ such that $\mathcal{I} \not\models C$
   randomly select a variable p in C
   $\mathcal{I} := flip(\mathcal{I}, p)$
 return *"don't know"*
end

WSAT example

0		0		1
p_1	$\vee$	$\neg p_2$	$\vee$	p_3
		$\neg p_2$	$\vee$	$\neg p_3$
$\neg p_1$			$\vee$	$\neg p_3$
$\neg p_1$	$\vee$	p_2		
p_1	$\vee$	p_2		

WSAT example

0		0		1	
p_1	$\vee$	$\neg p_2$	$\vee$	p_3	✓
		$\neg p_2$	$\vee$	$\neg p_3$	✓
$\neg p_1$			$\vee$	$\neg p_3$	✓
$\neg p_1$	$\vee$	p_2			✓
p_1	$\vee$	p_2			

flip no.	interpretation p_1	p_2	p_3	unsatisfied clauses	candidates for flipping	flipped variable
1	0	0	1			

WSAT example

0		0		1
p_1	$\vee$	$\neg p_2$	$\vee$	p_3
		$\neg p_2$	$\vee$	$\neg p_3$
$\neg p_1$			$\vee$	$\neg p_3$
$\neg p_1$	$\vee$	p_2		
p_1	$\vee$	p_2		

flip no.	interpretation			unsatisfied clauses	candidates for flipping	flipped variable
	p_1	p_2	p_3			
1	0	0	1	$p_1 \vee p_2$	p_1, p_2	

WSAT example

1		0		1	
p_1	$\vee$	$\neg p_2$	$\vee$	p_3	✓
		$\neg p_2$	$\vee$	$\neg p_3$	✓
$\neg p_1$			$\vee$	$\neg p_3$	
$\neg p_1$	$\vee$	p_2			
p_1	$\vee$	p_2			✓

flip no.	interpretation			unsatisfied clauses	candidates for flipping	flipped variable
	p_1	p_2	p_3			
1	0	0	1	$p_1 \vee p_2$	p_1, p_2	p_1
2	1	0	1			

WSAT example

1		0		1
p_1	$\vee$	$\neg p_2$	$\vee$	p_3
		$\neg p_2$	$\vee$	$\neg p_3$
$\neg p_1$			$\vee$	$\neg p_3$
$\neg p_1$	$\vee$	p_2		
p_1	$\vee$	p_2		

flip no.	interpretation			unsatisfied clauses	candidates for flipping	flipped variable
	p_1	p_2	p_3			
1	0	0	1	$p_1 \vee p_2$	p_1, p_2	p_1
2	1	0	1	$\neg p_1 \vee \neg p_3$ $\neg p_1 \vee p_2$	p_1, p_2, p_3	

WSAT example

1		1		1	
p_1	$\vee$	$\neg p_2$	$\vee$	p_3	✓
		$\neg p_2$	$\vee$	$\neg p_3$	
$\neg p_1$			$\vee$	$\neg p_3$	
$\neg p_1$	$\vee$	p_2			✓
p_1	$\vee$	p_2			✓

flip no.	interpretation			unsatisfied clauses	candidates for flipping	flipped variable
	p_1	p_2	p_3			
1	0	0	1	$p_1 \vee p_2$	p_1, p_2	p_1
2	1	0	1	$\neg p_1 \vee \neg p_3$ $\neg p_1 \vee p_2$	p_1, p_2, p_3	p_2
3	1	1	1			

WSAT example

1		1		1
p_1	$\vee$	$\neg p_2$	$\vee$	p_3
		$\neg p_2$	$\vee$	$\neg p_3$
$\neg p_1$			$\vee$	$\neg p_3$
$\neg p_1$	$\vee$	p_2		
p_1	$\vee$	p_2		

flip no.	interpretation			unsatisfied clauses	candidates for flipping	flipped variable
	p_1	p_2	p_3			
1	0	0	1	$p_1 \vee p_2$	p_1, p_2	p_1
2	1	0	1	$\neg p_1 \vee \neg p_3$ $\neg p_1 \vee p_2$	p_1, p_2, p_3	p_2
3	1	1	1	$\neg p_2 \vee \neg p_3$ $\neg p_1 \vee \neg p_3$	p_1, p_2, p_3	

WSAT example

1		1		0
p_1	$\vee$	$\neg p_2$	$\vee$	p_3
		$\neg p_2$	$\vee$	$\neg p_3$
$\neg p_1$			$\vee$	$\neg p_3$
$\neg p_1$	$\vee$	p_2		
p_1	$\vee$	p_2		

flip no.	interpretation p_1	p_2	p_3	unsatisfied clauses	candidates for flipping	flipped variable
1	0	0	1	$p_1 \vee p_2$	p_1, p_2	p_1
2	1	0	1	$\neg p_1 \vee \neg p_3$ $\neg p_1 \vee p_2$	p_1, p_2, p_3	p_2
3	1	1	1	$\neg p_2 \vee \neg p_3$ $\neg p_1 \vee \neg p_3$	p_1, p_2, p_3	p_3
	1	1	0			

WSAT example

1		1		0	
p_1	$\vee$	$\neg p_2$	$\vee$	p_3	
		$\neg p_2$	$\vee$	$\neg p_3$	✓
$\neg p_1$			$\vee$	$\neg p_3$	✓
$\neg p_1$	$\vee$	p_2			✓
p_1	$\vee$	p_2			✓

flip no.	interpretation			unsatisfied clauses	candidates for flipping	flipped variable
	p_1	p_2	p_3			
1	0	0	1	$p_1 \vee p_2$	p_1, p_2	p_1
2	1	0	1	$\neg p_1 \vee \neg p_3$ $\neg p_1 \vee p_2$	p_1, p_2, p_3	p_2
3	1	1	1	$\neg p_2 \vee \neg p_3$ $\neg p_1 \vee \neg p_3$	p_1, p_2, p_3	p_3
	1	1	0			