CS:4350 Logic in Computer Science

Inference Systems for Propositional Logic

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Credits

Part of these slides are based on Chap. 1 of *Logic in Computer Science* by M. Huth and M. Ryan, Cambridge University Press, 2nd edition, 2004.

Outline

Inference Systems for Propositional Logic

Semantic consequence/entailment

Derivability

Natural deduction

Soundness and completeness of natural deduction

- £, the language, is a class of sentences described by a formal grammar
- S, the semantics, is a formal specification for assigning meaning to sentences in L
- R, the inference system, is
 a set of axioms and inference rules to infer (i.e., generate
 sentences of £ from given sentences of £

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Propositional logic, formally

Propositional logic is a triple $(\mathcal{L}, \mathcal{S}, \mathcal{R})$ where

- \mathcal{L} is the set of all formulas built from Boolean variables and the propositional connectives $(\neg, \land, \lor, \ldots)$
- S is provided by interpretations of the variables as 0, 1 and the connectives as certain Boolean functions
- R is ??

There are many inference systems for PL We will study a few of them

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Formal properties of inference systems

A formal system is defined by a set of inference rules that allow us to generate formulas from given formulas

We will focus on these properties of our inference systems:

Soundness Every inferred formula is a semantic consequence of the given ones

Completeness Only semantic consequences are inferable

Termination Only finitely many inferences are needed to prove or disprove semantic consequence

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Given

- a set $U = \{A_1, \dots, A_n\}$ of formulas and
- a formula B

we write

$${A_1,\ldots,A_n} \models B$$

iff every interpretation that satisfies all formulas in U satisfies B too

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 $U \models B$ is read as B is a semantic/logical consequence of U, or B logically follows from U, or U entails B

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 $U \models A$ formally captures the notion of a fact A following from assumptions U

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Note 2: Do not confuse this use \models with that in $\mathcal{I} \models \mathcal{B}$ where \mathcal{I} is an interpretation

$$\begin{array}{lll} \{p\} & \models & p \lor q \\ \{p, p \to q\} & \models & q \\ \{p, q\} & \models & p \land q \\ \{\} & \models & r \to r \\ \{p, \neg r\} & \models & (p \lor q) \land (q \lor \neg r) \\ \{q\} & \models & (p \lor q) \land (q \lor \neg r) \\ \{p, q \lor \neg r\} & \not\models & p \land q \\ \{p \lor \neg p\} & \not\models & p \end{array}$$

	р	q	r	$\neg r$	$p \rightarrow q$	$p \lor q$	$p \wedge q$	$r \rightarrow r$	$q \vee \neg r$	$(p \lor q) \land (q \lor \neg r)$
1.	0	0	0	1	1	0	0	1	1	1
2.	0	0	1	0	1	0	0	1	0	1
3.	0	1	0	1	1	1	0	1	1	1
4.	0	1	1	0	1	1	0	1	1	1
5.	1	0	0	1	0	1	0	1	1	1
6.	1	0	1	0	0	1	0	1	0	0
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Exercise

Determine which of the following entailments hold

- $U \models A$ for all $A \in U$ (inclusion)
- if $U \models A$ then $V \models A$ for all $V \supseteq U$ (monotonicity)
- A is valid iff $\emptyset \models A$ (also written as $\models A$)
- A is unsatisfiable iff $A \models \bot$
- $U \models A$ iff $U \cup \{\neg A\}$ is unsatisfiable
- $\{A_1, \ldots, A_n\} \models B \text{ iff } \{A_1, \ldots, A_{n-1}\} \models A_n \rightarrow B \quad (deduction)$
- $\{A_1,\ldots,A_n\}\models B$ iff $\{A_1\wedge\cdots\wedge A_n\}\models B$ iff $\emptyset\models (A_1\wedge\cdots\wedge A_n)\to B$
- $A \equiv B$ iff $\{A\} \models B$ and $\{B\} \models A$

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Given

- a set $U = \{A_1, \dots, A_n\}$ of formulas (*premises*) and
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U \vdash_{l} A is read as U derives B in l, or B derives from U in l, or B is derivable from U in l
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We write just $U \vdash A$ when I is clear from context

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Ideally, I should be such that $U \vdash_I A$ if $U \models A$

All these symbols!

Note:

- $A \wedge B \rightarrow C$ is a formula, a sequence of symbols manipulated by an inference system /
- $A \wedge B \models C$ is a mathematical abbreviation for the statement: "every interpretation that satisfies $A \wedge B$, also satisfies C"
- $A \wedge B \vdash_{l} C$ is a mathematical abbreviation for the statement: "I derives C from $A \wedge B$ "

All these symbols!

In other words,

- $\bullet \ \to$ is a symbol of propositional logic, processed by inference systems
- |= denotes a relation from sets of formulas to formulas, based on their meaning in propositional logic
- ⊢_I denotes a relation from sets of formulas to formulas, based on their derivability in I

Implication vs. Entailment

The connective \rightarrow and the relation \models are related as follows:

$$A \rightarrow B$$
 is valid iff $A \models B$

Example: $p \to (p \lor q)$ is valid and $p \models (p \lor q)$

Implication vs. Entailment

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Example: $p \to (p \lor q)$ is valid and $p \models (p \lor q)$

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Soundness and completeness

The relations \models and \vdash are related as by these two properties of inference systems \vdash

Soundness / is sound if it can derive from any set *U* of formulas only formulas entailed by *U*:

if
$$U \vdash_{l} A$$
 then $U \models A$

Completeness / is *complete* if it can derive from any set *U* of formulas all formulas entailed by *U*:

if
$$U \models A$$
 then $U \vdash_i A$

Soundness and completeness

The relations \models and \vdash are related as by these two properties of inference systems \vdash

Soundness / is *sound* if it can derive from any set U of formulas only formulas entailed by U:

if
$$U \vdash_I A$$
 then $U \models A$

Completeness / is *complete* if it can derive from any set *U* of formulas all formulas entailed by *U*:

if
$$U \models A$$
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if
$$U \models A$$
 then $U \vdash_i A$

Natural deduction

There are many inference systems for propositional logic

Natural deduction is a family of inference systems with inference rules designed to mimic the way people reason deductively

Note

- "Natural" here is meant in contraposition to "mechanical / automated"
- Other inference systems for PL are more machine-oriented and so arguably not as natural for people
- Natural deduction is actually automatable but less conveniently than other, more machine-oriented inference systems

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$$\frac{A \quad B}{A \wedge B} \wedge i \qquad \qquad \frac{A \wedge B}{A} \wedge e_1$$

$$\frac{A \wedge B}{A} \wedge e_1$$

$$\frac{A \wedge B}{B} \wedge e_2$$

$$\frac{A \quad B}{A \wedge B} \wedge i \qquad \frac{A \wedge B}{A} \wedge e_1 \qquad \frac{A \wedge B}{B} \wedge e_2$$

$$\frac{A \wedge B}{A} \wedge e_1$$

$$\frac{A \wedge B}{B} \wedge e_2$$

Given: A set U of formulas Usage

 \wedge i: for any two formulas A and B in U, add $A \wedge B$ to U

 $\wedge e_1$: for any formula of the form $A \wedge B$ in U, add A to U

 $\triangle e_2$: for any formula of the form $A \triangle B$ in U, add A to U

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$p \wedge q, r \vdash q \wedge r$$

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$\underbrace{p \wedge q, r}_{\text{premises}} \vdash \underbrace{q \wedge r}_{\text{conclusion}}$$

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I like cats and (like) dogs, Jill likes birds ⊢ I like dogs and Jill likes birds

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$\underbrace{p \wedge q, r}_{\text{premises}} \vdash \underbrace{q \wedge r}_{\text{conclusion}}$$

Proof

$$p \wedge q$$
 premise

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$\underbrace{p \wedge q, r}_{\text{premises}} \vdash \underbrace{q \wedge r}_{\text{conclusion}}$$

Proof

- $p \wedge q$ premise
- ₂ r premise

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$\underbrace{p \wedge q, r}_{\text{premises}} \vdash \underbrace{q \wedge r}_{\text{conclusion}}$$

Proof

- $p \wedge q$ premise
- 2 r premise
- $_3$ q $\wedge e_2$ applied to 1

$$\frac{A \quad B}{A \wedge B} \wedge i \quad \frac{A \wedge B}{A} \wedge e_1 \quad \frac{A \wedge B}{B} \wedge e_2$$

Let's prove that we can derive $q \wedge r$ from $p \wedge q$ and r, i.e., that

$$\underbrace{p \land q, r}_{\text{premises}} \vdash \underbrace{q \land r}_{\text{conclusion}}$$

Proof

- $p \wedge q$ premise
- ₂ r premise
- $_3$ q $\wedge e_2$ applied to 1
- $_4$ $q \wedge r$ $\wedge i$ applied to 3, 2

Proof tree

$$\frac{p \wedge q}{q} \wedge e_2 r \wedge i$$

$$\frac{A}{\neg \neg A}$$
 $\neg \neg i$ $\frac{\neg \neg A}{A}$ $\neg \neg e$

Example Prove
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

$$\frac{A}{\neg \neg A} \neg \neg i \qquad \frac{\neg \neg A}{A} \neg \neg e$$

Example Prove $p, \neg \neg (q \land r) \vdash \neg \neg p \land r$

$$\frac{A}{\neg \neg A}$$
 $\neg \neg i$ $\frac{\neg \neg A}{A}$ $\neg \neg e$

Example Prove
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

premise
$$p \rightarrow q(q \land r)$$
 premise

$$\frac{A}{\neg \neg A}$$
 $\neg \neg i$ $\frac{\neg \neg A}{A}$ $\neg \neg e$

Example Prove
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

premise
$$p \rightarrow q \wedge r$$
 premise

$$_3$$
 $q \wedge r$ $\neg \neg e 2$

$$\frac{A}{\neg \neg A}$$
 $\neg \neg i$ $\frac{\neg \neg A}{A}$ $\neg \neg e$

Example Prove
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

$$\frac{A}{\neg \neg A}$$
 $\neg \neg i$ $\frac{\neg \neg A}{A}$ $\neg \neg e$

Example Prove
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

premise
$$\begin{array}{ccc}
 & p & \text{premise} \\
 & \neg \neg (q \land r) & \text{premise} \\
 & q \land r & \neg \neg e 2 \\
 & r & \land e 3 \\
 & 5 & \neg \neg p & \neg \neg i 1
\end{array}$$

$$\frac{A}{\neg \neg A}$$
 $\neg \neg i$ $\frac{\neg \neg A}{A}$ $\neg \neg e$

Example Prove
$$p, \neg \neg (q \land r) \vdash \neg \neg p \land r$$

\rightarrow elimination

$$\frac{A \qquad A \to B}{B} \to e$$

\rightarrow elimination

$$\frac{A \qquad A \to B}{B} \to e$$

Example Prove $p, p \rightarrow q, q \rightarrow r \vdash r$

ightarrow elimination

$$\frac{A \qquad A \to B}{B} \to e$$

Example Prove $p, p \rightarrow q, q \rightarrow r \vdash r$

$$p$$
 premise $p \rightarrow q$ premise $p \rightarrow q$ premise $p \rightarrow q$ premise

\rightarrow elimination

$$\frac{A \qquad A \to B}{B} \to e$$

Example Prove $p, p \rightarrow q, q \rightarrow r \vdash r$

$$\begin{array}{cccc} & p & & \text{premise} \\ & p \rightarrow q & \text{premise} \\ & q \rightarrow r & \text{premise} \\ & q & \rightarrow \text{e 1,2} \end{array}$$

ightarrow elimination

$$\frac{A \qquad A \to B}{B} \to e$$

Example Prove $p, p \rightarrow q, q \rightarrow r \vdash r$

\rightarrow elimination

$$\frac{A \quad A \to B}{B} \to e$$

$$\frac{A \qquad A \to B}{B} \to e \qquad \qquad \frac{A \to B \qquad \neg B}{\neg A} \text{ MT}$$

\rightarrow elimination

$$\frac{A \longrightarrow A \rightarrow B}{B} \rightarrow e$$

$$\frac{A \qquad A \to B}{B} \to e \qquad \qquad \frac{A \to B \qquad \neg B}{\neg A} \text{ MT}$$

- \rightarrow e is also known as *Modus Ponens*
- MT is known as Modus Tollens





Example Prove $p
ightarrow q dash
eg q
ightarrow \neg q
ightarrow \neg p$





Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$



$$\frac{\begin{bmatrix} A \\ \vdots \\ B \end{bmatrix}}{A \to B} \to i$$

Example Prove
$$p \rightarrow q \vdash \neg q \rightarrow \neg p$$

 $p \rightarrow q$ premise

$$\frac{A \to B \quad \neg B}{\neg A} \quad \mathsf{MT}$$

$$\begin{array}{c}
A \\
\vdots \\
B
\end{array}$$

$$A \to B \to i$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

- $\begin{array}{ccc} & p \rightarrow q & & \text{premise} \\ & \neg q & & \text{assumption} \end{array}$

$$\frac{A \to B \quad \neg B}{\neg A} \quad \mathsf{MT}$$

$$\begin{array}{c|c}
A \\
\vdots \\
B
\end{array}$$

$$A \to B \to i$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

- $_{\scriptscriptstyle 1}$ p o q premise



$$\begin{array}{c|c}
A \\
\vdots \\
B
\end{array}$$

$$A \to B \to i$$

Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

 $p \rightarrow q$ premise $q \rightarrow q$ assumption $q \rightarrow q$ MT 1,2

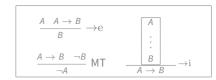




Example Prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$

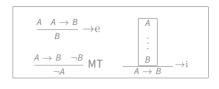
1	$p \rightarrow q$	premise	
2	$\neg q$	assumption	
3	$\neg p$	MT 1,2	
4	$\neg q \rightarrow \neg p$	→i 2-3	

Prove $\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$

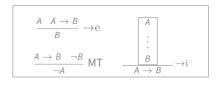


Prove
$$\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$

$$_{1} \quad q \rightarrow r$$



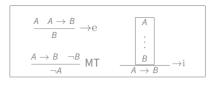
assumption



assumption assumption

Prove
$$\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$

- $_1$ $q \rightarrow r$
- $_2$ $\neg q \rightarrow \neg p$
- 3 P



assumption

assumption

assumption

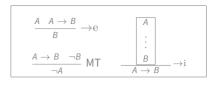
Prove
$$\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$

$$_1$$
 $q \rightarrow r$

$$_2$$
 $\neg q \rightarrow \neg p$

3 P

$$_{4}$$
 $\neg\neg p$



assumption

assumption

assumption

¬¬і з

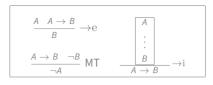
Prove
$$\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$

$$_1$$
 $q \rightarrow r$

$$_2$$
 $\neg q \rightarrow \neg p$

3 p

$$_{4}$$
 $\neg\neg p$



assumption

assumption

assumption

 $\neg \neg i \ 3$

MT 2,4

Prove
$$\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$

$$\begin{array}{ccc}
 & q \rightarrow r \\
 & \neg q \rightarrow \neg p \\
 & p \\
 & \neg \neg p
\end{array}$$

$$\frac{A \xrightarrow{A \to B}}{B} \to e \qquad \qquad \begin{vmatrix} A \\ \vdots \\ B \end{vmatrix} \\ \xrightarrow{-A} MT \qquad \frac{B}{A \to B} \to i$$

assumption assumption

assumption

 $\neg \neg i$ 3

MT 2,4

¬¬e 5

Prove
$$\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$

$$\begin{array}{cccc}
1 & q \rightarrow r \\
2 & \neg q \rightarrow \neg p \\
3 & p \\
4 & \neg \neg p \\
5 & \neg \neg q \\
6 & q \\
7 & r
\end{array}$$

$$\begin{array}{c|ccccc}
 & A & A \to B \\
\hline
 & B & & & & \\
\hline
 & A & B & \neg B \\
\hline
 & \neg A & MT & & & B \\
\hline
 & & A \to B & \neg i
\end{array}$$

assumption

assumption

assumption

 $\neg \neg i \ 3$

MT 2,4

¬¬е 5

 \rightarrow e 1,6

Prove
$$\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$

 $q \rightarrow r$

assumption

Prove
$$\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$

$$\frac{A \quad A \to B}{B} \to e$$

$$\frac{A}{B} \to B \to B \to B$$

$$\frac{A}{B} \to B \to B$$

$$\frac{A}{B} \to B \to B$$

$$\frac{B}{A \to B} \to B$$

$$q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$

$$1 \quad q \rightarrow r$$

$$2 \quad \neg q \rightarrow \neg p$$

$$3 \quad p$$

$$4 \quad \neg \neg p$$

$$5 \quad \neg \neg q$$

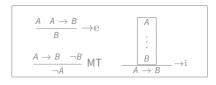
$$6 \quad q$$

$$7 \quad r$$

$$8 \quad p \rightarrow r$$

$$3 \quad \Rightarrow 1 \quad$$

Prove
$$\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$



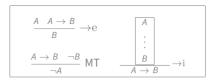
1	q o r	assumption
2	eg q o eg p	assumption
3	р	assumption
4	$\neg \neg p$	¬¬i 3
5	$\neg \neg q$	MT 2,4
6	q	¬¬e 5
7	r	→e 1,6
8	$p \rightarrow r$	→i 3-7

Prove
$$\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$

$$\frac{A \quad A \to B}{B} \to e \qquad \qquad \begin{vmatrix} A \\ \vdots \\ B \end{vmatrix} \\
\xrightarrow{-A} \qquad MT \qquad \qquad A \to B \longrightarrow i$$

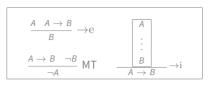
1	q o r	assumption
2	$\neg q ightarrow eg p$	assumption
3	p	assumption
4	$\neg \neg p$	¬¬і з
5	$\neg \neg q$	MT 2,4
6	q	¬¬e 5
7	r	→e 1,6
8	$p \rightarrow r$	→і 3-7
9	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	ightarrowi 2-8

Prove
$$\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$$

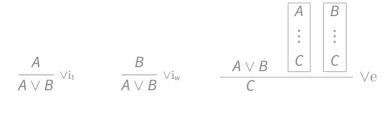


1	$q \rightarrow r$	assumption
2	$\neg q ightarrow eg p$	assumption
3	p	assumption
4	$\neg \neg p$	¬¬і з
5	$\neg \neg q$	MT 2,4
6	q	¬¬e 5
7	r	→e 1,6
8	$p \rightarrow r$	→і 3-7
9	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	→i 2-8

Prove $\vdash (q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$



1	$q \rightarrow r$	assumption
2	$\neg q ightarrow eg p$	assumption
3	р	assumption
4	$\neg \neg p$	¬¬i 3
5	$\neg \neg q$	MT 2,4
6	q	¬¬e 5
7	r	→e 1,6
8	$p \rightarrow r$	→i 3-7
9	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	→i 2-8
10	$(q \rightarrow r) \rightarrow (\neg q \rightarrow \neg p) \rightarrow (p \rightarrow q)$	→e 1-9



$$\frac{A}{A \vee B} \vee i_{1} \qquad \frac{B}{A \vee B} \vee i_{w} \qquad \frac{A \vee B}{C} \vee e$$

$$\frac{A \vee B}{C} \vee e$$

$$Ole 1 \quad \text{Prove } p \vee q \vdash q \vee p$$

$$\frac{A}{A \vee B} \vee_{i_1} \qquad \frac{B}{A \vee B} \vee_{i_w} \qquad \frac{A \vee B}{C} \stackrel{\vdots}{\cup} C \qquad \vee_{e}$$
The prove $p \vee q \vdash q \vee p$

Example 1 Prove
$$p \lor q \vdash q \lor p$$

 $p \lor q$ premise

$$\frac{A}{A \vee B} \vee_{i_1} \qquad \frac{B}{A \vee B} \vee_{i_w} \qquad \frac{A \vee B}{C} \stackrel{\vdots}{\cup} C \qquad \vee e$$

$$\frac{A}{\Box} \qquad \frac{B}{\Box} \qquad \vdots \qquad \vdots \qquad C \qquad \vee e$$

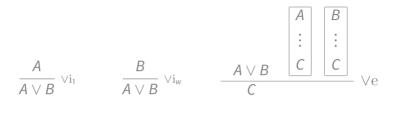
$$\frac{A \vee B}{\Box} \qquad C \qquad \vee e$$

- $p \lor q$ premise
- ₂ p assumption

$$\frac{A}{A \vee B} \vee_{i_{1}} \qquad \frac{B}{A \vee B} \vee_{i_{w}} \qquad \frac{A \vee B}{C} \stackrel{\stackrel{\scriptstyle B}{\vdots}}{C} \stackrel{\stackrel{\scriptstyle \vdots}{\vdots}}{C} \vee_{e}$$

Example 1 Prove
$$p \lor q \vdash q \lor p$$

- $p \lor q$ premise
- ₂ p assumption
- $_3$ $q \lor p \lor i_2 2$

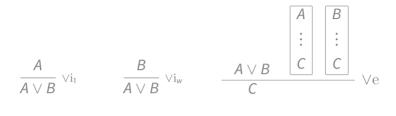


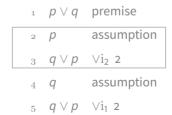
Example 1 Prove
$$p \lor q \vdash q \lor p$$

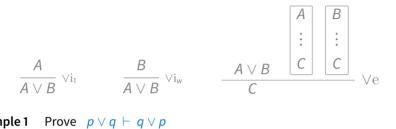
$$\begin{array}{cccc} & p \lor q & \text{premise} \\ & 2 & p & \text{assumption} \\ & _3 & q \lor p & \lor \mathbf{i_2} \ \mathbf{2} \end{array}$$

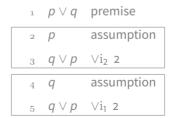
$$\frac{A}{A \vee B} \vee_{i_1} \qquad \frac{B}{A \vee B} \vee_{i_w} \qquad \frac{A \vee B}{C} \stackrel{\vdots}{\subset} C \qquad \vee_{e}$$
ple 1 Prove $p \vee q \vdash q \vee p$

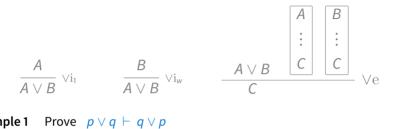
$$\begin{array}{cccc} & p \lor q & \text{premise} \\ & 2 & p & \text{assumption} \\ & 3 & q \lor p & \lor i_2 & 2 \\ & 4 & q & \text{assumption} \end{array}$$

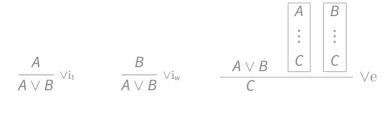


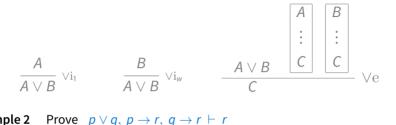












Example 2 Prove $p \lor q$, $p \to r$, $q \to r \vdash r$

$$\frac{A}{A \vee B} \vee_{i_{1}} \qquad \frac{B}{A \vee B} \vee_{i_{w}} \qquad \frac{A \vee B}{C} \stackrel{\vdots}{ } \stackrel{\vdots}{ } \stackrel{\vdots}{ } \qquad \vee e$$

$$\frac{A}{A \vee B} \vee_{i_{1}} \qquad \frac{B}{A \vee B} \vee_{i_{w}} \qquad \frac{A \vee B}{C} \vee_{i_{w}} \vee_{i_{w}}$$

Example 2 Prove
$$p \lor q, p \to r, q \to r \vdash r$$

- $p \lor q$ premise
- $_2$ $p \rightarrow r$ premise
- $g \to r$ premise

$$\frac{A}{A \vee B} \vee i_1 \qquad \frac{B}{A \vee B} \vee i_w \qquad \frac{A \vee B}{C} \stackrel{\stackrel{\textstyle (B)}{\stackrel{}{:}}}{\stackrel{\textstyle (C)}{\stackrel{}{:}}} \vee e$$

Example 2 Prove $p \lor q$, $p \to r$, $q \to r \vdash r$

$$\frac{1}{A} \perp e$$
 $\frac{A}{A} \neg A \neg A$

$$\frac{\perp}{A}$$
 \perp e $\frac{A}{\perp}$ \neg e

Example Prove
$$\neg p \lor q \vdash p \rightarrow q$$

$$\frac{\perp}{A}$$
 \pm e $\frac{A}{\perp}$ \pm e

Example Prove
$$\neg p \lor q \vdash p \rightarrow q$$

 $\neg p \lor q$ premise

$$\frac{\perp}{A}$$
 \pm e $\frac{A}{\perp}$ \pm e

Example Prove
$$\neg p \lor q \vdash p \rightarrow q$$

- $\neg p \lor q$ premise
- $_{2}$ $\neg p$ assumption

$$\frac{\perp}{A}$$
 \perp e $\frac{A}{\perp}$ \neg e

Example Prove
$$\neg p \lor q \vdash p \rightarrow q$$

- $\neg p \lor q$ premise
- $_2$ $\neg p$ assumption
- ₃ *p* assumption

$$\frac{\perp}{A}$$
 \perp e $\frac{A}{\perp}$ \neg e

Example Prove $\neg p \lor q \vdash p \rightarrow q$

$$\neg p \lor q$$
 premise

- $_2$ $\neg p$ assumption
- ₃ *p* assumption
- ₄ ⊥ ¬e 3,2

$$\frac{\perp}{A}$$
 \perp e $\frac{A}{\perp}$ \neg e

Example Prove $\neg p \lor q \vdash p \rightarrow q$

$$\neg p \lor q$$
 premise

- $_2$ $\neg p$ assumption
- ₃ p assumption
- ₄ ⊥ ¬e 3,2
- 5 *q* ⊥e 4

$$\frac{\perp}{A}$$
 \perp e $\frac{A}{\perp}$ \neg e

Example Prove
$$\neg p \lor q \vdash p \rightarrow q$$

- $\neg p \lor q$ premise
- $_2$ $\neg p$ assumption

$$\frac{\perp}{A}$$
 \perp e $\frac{A}{\perp}$ \neg e

Example Prove
$$\neg p \lor q \vdash p \rightarrow q$$

- $\neg p \lor q$ premise
- $_2$ $\neg p$ assumption
- - 6 $p \rightarrow q$ \rightarrow i 3-5

$$\frac{A}{\bot}$$
 $\neg \epsilon$

Example Prove $\neg p \lor q \vdash p \rightarrow q$

 $\neg p \lor q$ assumption

premise

2	P	assumption
3	р	assumption
4	\perp	¬е 3,2
5	9	⊥е 4
6	$p \rightarrow q$	→i 3-5

q assumption

 $_{1}$ $\neg p \lor q$

$$\frac{\perp}{\Lambda}$$
 \perp e

$$\frac{A}{\bot}$$
 $\neg \epsilon$

Example Prove $\neg p \lor q \vdash p \rightarrow q$

2	$\neg p$	assumption
3	р	assumption
4	\perp	¬e 3,2
5	9	⊥е 4
6	$p \rightarrow q$	→і 3-5

q assumptionp assumption

premise

 $_{1}$ $\neg p \lor q$

$$\frac{\perp}{\Lambda}$$
 \perp e

Example Prove $\neg p \lor q \vdash p \rightarrow q$

n
n

premise

q assumptionp assumptionq copy 2

 $\neg p \lor q$

$$\frac{\perp}{A}$$
 \perp e $\frac{A}{\Box}$

Example Prove $\neg p \lor q \vdash p \rightarrow q$

6 $p \rightarrow q \rightarrow i$ 3-5

q assumptionp assumptionq copy 2

premise

$$\frac{\bot}{A}$$
 \text{ Le } $\frac{A}{\bot}$ \text{ ¬A} \text{ ¬e}

Example Prove $\neg p \lor q \vdash p \rightarrow q$

$$1 \neg p \lor q$$

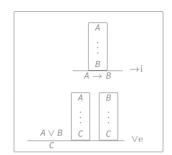
2	P	assumption
3	p	assumption
4	\perp	¬е 3,2
5	q	⊥e 4
6	$p \rightarrow q$	→i 3-5

q	assumption
p	assumption
q	сору 2
$p \rightarrow q$	→i 3-4

premise

$$\frac{\perp}{\Delta}$$
 \perp e

$$\frac{A}{---}$$
 ¬e



Example Prove $\neg p \lor q \vdash p \rightarrow q$

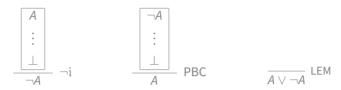
$$\neg p \lor q$$

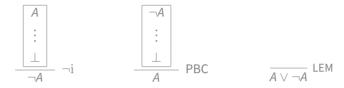
premise

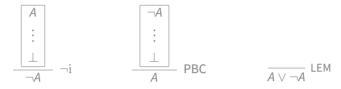
2	$\neg p$	assumption
3	р	assumption
4	\perp	¬е з,2
5	q	⊥е 4
6	$p \rightarrow q$	→i 3-5

9	assumption
p	assumption
9	сору 2
$p \rightarrow q$	ightarrowi 3-4

∨e 1, 2-6

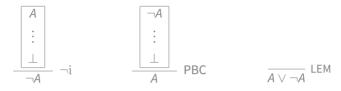






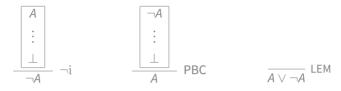
Example 1 Prove
$$p \rightarrow q$$
, $p \rightarrow \neg q \vdash \neg p$

- p o q premise
- $p \rightarrow \neg q$ premise



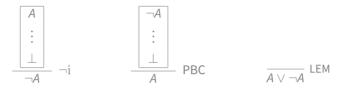
Example 1 Prove
$$p \rightarrow q$$
, $p \rightarrow \neg q \vdash \neg p$

- $p \rightarrow q$ premise
- $_{2}$ $p
 ightarrow \neg q$ premise
- ₃ *p* assumption

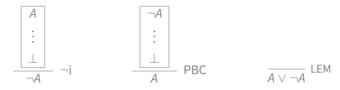


$$p \rightarrow q$$
 premise $p \rightarrow \neg q$ premise

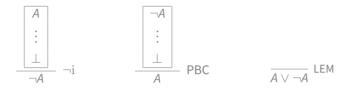
$$\begin{array}{ccc} _3 & p & & \text{assumption} \\ _4 & q & & \rightarrow \text{e 1, 3} \end{array}$$



$$p \rightarrow q$$
 premise
 $p \rightarrow \neg q$ premise
 $p \rightarrow \neg q$ premise
 $p \rightarrow q$ assumption
 $p \rightarrow q$ premise
 $p \rightarrow q$ premise
 $p \rightarrow q$ premise
 $p \rightarrow q$ premise



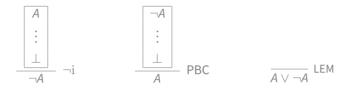
$$p \rightarrow \neg q \qquad p$$
 $1 \quad p \rightarrow q \qquad premise$
 $2 \quad p \rightarrow \neg q \qquad premise$
 $3 \quad p \qquad assumption$
 $4 \quad q \qquad \rightarrow e \quad 1, 3$
 $5 \quad \neg q \qquad \rightarrow e \quad 2, 3$
 $6 \quad \bot \qquad \neg e \quad 4, 5$



$$_{\scriptscriptstyle 1}$$
 $p o q$ premise

$$_{2}$$
 $p \rightarrow \neg q$ premise

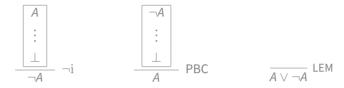
3 p	assumption
4 q	ightarrowe 1,3
$_{5}$ $\neg q$	ightarrowe 2,3
6	¬e 4,5

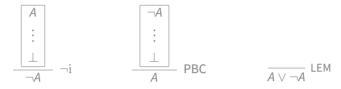


Example 1 Prove $p \rightarrow q$, $p \rightarrow \neg q \vdash \neg p$

 $\begin{array}{cccc} & p \rightarrow q & \text{premise} \\ & p \rightarrow \neg q & \text{premise} \\ & & p & \text{assumption} \\ & & q & \rightarrow \text{e 1, 3} \\ & & 5 & \neg q & \rightarrow \text{e 2, 3} \\ & & 6 & \bot & \neg \text{e 4, 5} \\ \end{array}$

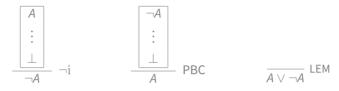
¬i 2-4



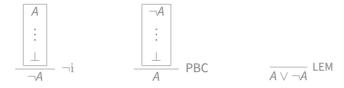


Example 2 Prove $\neg p \rightarrow \bot \vdash p$

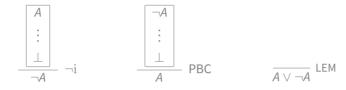
 $\neg p \rightarrow \bot$ premise

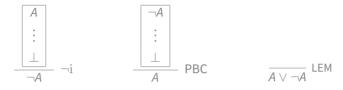


- $\neg p \rightarrow \bot$ premise
- ₂ ¬p assumption

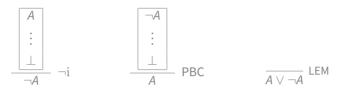


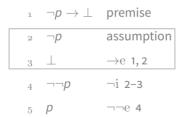
- $\neg p \rightarrow \bot$ premise
- 2 ¬p assumption
- $_3$ \perp \rightarrow e 1,2

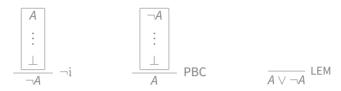




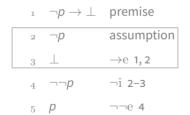
$$\begin{array}{ccc}
 & \neg p \rightarrow \bot & \text{premise} \\
\hline
 & 2 & \neg p & \text{assumption} \\
 & 3 & \bot & \rightarrow \text{e 1, 2} \\
\hline
 & 4 & \neg \neg p & \neg \text{i 2-3}
\end{array}$$



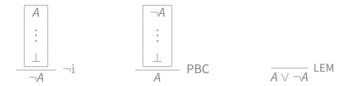




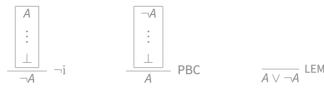
Example 2 Prove $\neg p \rightarrow \bot \vdash p$



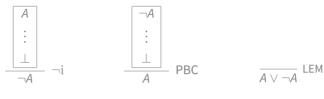
PBC can be simulated



Example 3 Prove $\vdash p \lor \neg p$



Example 3 Prove $\vdash p \lor \neg p$ $_1 \neg (p \lor \neg p)$ assumption



Example 3 Prove $\vdash p \lor \neg p$ $_1 \neg (p \lor \neg p)$ assumption

₂ p assumption





Example 3 Prove
$$\vdash p \lor \neg p$$
 1 $\neg (p \lor \neg p)$ assumption

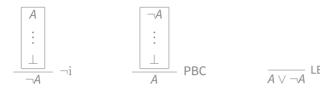
$$\neg (p \lor \neg p)$$

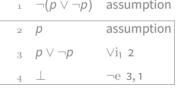


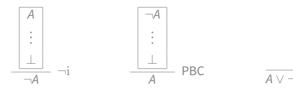
Example 3 Prove
$$\vdash p \lor \neg p$$
 1 $\neg (p \lor \neg p)$ assumption

$$\neg (p \lor \neg p)$$
 ass

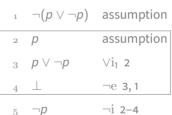
$$_3$$
 $p \lor \neg p$ $\lor i_1$

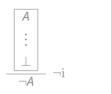






Example 3 Prove $\vdash p \lor \neg p$ $_1 \neg (p \lor \neg p)$ assumption

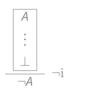




$$\neg (p \lor \neg p)$$
 assumption

2	р	assumption
3	$p \vee \neg p$	$\vee i_1$ 2
4	上	¬e 3,1
	-n	_i 2_4

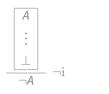
$$_{5}$$
 $\neg p$ $\neg 1$ 2-



Example 3 Prove
$$\vdash p \lor \neg p$$
 $_1 \neg (p \lor \neg p)$ assumption

$$\neg (p \lor \neg p)$$
 assumption

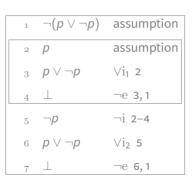
$$6 \quad p \lor \neg p \qquad \lor i_2 5$$
 $7 \quad \bot \qquad \neg e 6, 1$

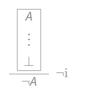




LEM

Example 3 Prove
$$\vdash p \lor \neg p$$

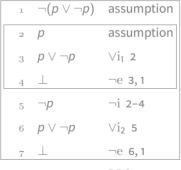






LEN

Example 3 Prove $\vdash p \lor \neg p$







LEN

Example 3 Prove
$$\vdash p \lor \neg p$$

LFM can be simulated too

 $\neg(p \lor \neg p)$ assumption

2 *p*

assumption

3 DV-

 $\forall i_1$ 2

4 -

¬e 3,1

5 7

¬i 2-4

6 p

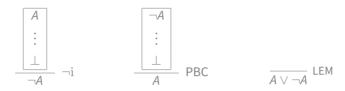
√i₂ 5

7 -

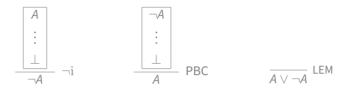
¬e 6,1

8 $p \vee \neg p$

PBC 7



PBC and LEM are derived rules



PBC and LEM are derived rules

MT and $\neg \neg i$ are derived rules too

Soundness of natural deduction

We will prove a crucial property of natural deduction:

any formula A derived from a set U of premises is a logical consequence of U

Theorem 1 (Soundness)

For all formulas A_1, \ldots, A_n and A such that $A_1, \ldots, A_n \vdash A$, we have that

 $A_1,\ldots,A_n\models A$.

For the proof of the theorm, we will rely on this lemma

Lemma 2

For all formulas A_1, \ldots, A_n , A and B,

- 1. $A_1, \ldots, A_n, A \models B \text{ iff } A_1, \ldots, A_n \models A \rightarrow B$
- 2. $A_1, \ldots, A_n, \neg B \models \bot iff A_1, \ldots, A_n \models B$

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2.
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The proof of Theorem 1 is by induction on proof length

The *length* of a natural deduction proof is the number of lines in it

Let P be the a proof of $A_1, \ldots, A_n \vdash A$, seen as a sequence of formulas

Assume, without loss of generality, that A is the last formula in the sequence

By induction on the length *l* of *P*.

(l = 1)

Then $A = A_i$ for some $i \in \{1, \dots, n\}$. Trivially, $A_1, \dots, A_n \models A_{j,1} = A_{j,1} = A_{j,2} = A_{j,3} = A_{j,4} = A_{j,4}$

Proof of Theorem 1.

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Proof of Theorem 1.

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By induction on the length l of P.

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By induction on the length l of P.

$$(l = 1)$$

Then $A = A_i$ for some $i \in \{1, ..., n\}$. Trivially, $A_1, ..., A_n \models A_i$.

(l > 1)

Assume by induction that the theorem holds for all proofs of length l' < l.

The proof depends on the final rule used to derive A.

 (Ae_1) If A was derived by Ae_1 , then P looks like.

$$A_1$$
 premise $A \wedge B \dots$ $A \wedge A \wedge B \dots$

for some formula B

Note that the subsequence of P from A_1 to $A \wedge B$ is a proof of $A \wedge B$ of length < l.

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$$A_1$$
 premise \vdots $A \wedge B$... \vdots $A \wedge e_1$

for some formula B.

Note that the subsequence of P from A_1 to $A \wedge B$ is a proof of $A \wedge B$ of length $A_2 \wedge B$. Then, by inductive hypothesis, $A_1, \ldots, A_n \models A \wedge B$. Hence, $A_1, \ldots, A_n \models A$.

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$$A_1$$
 premise \vdots $A \wedge B$... \vdots $A \wedge e_1$

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$$A_1$$
 premise \vdots $A \wedge B$... \vdots $A \wedge e_1$

for some formula B.

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(∧i)

(\wedge i) Then *A* has the form $B_1 \wedge B_2$

(\wedge i) Then A has the form $B_1 \wedge B_2$ and P looks like:

A_1	premise		A_1	premise
:			:	
B_1			B_2	
:		or	:	
B_2			B_1	
:			:	
$B_1 \wedge B_2$	$\wedge i$		$B_1 \wedge B_2$	$\wedge i$

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A_1	premise		A_1	premise
:			:	
B_1			B_2	
:		or	:	
B_2			B_1	
:			:	
$B_1 \wedge B_2$	$\wedge i$		$B_1 \wedge B_2$	$\wedge i$

This implies that P contains a (shorter) proof of B_1 and of B_2 .

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A_1	premise		A_1	premise
:			:	
B_1			B_2	
:		or	:	
B_2			B_1	
:			:	
$B_1 \wedge B_2$	$\wedge i$		$B_1 \wedge B_2$	$\wedge i$

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Then, by inductive hypothesis, $A_1, \ldots, A_n \models B_i$ for i = 1, 2.

(\wedge i) Then *A* has the form $B_1 \wedge B_2$ and *P* looks like:

A_1	premise		A_1	premise
:			:	
B_1			B_2	
:		or	:	
B_2			B_1	
:			:	
$B_1 \wedge B_2$	$\wedge i$		$B_1 \wedge B_2$	$\wedge i$

This implies that P contains a (shorter) proof of B_1 and of B_2 .

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Hence, $A_1, \ldots, A_n \models B_1 \wedge B_2$.

(→i)

 $(\rightarrow i)$ Then A has the form $B_1 \rightarrow B_2$ and

 $(\rightarrow i)$ Then A has the form $B_1 \rightarrow B_2$ and

P looks like:

```
A_1 premise A_2 :
```

6
$$B_1 \rightarrow B_2 \rightarrow i$$

 $(\rightarrow i)$ Then A has the form $B_1 \rightarrow B_2$ and

P looks like:

5	B_2	
4	:	
3	B ₁	assumption
2	:	
1	A_1	premise

but then

30 / 39

 $(\rightarrow i)$ Then A has the form $B_1 \rightarrow B_2$ and

P looks like:	1	A_1	premise	but then	1	A_1	premise
	2	:			2	:	
	3	B ₁	assumption		3	B_1	premise
	4	:			4	:	
	5	B_2			5	B_2	
	6	$B_1 \rightarrow B_2$	\rightarrow i				

is a proof of B_2 from A_1, \ldots, A_n, B_1 that is shorter than P.

 $(\rightarrow i)$ Then A has the form $B_1 \rightarrow B_2$ and

₽ looks like:	1	A_1	premise	but then	1	A_1	premise
	2	:			2	:	
	3	B ₁	assumption		3	B_1	premise
	4	:			4	:	
	5	B_2			5	B_2	
	6	$B_1 \rightarrow B_2$	\rightarrow i				

is a proof of B_2 from A_1, \ldots, A_n, B_1 that is shorter than P.

Then, by inductive hypothesis, $A_1, \ldots, A_n, B_1 \models B_2$.

 $(\rightarrow i)$ Then A has the form $B_1 \rightarrow B_2$ and

P looks like:	1	A_1	premise	but then	1	A_1	premise
	2	:			2	:	
	3	B ₁	assumption		3	B_1	premise
	4	:			4	:	
	5	B ₂			5	B_2	
	6	$B_1 \rightarrow B_2$	\rightarrow i				

is a proof of B_2 from A_1, \ldots, A_n, B_1 that is shorter than P.

Then, by inductive hypothesis, $A_1, \ldots, A_n, B_1 \models B_2$.

It follows from Lemma 2(1) that $A_1, \ldots, A_n \models B_1 \rightarrow B_2$.

(¬i)

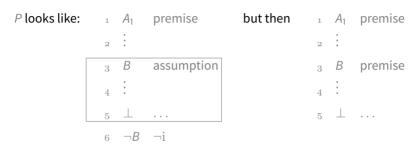
 $(\neg i)$ Then A has the form $\neg B$ and

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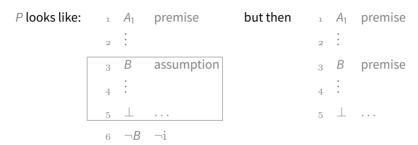
but then

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is a proof of \perp from A_1, \ldots, A_n, B that is shorter than P.

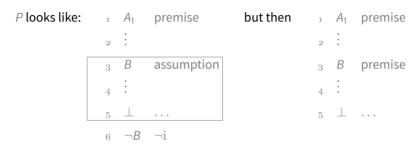
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Then, by inductive hypothesis, $A_1, \ldots, A_n, B \models \bot$.

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Then, by inductive hypothesis, $A_1, \ldots, A_n, B \models \bot$.

It follows from Lemma 2 that $A_1, \ldots, A_n \models \neg B$.

```
(\wedge i_2) Analogous to \wedge i_2 case.
(\vee i_1) Exercise.
(\vee i_1) Exercise.
(∨e) Exercise.
(\rightarrow e) Exercise.
(\neg e) Exercise.
(\perp e) Exercise.
(\neg \neg e) Exercise.
```

We will now prove another important property of natural deduction:

any logical consequence A of a set U of formulas has a proof with premises U

Assumption: We remove op from the language and simulate it with $pee \neg p$

Theorem 3 (Completeness

For all formulas A_1, \ldots, A_n and A such that $A_1, \ldots, A_n \models A$, we have that

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Lemma 4

For all formulas A_1, \ldots, A_n and A the following holds:

1.
$$A_1, A_2, \ldots, A_n \models A \text{ implies } \models A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$$
.

2.
$$\vdash A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$$
 implies $A_1, A_2, \dots, A_n \vdash A$.

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For all formulas A_1, \ldots, A_n and A the following holds:

- 1. $A_1, A_2, \ldots, A_n \models A \text{ implies } \models A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$.
- **2.** $\vdash A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$ implies $A_1, A_2, \dots, A_n \vdash A$.

Proof.

By induction on n in both cases (see Huth & Ryan).

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Theorem 5 (Completeness for validity)

All valid formulas B are provable in natural deduction: if \models B then \vdash B.

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Theorem 5 (Completeness for validity)

All valid formulas B are provable in natural deduction: if \models B then \vdash B.

Proof of Theorem 3
$$(A_1, \ldots, A_n \models A \text{ implies } A_1, \ldots, A_n \vdash A)$$
.

Assume $A_1, \ldots, A_n \models A$, prove $A_1, A_2, \ldots, A_n \vdash A$.

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For all formulas A_1, \ldots, A_n and A the following holds:

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Proof of Theorem 3 $(A_1, \ldots, A_n \models A \text{ implies } A_1, \ldots, A_n \vdash A)$.

Assume $A_1, \ldots, A_n \models A$, prove $A_1, A_2, \ldots, A_n \vdash A$.

By Lemma 4(1), \models $A_1 \to (A_2 \to (\cdots (A_n \to A) \cdots))$.

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By Theorem 5, $\vdash A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$.

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Assume $A_1, \ldots, A_n \models A$, prove $A_1, A_2, \ldots, A_n \vdash A$.

By Lemma 4(1), $\models A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$.

By Theorem 5, $\vdash A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow A) \cdots))$.

By Lemma 4(2), $A_1, A_2, ..., A_n \vdash A$.

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Theorem 5 (Completeness for validity)

All valid formulas B are provable in natural deduction: if \models B then \vdash B.

So we are left with proving Theorem 5

Lemma 6

Let A be a formula over variables p_1, \ldots, p_n with $n \ge 0$ and let \mathcal{I} be an interpretation. Let $\hat{p}_i = p$ if $\mathcal{I} \models p$ and $\hat{p}_i = \neg p$ otherwise. Then,

$$\hat{p}_1, \ldots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \ldots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A.$$

Proof of Lemma 6. By structural induction on A

(Base case)

If A is just a variable, say p_1 , then it is immediate that $p_1 \vdash p_1$ and $\neg p_1 \vdash \neg p_1$. If A is \bot then n=0 and $\mathcal{I} \not\models A$. We can prove $\neg\bot$ from no premises by \neg i.

(Inductive Step) If A is not a variable or \bot , assume the result holds for all proper subformulas of A.

We reason by cases on the form of A

Lemma 6

Let A be a formula over variables p_1, \ldots, p_n with $n \ge 0$ and let \mathcal{I} be an interpretation. Let $\hat{p}_i = p$ if $\mathcal{I} \models p$ and $\hat{p}_i = \neg p$ otherwise. Then,

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If *A* is just a variable, say p_1 , then it is immediate that $p_1 \vdash p_1$ and $\neg p_1 \vdash \neg p_1$.

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Proof of Lemma 6. ($\hat{p}_1, \dots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \dots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A$) (continued)

 $(A = \neg B)$ (that is, suppose A has the form $\neg B$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B$. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg B$.
- If I ⊭ A then I ⊨ B. By inductive hypothesis, p̂₁,...,p̂_n ⊢ B.
 Take a proof of B from p̂₁,...,p̂_n and apply ¬¬i to B.
 The resulting proof is a proof of ¬A.

Proof of Lemma 6. $(\hat{p}_1, \dots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \dots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A)$ (continued)

 $(A = \neg B)$ (that is, suppose A has the form $\neg B$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B$. By inductive hypothesis, $\hat{p}_1, \dots, \hat{p}_n \vdash \neg B$.
- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B$. By inductive hypothesis, $\hat{\rho}_1, \dots, \hat{\rho}_n \models B$. Take a proof of B from $\hat{\rho}_1, \dots, \hat{\rho}_n$ and apply $\neg\neg i$ to B. The resulting proof is a proof of $\neg A$.

Proof of Lemma 6. ($\hat{p}_1, \ldots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \ldots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A$) (continued)

 $(A = \neg B)$ (that is, suppose A has the form $\neg B$)

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B$. By inductive hypothesis, $\hat{p}_1, \dots, \hat{p}_n \vdash \neg B$.
- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B$. By inductive hypothesis, $\hat{p}_1, \dots, \hat{p}_n \vdash B$. Take a proof of B from $\hat{p}_1, \dots, \hat{p}_n$ and apply $\neg \neg i$ to B. The resulting proof is a proof of $\neg A$.

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Proof of Lemma 6. (\hat{p}_1, \dots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \dots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A) (continued)
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$$(A = B_1 \wedge B_2)$$

Proof of Lemma 6. ($\hat{p}_1, \dots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \dots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A$) (continued)

$$(A = B_1 \wedge B_2)$$

• If $\mathcal{I} \models A$ then $\mathcal{I} \models B_1$ and $\mathcal{I} \models B_2$.

By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash B_1$ and $\hat{p}_1, \ldots, \hat{p}_n \vdash B_2$.

A proof of A from $\hat{p}_1, \ldots, \hat{p}_n$ is obtained by chaining a proof of B_1 and a proof of B_2 and applying $\triangle i$ to B_1 and B_2 .

Proof of Lemma 6. $(\hat{p}_1, ..., \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, ..., \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A)$ (continued)

$$(A = B_1 \wedge B_2)$$

If \(\mu \) \(\mu \) A then \(\mu \) \(\mu \) B_k for some \(k \in \) {1, 2}. Say \(k = 1 \) (the other case is similar).
 By inductive hypothesis, \(\hat{\rho}_1, \ldots, \hat{\rho}_n \) ⊢ \(B_1. \)
 A proof of \(\neg B_1 \) can be extended to a proof of \(\neg A \) as follows:

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Proof of Lemma 6. (\hat{p}_1, \dots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \dots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A) (continued)
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$$(A = B_1 \vee B_2)$$

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Proof of Lemma 6. $(\hat{p}_1, \dots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \dots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A)$ (continued)

$$(A = B_1 \vee B_2)$$

If \(\mathcal{I} \) |= \(A\) then \(\mathcal{I} \) |= \(B_k\) for some \(k \) ∈ \(\{1,2 \) .
 A proof of \(A\) from \(\hat{p}_1, \ldots, \hat{p}_n\) is obtained from a proof of \(B_k\) by applying \(\neq i_k\) to \(B_k\) to get \(B_1 \lor B_2\).

Proof of Lemma 6. $(\hat{p}_1, ..., \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, ..., \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A)$ (continued)

$$(A = B_1 \vee B_2)$$

If \(\mathcal{I} \overline{\mathcal{B}} \) A then \(\mathcal{I} \overline{\mathcal{B}} \) B₁ and \(\mathcal{I} \overline{\mathcal{B}} \) B₂.

A proof of ¬A from \(\hat{\hat{\hat{\hat{D}}}}_1, \ldots, \hat{\hat{\hat{\hat{D}}}_n} \) is obtained by chaining a proof of ¬B₁ and a proof of ¬B₂ and continuing as follows:

Proof of Lemma 6. ($\hat{p}_1, \dots, \hat{p}_n \vdash A \text{ if } \mathcal{I} \models A \text{ and } \hat{p}_1, \dots, \hat{p}_n \vdash \neg A \text{ if } \mathcal{I} \not\models A$) (continued)

$$(A = B_1 \rightarrow B_2)$$

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B_1$ or $\mathcal{I} \models B_2$. (exercise)
- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B_1$ and $\mathcal{I} \not\models B_2$. (exercise)

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$$(A = B_1 \to B_2)$$

- If $\mathcal{I} \models A$ then $\mathcal{I} \not\models B_1$ or $\mathcal{I} \models B_2$. (exercise)
- If $\mathcal{I} \not\models A$ then $\mathcal{I} \models B_1$ and $\mathcal{I} \not\models B_2$. (exercise)

Lemma 7

Let L_2, \ldots, L_n , A be formulas and let p one of A's variables. If $p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ then $L_2, \ldots, L_n \vdash A$.

Proof of Lemma 7. $(p, L_2, \ldots, L_n \vdash A \text{ and } \neg p, L_2, \ldots, L_n \vdash A \text{ implies } L_2, \ldots, L_n \vdash A)$

Suppose we have the proofs:

1
$$p$$
 premise and 1 $\neg p$ premises 2 L_2 premise 3 \vdots 3 \vdots 4 A ...

Proof of Lemma 7. ($p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$)

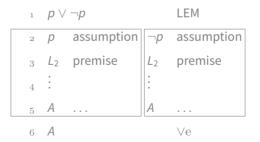
Suppose we have the proofs:

Proof of Lemma 7. ($p, L_2, \ldots, L_n \vdash A$ and $\neg p, L_2, \ldots, L_n \vdash A$ implies $L_2, \ldots, L_n \vdash A$)

Suppose we have the proofs:

Proof of Lemma 7. $(p, L_2, ..., L_n \vdash A \text{ and } \neg p, L_2, ..., L_n \vdash A \text{ implies } L_2, ..., L_n \vdash A)$

Suppose we have the proofs:



Let p_1, \ldots, p_n be all of A's variables and consider the set

$$S = \{ p_1, \neg p_1 \} \times \cdots \times \{ p_n, \neg p_n \},$$

of all tuples $(\hat{p}_1, \dots, \hat{p}_n)$ where each \hat{p}_i is either p_i or $\neg p_i$. We prove by induction on $i = 1, \dots, n+1$ that

$$\hat{p}_i, \dots, \hat{p}_n \vdash A \text{ for every } (\hat{p}_1, \dots, \hat{p}_n) \in S$$
 . (1)

The theorem then follows from Property (1) for i=n+1.1

(i=1) Property (1) holds by Lemma 6 since every $(\hat{p}_1,\ldots,\hat{p}_n)\in S$ corresponds to an interpretation of A and all interpretations satisfy A (by def. of validity).

$$(i>1)$$
 Suppose $\hat{p}_i,\ldots,\hat{p}_n\vdash A$ for all $(\hat{p}_1,\ldots,\hat{p}_n)\in S$. We prove that $\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$ for all $(\hat{p}_1,\ldots,\hat{p}_n)\in S$. Let $(\hat{p}_1,\ldots,\hat{p}_i,\hat{p}_{i+1},\ldots,\hat{p}_n),(\hat{p}_1,\ldots,-p_i,\hat{p}_{i+1},\ldots,\hat{p}_n)\in S$. By induction hypothesis, $p_i,\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$ and $-p_i,\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$ Then $\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$ by Lemma 7.

Let p_1, \ldots, p_n be all of A's variables and consider the set

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of all tuples $(\hat{p}_1, \dots, \hat{p}_n)$ where each \hat{p}_i is either p_i or $\neg p_i$.

We prove by induction on i = 1, ..., n + 1 that

$$\hat{p}_i, \dots, \hat{p}_n \vdash A \text{ for every } (\hat{p}_1, \dots, \hat{p}_n) \in S.$$
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The theorem then follows from Property (1) for i=n+1.

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The theorem then follows from Property (1) for i = n + 1.

(i=1) Property (1) holds by Lemma 6 since every $(\hat{p}_1,\ldots,\hat{p}_n)\in S$ corresponds to an interpretation of A and all interpretations satisfy A (by def. of validity).

 $\begin{array}{l} (i>1) \ {\rm Suppose} \ \hat{\rho}_i, \dots, \hat{\rho}_n \ \vdash \ A \ {\rm for \ all} \ (\hat{\rho}_1, \dots, \hat{\rho}_n) \in S. \\ {\rm We \ prove \ that} \ \hat{\rho}_{l+1}, \dots, \hat{\rho}_n \ \vdash \ A \ {\rm for \ all} \ (\hat{\rho}_1, \dots, \hat{\rho}_n) \in S. \\ {\rm Let} \ (\hat{\rho}_1, \dots, \rho_l, \hat{\rho}_{l+1}, \dots, \hat{\rho}_n), (\hat{\rho}_1, \dots, \neg p_l, \hat{\rho}_{l+1}, \dots, \hat{\rho}_n) \in S. \\ {\rm By \ induction \ hypothesis,} \ \rho_l, \hat{\rho}_{l+1}, \dots, \hat{\rho}_n \ \vdash \ A \ {\rm and} \ \neg \rho_l, \hat{\rho}_{l+1}, \dots, \hat{\rho}_n \ \vdash \ A \\ {\rm Then} \ \hat{\rho}_{l+1}, \dots, \hat{\rho}_n \ \vdash \ A \ {\rm by \ Lemma \ 7.} \end{array}$

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We prove that $\hat{p}_{i+1}, \ldots, \hat{p}_n \vdash A$ for all $(\hat{p}_1, \ldots, \hat{p}_n) \in S$. Let $(\hat{p}_1, \ldots, p_i, \hat{p}_{i+1}, \ldots, \hat{p}_n), (\hat{p}_1, \ldots, \neg p_i, \hat{p}_{i+1}, \ldots, \hat{p}_n) \in S$. By induction hypothesis, $p_i, \hat{p}_{i+1}, \ldots, \hat{p}_n \vdash A$ and $\neg p_i, \hat{p}_{i+1}, \ldots, \hat{p}_n \vdash A$ by Lemma 7.

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Let $(p_1,\ldots,p_l,p_{l+1},\ldots,p_n),(p_1,\ldots,\neg p_l,p_{l+1},\ldots,p_n)\in S.$ By induction hypothesis, $p_l,\hat{p}_{l+1},\ldots,\hat{p}_n\vdash A$ and $\neg p_l,\hat{p}_{l+1},\ldots,\hat{p}_n\vdash A$ Then $\hat{p}_{l+1},\ldots,\hat{p}_n\vdash A$ by Lemma 7.

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By induction hypothesis, p_i , \hat{p}_{i+1} , ..., $\hat{p}_n \vdash A$ and $\neg p_i$, \hat{p}_{i+1} , ..., $\hat{p}_n \vdash A$

Then $p_{i+1}, \ldots, p_n \vdash A$ by Lemma 7.

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Let $(\hat{p}_1,\ldots,p_i,\hat{p}_{i+1},\ldots,\hat{p}_n),(\hat{p}_1,\ldots,\neg p_i,\hat{p}_{i+1},\ldots,\hat{p}_n)\in S$.
By induction hypothesis, $p_i,\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$ and $\neg p_i,\hat{p}_{i+1},\ldots,\hat{p}_n\vdash A$.
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Let p_1, \ldots, p_n be all of A's variables and consider the set

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