Chapter 6
Symbolic Differentiation

This chapter presents the "method of computing" or "calculus" of derivatives by giving symbolic rules for finding formulas for derivatives when we are given formulas for the functions.

When we compute a derivative, we want to know that the increment approximation is valid. You must use high school algebra and trig, but you do not have to establish the increment approximation directly as we did in Chapter 5. The graphical and symbolic theorems of 1-variable differentiation say the following:

If we can compute the derivative $f'[x]$ of a function $f[x]$ using the rules from this chapter, then a sufficiently magnified view of the graph $y = f[x]$ appears linear at each point of the interval where the formulas for $f[x]$ and $f'[x]$ are valid.

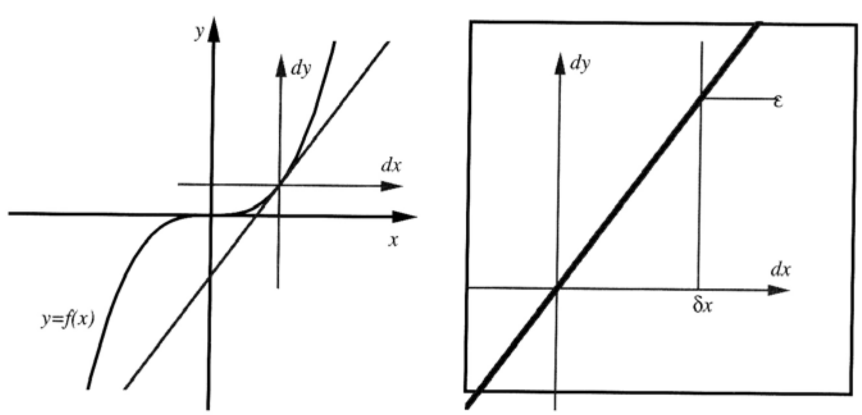

Figure 6.0:1: $y = f\,[x]$ and $dy = m \cdot dx$ through a powerful microscope

The line with local coordinates $dy = m\ dx$ "looks" the same as $y = f[x]$ under magnification when $m = f'[x]$. The slope $f'[x]$ depends on the center of magnification,

x.

This condition of "tangency" is expressed symbolically by the approximation formula that says the nonlinear change is a linear term plus something small compared to the change:

$$f[x + \delta x] - f[x] = f'[x] \cdot \delta x + \varepsilon \cdot \delta x$$

with the magnified error ε small, $\varepsilon \approx 0$, whenever the input perturbation is small, $\delta x \approx 0$, and x lies in an interval $[\alpha, \beta]$ where both $f[x]$ and $f'[x]$ are defined.

The microscope equation above expresses the nonlinear change, $f[x + dx] - f[x]$, in terms of a change dx or "local variable" dx, with x fixed. The linear term in dx is called the differential,

$$dy = f'[x] \cdot dx \qquad \text{or} \qquad dy = m \cdot dx$$

in (dx, dy) coordinates (with x fixed), where dy represents the change from $f[x]$. When $dx = \delta x \approx 0$ is small, the difference between these terms is small compared to δx because the difference is a product of a small term ε and the small change δx. On magnification by $1/\delta x$, the term $\varepsilon \cdot \delta x$ appears to be the size of ε. If this is small enough (by virtue of large enough magnification), we do not see it and the graph appears linear.

The results of this chapter ensure that the error is small whenever x lies in an interval $[\alpha, \beta]$ where both $f[x]$ and $f'[x]$ are defined. (At a fixed high magnification the graph appears straight simultaneously for every microscope at an x-focus point in $[\alpha, \beta]$.) The rules of calculus are theorems which guarantee that this approximation is valid, provided the resulting formulas are defined on intervals. This is a powerful yet practical theory. Here is a brief example of how it is used.

Example 6.1 $f'[x_0]$ *is the Slope of* $y = f[x]$

The slope of the line (in local coordinates)

$$dy = m \cdot dx$$

is m and the line points upward if $m > 0$. Because a microscopic image of the graph $y = f[x]$ cannot be distinguished from the graph of the linear equation $dy = m \cdot dx$ when $m = f'[x_0]$, the graph $y = f[x]$ is increasing at the approximate rate $f'[x_0]$ near x_0.

Example 6.2 *Using the Theory*

The theory is easy to use once you learn the rules from this chapter. Here are two examples where the theory breaks down. The breakdown is easy to detect. By the end of the chapter, you will be able to apply rules and compute the following two derivatives for

$$f[x] = \sqrt{x^2 + 2x + 1} \qquad \text{and} \qquad y = x^{\frac{2}{3}}$$

obtaining

$$f'[x] = \frac{x+1}{\sqrt{x^2+2x+1}} \qquad \text{and} \qquad \frac{dy}{dx} = \frac{2}{3\sqrt[3]{x}}$$

After computing without fear, you need to check the formulas to see that

$$f'[-1] = \frac{-1+1}{\sqrt{(-1)^2-2+1}} = \frac{0}{0} \qquad \text{and} \qquad \frac{dy}{dx} = \frac{2}{3\sqrt[3]{0}} = \frac{2}{0}$$

are undefined. When the formulas are not valid, the theory does not predict anything; but, in this case, we have seen that there is a kink in the graph of $f[x]$ at $x = -1$ (see Exercise 3.2.4). There is a vertical cusp on the graph of $y = x^{\frac{2}{3}}$ at $x = 0$ (see Problem 6.1).

The rules of this chapter guarantee that the increment approximation for tangency holds when the resulting formulas are valid on intervals. Of course, your first task now is to learn:

Example 6.3 *All the Rules of Differentiation*

There are only eight rules in this chapter and you must memorize them:

$$
\begin{aligned}
y &= x^p & &\Rightarrow & \tfrac{dy}{dx} &= p x^{p-1}, \quad p \text{ constant}\\
y &= \operatorname{Sin}[\theta] & &\Rightarrow & \tfrac{dy}{d\theta} &= \operatorname{Cos}[\theta]\\
y &= \operatorname{Cos}[\theta] & &\Rightarrow & \tfrac{dy}{d\theta} &= -\operatorname{Sin}[\theta]\\
y &= e^x & &\Rightarrow & \tfrac{dy}{dx} &= y = e^x\\
x &= \operatorname{Log}[y] & &\Rightarrow & \tfrac{dx}{dy} &= \tfrac{1}{y}
\end{aligned}
$$

$$\tfrac{d(a\, f[x] + b\, g[x])}{dx} = a\tfrac{df[x]}{dx} + b\tfrac{dg[x]}{dx}$$

$$\tfrac{d(f[x]\, g[x])}{dx} = \tfrac{df[x]}{dx} \cdot g[x] + f[x] \cdot \tfrac{dg[x]}{dx}$$

$$\tfrac{dy}{dx} = \tfrac{dy}{du} \cdot \tfrac{du}{dx}, \qquad \text{when } y = f[u] \quad \& \quad u = g[x]$$

The hard thing is to learn to combine these rules with high school algebra and trig.

6.1 Rules for Special Functions

This section gives the five specific differentiation rules for basic functions.

The algebra of exponents together with the derivatives we computed in Chapter 5 suggest a single rule that includes the examples of Exercise 5.1 and before. To understand why this rule covers the cases of roots and reciprocals, you must understand the laws of exponents in the review Chapter 28, especially Exercise 28.4.

Theorem 6.1 *The Power Rule*
For any constant p,

$$y = x^p \quad \Rightarrow \quad \frac{dy}{dx} = p\, x^{p-1}$$

In other words, functions that can be expressed as powers are locally linear with derivative as above – provided the formulas on both sides of the implication are defined on an interval.

We showed directly in the last chapter that if $y = \sqrt{x}$, then $dy = \frac{1}{2\sqrt{x}}\, dx$.

Example 6.4 $\frac{d(\sqrt{x})}{dx}$ *by Rules*

This is one special case of the Power Rule with $p = 1/2$, because

$$y = \sqrt{x} = x^{\frac{1}{2}}, \quad \text{so} \quad \frac{dy}{dx} = \frac{1}{2}\, x^{\frac{1}{2}-1} = \frac{1}{2}\, x^{-\frac{1}{2}}$$
$$= \frac{1}{2}\,\frac{1}{x^{\frac{1}{2}}} = \frac{1}{2}\,\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Notice that our final formula is only valid on the open interval $(0, \infty) = \{x : 0 < x < \infty\}$. The open interval of validity is part of the Power Rule, but you compute first and then think. Do not forget the second step.

Example 6.5 $\frac{d(1/x^2)}{dx}$ *by Rules*

This is a special case of the Power Rule with $p = -2$, because

$$y = \frac{1}{x^2} = x^{-2}, \quad \text{so} \quad \frac{dy}{dx} = -2\,x^{-2-1} = -2\,x^{-3}$$
$$= \frac{-2}{x^3}$$

Notice that our final formula is only valid on the open interval $(0, \infty) = \{x : 0 < x < \infty\}$ or the interval $(-\infty, 0)$ but not on any interval of the form $[a, b]$ with $a < 0 < b$.

Example 6.6 $\frac{d(x\sqrt{x})}{dx}$ *by Rules*

This is a special case of the Power Rule with $p = 3/2$ because

$$y = x\sqrt{x} = x^1\ x^{1/2} = x^{1+\frac{1}{2}} = x^{\frac{3}{2}}$$
$$\frac{dy}{dx} = \frac{3}{2}\,x^{\frac{3}{2}-1} = \frac{3}{2}\ x^{\frac{1}{2}}$$

Notice that our final formula is valid only for $x \geq 0$. The largest open interval where the function and derivative are defined is $(0, \infty)$.

In the last chapter we directly proved derivative formulas for the sine and cosine using a microscopic view of the circle. The angles must be measured in radians in order to compare differences in sine and cosine with length along the unit circle. Here are the formulas:

Theorem 6.2 *The Sine and Cosine Rules*
For θ in radians,

$$y = \mathrm{Sin}[\theta] \qquad \Rightarrow \qquad \frac{dy}{d\theta} = \mathrm{Cos}[\theta]$$
$$y = \mathrm{Cos}[\theta] \qquad \Rightarrow \qquad \frac{dy}{d\theta} = -\mathrm{Sin}[\theta]$$

The sine and cosine rules are valid for all real θ. This means that the increment approximation holds on $(-\infty, \infty)$.

We will postpone the proof of the exponential and log rules but include them here because they are the only other special function rules you need to learn.

Theorem 6.3 *The Log and Exponential Rules*

$$y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = y = e^x$$

$$x = \mathrm{Log}\,[y] \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{y}$$

The exponential rule is valid for all real x or, in other words, the increment approximation holds on $(-\infty, \infty)$. The natural logarithm rule makes sense only if the log and the formula for the derivative are both defined, so the increment approximation for Log$[y]$ is valid on $(0, \infty)$.

Exercise Set 6.1

1. You cannot divide by zero or take even roots of negative numbers (as real functions). Show that the Power Rule does not apply at $x = 0$ for $p = \frac{1}{3}$ or at $x = -2$ for $p = \frac{1}{4}$.

$$y = x^p = x^{1/3} \quad \Rightarrow \quad \frac{dy}{dx} = p\,x^{p-1} = ?$$

and

$$y = x^p = x^{1/4} \quad \Rightarrow \quad \frac{dy}{dx} = p\,x^{p-1} = ?$$

Use the computer to graph the two functions $y = x^{1/3} = \sqrt[3]{x}$ and $y = x^{1/4} = \sqrt[4]{x}$ for $-3 \le x \le 3$. Explain your "bad" analytical result above in terms of the graph.

We will not prove the general Power Rule Theorem now, but ask you to check it for the cases we already know in the next exercise.

2.

   (a) Show that the Power Rule agrees with all the derivatives we computed directly in and before Exercise 5.1.1 as well as those that you computed in Problem 5.1. They are the following:

$$y = x \quad \Rightarrow \quad \frac{dy}{dx} = 1 = 1x^{1-1} = x^0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-1}{x^2} = -1x^{-2}$$

$$y = x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 2x^1 = 2x^{2-1} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2}{x^3} = -2x^{-3}$$

$$y = x^3 \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2 = 3x^{3-1} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{\frac{1}{2}}$$

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}} = \frac{1}{3}x^{-\frac{2}{3}}$$

(b) Differentiate the following by first converting to power form and then applying the Power Rule. Convert your derivatives back to radical notation.

$$\begin{array}{lll} a)\quad y=\frac{1}{x^5} & b)\quad y=\sqrt[5]{x} & c)\quad y=\frac{1}{\sqrt[3]{x^2}} \\ d)\quad y=x^2\sqrt{x} & e)\quad y=x^4\sqrt{x^3} & f)\quad y=\frac{x^2}{\sqrt[3]{x^2}} \\ g)\quad y=x\sqrt[3]{x} & h)\quad y=x^5\sqrt[3]{x^2} & i)\quad y=\frac{x^2}{\sqrt[3]{x^2}} \end{array}$$

(c) The derivative of a constant function is zero. Why? The Power Rule also includes a case of this in the form,

$$y=x^0 \quad\Rightarrow\quad \frac{dy}{dx}=0=0\,x^{0-1}$$

What is the value of x^0? Is 0^0 defined?

It is important to be able to apply the differentiation rules to functions defined in terms of letters other than x. At first, it is simplest to learn the manipulations with one letter, that's true, but it is also important to move beyond that. Here is some practice:

3. Other Variables

$$\begin{array}{ll} a)\quad y=x^2\Rightarrow\frac{dy}{dx}=? & b)\quad u=\frac{1}{v^2}\Rightarrow\frac{du}{dv}=? \\ c)\quad y=\sqrt{x}\Rightarrow\frac{dy}{dx}=? & d)\quad u=\frac{1}{\sqrt{v}}\Rightarrow\frac{du}{dv}=? \\ e)\quad y=x^3\sqrt{x^3}\Rightarrow\frac{dy}{cx}=? & f)\quad u=\frac{v^2}{\sqrt[3]{v^2}}\Rightarrow\frac{du}{dv}=? \\ g)\quad y=\mathrm{Sin}\,[x]\Rightarrow\frac{dy}{dx}=? & h)\quad u\,\mathrm{Cos}\,[v]\Rightarrow\frac{du}{dv}=? \\ i)\quad y=\mathrm{Log}\,[x]\Rightarrow\frac{dy}{dx}=? & j)\quad u=e^v\Rightarrow\frac{du}{dv}=? \end{array}$$

Problem 6.1 A CUSP

*When the graph of $y=x^{2/3}$ is magnified at $x=y=0$, what do we see? The Power Rule does not apply at $x=0$. Why? Still, we can either use small increments directly or look at microscopic views for smaller and smaller values of δx. The question is th following: In an tiny microscope, do we see a "VEE" or a vertical straight line segment? (HINT: Run the animation in the computer program **Zoom**, then explain what you see analytically.)*

The Power Rule, the Sine and Cosine Rules, and the Log and Exponential Rules are the only particular function rules you need to learn. The other general rules of differentiation allow you to use these to build a host of formulas that you can differentiate. The general function combination rules take up the next three sections.

6.2 The Superposition Rule

This section shows that the sum of the derivatives is the derivative of the sum. The physical Superposition Principle says that the response to a sum of stimuli is the sum of the responses to the separate stimuli. These are closely related ideas.

Another way to express the physical Superposition Rule is

$$\text{Output[stimulus 1 + stimulus 2] = Output[stimulus 1] + Output[stimulus 2]}$$

This is a simple property that is often violated in real life. For example, the combined effect of a cup of coffee and an aspirin is not the same as the two separate effects. Systems that satisfy the Superposition Principle are often called "linear systems," because you can apply the "output" to a linear combination $af[x] + bg[x]$ or form the same linear combination of the separate outputs,

$$\mathrm{Out}[af[x] + bg[x]] = a\mathrm{Out}[f[x]] + b\mathrm{Out}[g[x]]$$

Chapter 23 and the associated projects develop important applications where physical superposition does apply and "linearity" of the derivative is at the heart of the matter. In the case of differentiation, "Output" means derivative and "stimulus" means input function.

Theorem 6.4 *The Superposition Rule (or Linearity of Differentiation)*
If $f[x]$ and $g[x]$ are smooth real functions for $\alpha < x < \beta$ and a and b are real constants, then the linear combination function $h[x] = a\,f[x] + b\,g[x]$ is also smooth for $\alpha < x < \beta$ and

$$\frac{d\,(a\,f[x] + b\,g[x])}{dx} = a\frac{df[x]}{dx} + b\frac{dg[x]}{dx}$$

In words the theorem says that the derivative of a linear combination of functions is the same linear combination of their derivatives.

Proof of Superposition for Differentiation:

The general proof of the Superposition Rule is little more than algebra. We have the Increment Formula for $f[x]$ and $g[x]$, so

$$\begin{aligned} h[x + \delta x] - h[x] &= (a\,f[x + \delta x] + b\,g[x + \delta x]) - (a\,f[x] + b\,g[x]) \\ &= a\,(f[x + \delta x] - f[x]) + b\,(g[x + \delta x] - g[x]) \\ &= a\left(f'[x]\delta x + \varepsilon_1 \delta x\right) + b\left(g'[x]\delta x + \varepsilon_2 \delta x\right) \\ &= \left(a\,f'[x] + b\,g'[x]\right)\delta x + (a\varepsilon_1 + b\varepsilon_2)\delta x \end{aligned}$$

Because $\varepsilon_1 \approx 0$ and $\varepsilon_2 \approx 0$ (by the Increment Formula for f and g), we know $(a\varepsilon_1 + b\varepsilon_2) \approx 0$, thus $h'[x] = a\,f'[x] + b\,g'[x]$ and the theorem is proved.

Example 6.7 $\frac{d(5\cdot\sqrt{x}-\pi\cdot x^2)}{dx}$

Let

$$f[x]=\sqrt{x}, \qquad a=5, \qquad g[x]=x^2, \qquad b=-\pi$$

then

$$a\,f[x]+b\,g[x]=5\cdot\sqrt{x}-\pi\cdot x^2$$

and the derivatives of the pieces are $\frac{df}{dx}=\frac{1}{2\sqrt{x}}$ and $\frac{dg}{dx}=2x$, so

$$\boxed{\frac{d(af[x]+bg[x])}{dx}=a\frac{df[x]}{dx}+b\frac{dg[x]}{dx}}$$

$$\frac{d(5\cdot\sqrt{x}-\pi\cdot x^2)}{dx}=5\cdot\frac{1}{2\sqrt{x}}-\pi\cdot 2x$$

A shortcut way to write this computation is

$$\begin{aligned}\frac{d(5\sqrt{x}-\pi\cdot x^2)}{dx}&=5\frac{d(\sqrt{x})}{dx}-\pi\frac{d(x^2)}{dx}\\&=5\frac{d(x^{1/2})}{dx}-\pi\frac{d(x^2)}{dx}\\&=5\cdot\frac{1}{2}\cdot x^{\frac{1}{2}-1}-\pi\cdot 2\cdot x^{2-1}=\frac{5}{2}x^{-\frac{1}{2}}-2\pi\,x\\&=\frac{5}{2}\frac{1}{x^{1/2}}-2\pi\,x\\&=\frac{5}{2\sqrt{x}}-2\pi\,x\end{aligned}$$

Example 6.8 *The Constant Multiple Rule,* $\frac{d(a\,f[x])}{dx}=a\frac{df}{dx}[x]$

We may take $b=0$ and $g[x]=0$ in the Superposition Rule. If $y=\frac{e^\pi}{\sqrt[3]{x}}$, let $a=e^\pi$ and $f[x]=\frac{1}{x^{1/3}}$, $b=g[x]=0$, so

$$\boxed{\frac{d(a\,f[x])}{dx}=a\frac{df[x]}{dx}}$$

$$\begin{aligned}\frac{d(\frac{e^\pi}{\sqrt[3]{x}})}{dx} &= e^\pi \frac{d(\frac{1}{\sqrt[3]{x}})}{dx} \quad = e^\pi \frac{d(\frac{1}{x^{1/3}})}{dx} = e^\pi \frac{d(x^{-1/3})}{dx} \\ &= e^\pi \frac{-1}{3} x^{-\frac{1}{3}-1} = e^\pi \frac{-1}{3} x^{-\frac{1}{3}-\frac{3}{3}} = e^\pi \frac{-1}{3} x^{-\frac{4}{3}} \\ &= -\frac{e^\pi}{3} \frac{1}{x^{\frac{4}{3}}} = -\frac{e^\pi}{3} \frac{1}{\sqrt[3]{x^4}}\end{aligned}$$

Notice that e^π is a constant, $e^\pi = (2.71828\cdots)^{(3.14159\cdots)} \approx 23.1407$

6.2.1 Symbolic Differentiation with the Computer

The computer can solve all the symbolic differentiation exercises in this chapter. At first this fact might discourage you, but it should not. We want you to learn to use the rules of differentiation well enough to be confident that you understand them. The computer cannot think or understand the meaning of the result of these computations, and the input syntax needed to make it solve the exercises is troublesome itself.

WARNING: If you do NOT learn to compute derivatives without the computer, you probably will not understand the rules well enough to succeed in calculus, even with a computer. Combining your basic ability with the computer will lead to greater success than used to be possible, because once you understand the rules, the computer can become a mental "lever." It can do complicated symbolic computations for you with great reliability. You are then left with the important and interesting job of formulating the problem, programming it into the computer, and interpreting the result.

You are also welcome to use the computer to check all of your work from this chapter. Use the built-in computer differentiation command (in **DfDx**) once you have learned all the differentiation rules. The **DiffRules** program is only intended to show you what cannot be done without some of the rules. Knowing what cannot be done is part of understanding the strength of the general functional rules.

Exercise Set 6.2

1. Basic Superposition Drill
Find $\frac{dy}{dx}$ for each of the following functions $y = y[x]$. (The letters a, b, c, and h denote constants or parameters, and e is the natural base for logs and exponentials.)

a) $y = 7x^4$

b) $y = -5\,x^2$

c) $y = 7x + x^4$

d) $y = 8\,x^3 - 5\,x^2 + 4\,x + 1$

e) $y = \frac{7}{x^4}$

f) $y = -5\sqrt[3]{x}$

g) $y = 7\sqrt[5]{x} - 3\,\frac{1}{x^4}$

h) $y = 7\,\frac{\sqrt{x}}{\sqrt[3]{x}} - 5\,\frac{1}{\sqrt{x}}$

i) $y = 7\,e^x - 3\,\mathrm{Log}[x]$

j) $y = 7\,\frac{1}{x^e} + 3\,\mathrm{Log}[1/x]$

k) $y = e\sqrt[5]{x} - \pi\,\frac{1}{x^4}$

l) $y = 3\,\mathrm{Sin}[x] - \mathrm{Cos}[7e]$

m) $y = f[x] = a + b\,x + c\,x^2 + h\,x^3$

n) $y = f[x] = 7\ \mathrm{Cos}[x] - 3\,\mathrm{Sin}[x]$

The proof of "linearity of differentiation" is easy provided you understand the function notation. Unfortunately, the notation gives many students trouble at first. The next exercise helps you understand the general notation by working a specific example. One payoff to understanding the function formulas is that you will be able to write better computer programs because much of modern computing uses function notation.

2. Let $f[x] = \sqrt{x}$, let $g[x] = x^2$, let $a = 5$, and let $b = \pi$. Write out each step of the proof of the Superposition Rule that appears above, except write the steps with these specific functions and constants.

It is important, especially in applications, to be able to apply the differentiation rules to functions defined in terms of letters other than x.

3. Other Variables

a) $v = u^2 + 2\,u + 2 \quad \Rightarrow \quad \frac{dv}{du} = ?$

b) $u = 4\,\sqrt{v} - \pi\,\frac{1}{v^2} \quad \Rightarrow \quad \frac{du}{dv} = ?$

c) $y = 3\,u\,\sqrt{u} + 5/u^2 \quad \Rightarrow \quad \frac{dy}{du} = ?$

d) $u = v^{1/4} - \frac{1}{v^{1/3}} \quad \Rightarrow \quad \frac{du}{dv} = ?$

e) $y = \mathrm{Cos}[\theta] - \mathrm{Sin}[\theta] \quad \Rightarrow \quad \frac{dy}{d\theta} = ?$

f) $u = \mathrm{Log}[v] - e^v \quad \Rightarrow \quad \frac{du}{dv} = ?$

The computer program **DiffRules** contains exercises to show you how the rules of differentiation can be used to build a the computer program for differentiation. The computer program **DfDx** shows you how to use the computer's built-in differentiation.

4. Differentiation by Computer
   Run the **DiffRules** program and work through it line by line so that you can see how the addition of each of the Superposition Rule, the Product Rule, and the Chain Rule makes symbolic differentiation more powerful. We cannot differentiate $\sqrt{x}\,\mathrm{Cos}[x]$ without the Product Rule, and neither can the computer.

Problem 6.2

Using only the Superposition Rule and derivatives of sine, cosine, natural log and exponential; find the following or write brief explanations why they cannot be done this way. The letters a, b, c, and m denote parameters (or unknown constants).

a) $y = mx + b, y' = ?$

b) $y = uv + w, \frac{dy}{dv} = ?$

c) $y = uv + w, \frac{dy}{dw} = ?$

d) $f[x] = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}, f'[x] = ?$

e) $f[x] = a + bx + cx^2 + mx^3, f'[x] = ?$

f) $u = 3\,\mathrm{Sin}(\theta) - \mathrm{Cos}(\theta), \frac{du}{d\theta} = ?$

g) $y = \frac{1}{\sqrt[3]{x^5}}, \frac{dy}{dx} = ?$

h) $y = \sqrt{x} + \mathrm{Sin}[x], \frac{dy}{dx} = ?$

i) $\sqrt{x}\,\mathrm{Sin}[x], \frac{dy}{dx} = ?$

j) $y = \sqrt{\mathrm{Sin}[x]}, \frac{dy}{dx} = ?$

k) $y = \mathrm{Sin}(\sqrt{x}), \frac{dy}{dx} = ?$

l) $y = \sqrt{x}, f[x] = \frac{dy}{dx}, f'[x] = ?$

m) $y = e^x + \mathrm{Log}[x], \frac{dy}{dx} = ?$

n) $y = e^x\ \mathrm{Sin}[x], \frac{dy}{dx} = ?$

What is the slope of each of the above graphs when the independent variable equals -1? 0? (What are the tricks in this question?)

Once you have the Product Rule, you will be able to do some additional parts of the previous problem and when you also have the Chain Rule, you will be able to do all the parts. The computer program **DiffRules** can be used to check your work. Only enter the specific function rules and the Superposition Rule. If the computer cannot find the derivative with these rules, it will return your question as its output.

The next problem shows two practical ways that superposition of derivatives arise. You only need to link those obvious applications with the symbolic expressions to solve the problems.

Problem 6.3

Express the conditions in the following scenarios in terms of three functions yielding positions as a function of time and the time derivatives of these functions. Write the general function rules that yield the answer to the questions, and verify that the mathematical rules agree with the intuitively obvious answers.

1. (a) *A man and a woman are riding on a train that is travelling at the rate of 75 mph. Inside the train, the woman is walking forward at the rate of 4 mph and the man is walking backward at the rate of 3 mph. How fast is the man traveling relative to the ground? How fast is the woman traveling relative to the ground?*

   Let $T[t]$ equal the distance (in miles) that the train has traveled along the ground. Let $m[t]$ equal the distance the man has traveled forward on the train. Let $w[t]$ equal the distance the woman has traveled forward on the train. How much are $\frac{dT}{dt}$, $\frac{dm}{dt}$, and $\frac{dw}{dt}$, including sign? What does the function $f[t] = T[t] + m[t]$ represent? What is $\frac{df}{dt}$? How does this compare to $\frac{dT}{dt} + \frac{dm}{dt}$? What are the similar constructions for the woman?

   (b) *A U.S. tourist is driving in Canada at 90 kilometers per hour, but her odometer and speedometer read in the archaic English units of her home country. We want to see the functional relationship between English and metric measurements of speed and distance. Let the odometer reading be the numerical function $E[t]$ = distance traveled in miles, where t = time in hours. The distance traveled in kilometers is a function $M[t]$. There are approximately 1.609 kilometers in a mile. Express E in terms of a constant and M. Express $\frac{dE}{dt}$ in terms of this same constant and $\frac{dM}{dt}$. What is her speed in miles per hour?*

Problem 6.4 SPHERICAL SHELL

The formula for the volume of a sphere is $V[r] = \pi\frac{4}{3}r^3$ and for the surface area is $S[r] = \pi 4r^2$. Compute the derivative $\frac{dV}{dr}$ and explain its connection with $S[r]$ by considering small changes in the sphere.

6.3 The Product Rule

The derivative of a product is NOT the product of the derivatives.

Let m, n, b, and c denote unknown constants or parameters. We cannot differentiate the product $y = (mx + b)(nx + c)$ directly using the rules we have so far. However, we could do some algebra,

$$(mx+b)(nx+c) = mnx^2 + mcx + bnx + bc = mnx^2 + (mc+nb)x + bc$$

so,

$$\frac{dy}{dx} = 2mnx + mc + nb$$

To verify that this agrees with the Product Rule given below, take $f[x] = mx+b$ and $g[x] = nx+c$. The derivatives are

$$\frac{df}{dx}[x] = \frac{d(mx+b)}{dx} = m \quad \& \quad \frac{dg}{dx}[x] = \frac{d(nx+c)}{dx} = n$$

The formula from the Product Rule below gives
This is the same answer because

$$m(nx+c) + (mx+b)n = mnx + mc + mnx + nb = 2mnx + mc + nb$$

Algebra can rearrange products of many power functions and linear combinations of power functions into forms we can differentiate, but often it is easier to use the Product Rule. A product like $\sqrt{x}\,\mathrm{Cos}[x]$ really requires a new rule.

Theorem 6.5 *The Product Rule*
If $f[x]$ and $g[x]$ are smooth for $\alpha < x < \beta$, then the product $h[x] = f[x] \cdot g[x]$ is also locally linear for $\alpha < x < \beta$ and

$$\frac{d(f[x]\,g[x])}{dx} = \frac{df[x]}{dx} \cdot g[x] + f[x] \cdot \frac{dg[x]}{dx}$$

In words, the Product Rule says,"Differentiate the terms of a product one at a time, multiply by the other undisturbed term, and add these together."

Proof of the Product Rule

The proof of the product rule is another straightforward computation with tiny increments. All we do is add and subtract a term.

$$\begin{aligned}
h[x+\delta x] - h[x] &= f[x+\delta x]g[x+\delta x] - f[x]g[x] \\
&= f[x+\delta x]g[x+\delta x] - f[x]g[x+\delta x] + f[x]g[x+\delta x] - f[x]g[x] \\
&= (f[x+\delta x] - f[x])\, g[x+\delta x] + f[x]\,(g[x+\delta x] - g[x]) \\
&= \left(f'[x]\delta x + \varepsilon_1 \delta x\right) g[x+\delta x] + f[x]\left(g'[x]\delta x + \varepsilon_2 \delta x\right) \\
&= \left(f'[x]\delta x + \varepsilon_1 \delta x\right)(g[x] + \varepsilon_3) + f[x]\left(g'[x]\delta x + \varepsilon_2 \delta x\right) \\
&= f'[x]g[x]\delta x + f[x]g'[x]\delta x + \delta x \cdot (\varepsilon_4) \\
&= \left(f'[x]g[x] + f[x]g'[x]\right) \cdot \delta x + \varepsilon_4 \cdot \delta
\end{aligned}$$

Exercise SmallTerms below asks you to show that ε_3 and ε_4 are small. You can do this by simple estimates.

Example 6.9 $\frac{d(\sqrt{x}\,\mathrm{Cos}[x])}{dx}$

Let $f[x] = \sqrt{x}$ and $g[x] = \mathrm{Cos}[x]$, then $\frac{df}{dx} = \frac{1}{2\sqrt{x}}$ and $\frac{dg}{dx} = -\mathrm{Sin}[x]$, so

$$\boxed{\frac{d(f[x]\cdot g[x])}{dx} = \frac{df[x]}{dx}\cdot g[x] + f[x]\cdot\frac{dg[x]}{dx}}$$

$$\frac{d(\sqrt{x}\,\mathrm{Cos}[x])}{dx} = \frac{1}{2\sqrt{x}}\cdot\mathrm{Cos}[x] - \sqrt{x}\cdot\mathrm{Sin}[x]$$

We can also combine the Superposition Rule and Product Rule.

Example 6.10 $\frac{d((\frac{a}{\sqrt{x}} - b\ \mathrm{Log}[x])(e^x - c))}{dx}$

Let $f[x] = \frac{a}{\sqrt{x}} - b\ \mathrm{Log}[x]$ and $g[x] = e^x - c$, for constants a, b, and c. Then superposition says

$$\boxed{\begin{array}{c}\frac{df}{dx} = a\frac{d\left(\frac{1}{\sqrt{x}}\right)}{dx} - b\frac{d(\mathrm{Log}[x])}{dx} = a\frac{d\left(x^{-1/2}\right)}{dx} - b\frac{d\,\mathrm{Log}[x]}{dx} \\ = -\frac{a}{2}x^{-3/2} - \frac{b}{x} = -\frac{1}{x}\left(\frac{a}{\sqrt{x}} + b\right)\end{array}}$$

and

$$\frac{dg}{dx} = \frac{d\left(e^x\right)}{dx} - \frac{dc}{dx} = e^x + 0 = e^x$$

so

$$\boxed{\frac{d(f[x]\cdot g[x])}{dx} = \frac{df[x]}{dx}\cdot g\,[x] + f\,[x]\cdot\frac{dg[x]}{dx}}$$

$$\frac{d(f[x]\,g[x])}{dx} = \left(-\frac{1}{x}\left(\frac{a}{\sqrt{x}} + b\right)\right)\cdot(e^x - c) + \left(\frac{a}{\sqrt{x}} - b\ \mathrm{Log}[x]\right)(e^x)$$

We cannot differentiate $y = \mathrm{Sin}[2\,x]$ with the rules we have developed so far. Even the simple expression $2\,x$ inside the sine makes this problem outside our present rules.

Example 6.11 *An Impossible Problem Made Possible*

We can use the addition formula for sine to show that $\text{Sin}[2\,x] = 2\ \text{Sin}[x] \times \text{Cos}[x]$,

$$\text{Sin}[\alpha + \beta] = \text{Sin}[\alpha]\,\text{Cos}[\beta] + \text{Sin}[\beta]\,\text{Cos}[\alpha]$$

$$\text{Sin}[x + x] = \text{Sin}[x]\,\text{Cos}[x] + \text{Sin}[x]\,\text{Cos}[x]$$

$$\text{Sin}[2\,x] = 2\ \text{Sin}[x]\,\text{Cos}[x]$$

This form of the expression can be differentiated using the Product Rule as follows:

Example 6.12 $\frac{d(\text{Sin}[x]\,\text{Cos}[x])}{dx}$

Let $f[x] = \text{Sin}[x]$ and $g[x] = \text{Cos}[x]$, so $\frac{df}{dx} = \text{Cos}[x]$ and $\frac{dg}{dx} = -\,\text{Sin}[x]$, and

$$\boxed{\frac{d(f[x]\cdot g[x])}{dx} = \frac{df[x]}{dx}\cdot g[x] + f[x]\cdot\frac{dg[x]}{dx}}$$

$$\frac{d(\text{Sin}[x]\,\text{Cos}[x])}{dx} = \text{Cos}[x]\cdot\text{Cos}[x] - \text{Sin}[x]\cdot\text{Sin}[x]$$

Together, the two examples mean that

$$\begin{aligned}\frac{d(\text{Sin}[2x])}{dx} &= 2\frac{d(\text{Sin}[x]\,\text{Cos}[x])}{dx}\\ &= 2\left(\text{Cos}[x]^2 - \text{Sin}[x]^2\right)\\ &= 2\,\text{Cos}[2x]\end{aligned}$$

by the addition formula for cosine, $\text{Cos}[\alpha + \beta] = \text{Cos}[\alpha]\,\text{Cos}[\beta] - \text{Sin}[\alpha]\,\text{Sin}[\beta]$. We can do this directly with the Chain Rule below.

6.3.1 The Microscope Approximation and Rules of Differentiation

The "microscope equation" defining the differentiability (Definition derivable) of a function $f[x]$,

$$\boxed{f[x+\delta x] = f[x] + f'[x]\cdot\delta x + \varepsilon\cdot\delta x}$$

with $\varepsilon \approx 0$ if $\delta x \approx 0$, is similar to a functional identity in that it involves an unknown function $f[x]$ and its related unknown derivative function $f'[x]$. If we plug in the "input" function $f[x] = x^2$ into this equation, the output is $f'[x] = 2x$. If we plug in the "input" function $f[x] = \mathrm{Log}[x]$, the output is $f'[x] = \frac{1}{x}$.

The microscope equation involves unknown functions, but, strictly speaking, it is not a functional identity because of the error term ε (or the limit that can be used to formalize the error). It is only an approximate identity.

The various "differentiation rules," the Superposition Rule, the Product Rule, and the Chain Rule are functional identities relating functions and their derivatives. For example, the Product Rule states

$$\boxed{\frac{d(f[x]\cdot g[x])}{dx} = \frac{df[x]}{dx}\cdot g[x] + f[x]\cdot\frac{dg[x]}{dx}}$$

We urge you to write out this identity with general functions each time you use the Product Rule in a differentiation computation.

We can think of $f[x]$ and $g[x]$ as "variables" that vary by simply choosing different functions for $f[x]$ and $g[x]$. Then the Product Rule yields an identity by "plugging in" the choices of $f[x]$, $g[x]$, and their derivatives.

Example 6.13 *More Examples of the Product Rule*

Choosing $f[x] = x^2$ and $g[x] = \mathrm{Log}[x]$ and plugging into the Product Rule yields

$$\boxed{\frac{d(f[x]\cdot g[x])}{dx} = \frac{df[x]}{dx}\cdot g[x] + f[x]\cdot\frac{dg[x]}{dx}}$$

$$\frac{d(x^2\,\mathrm{Log}[x])}{dx} = 2x\,\mathrm{Log}[x] + x^2\frac{1}{x}$$

Choosing $f[x] = x^3$ and $g[x] = \mathrm{Exp}[x]$ and plugging into the Product Rule yields

$$\boxed{\frac{d(f[x]\cdot g[x])}{dx} = \frac{df[x]}{dx}\cdot g[x] + f[x]\cdot\frac{dg[x]}{dx}}$$

$$\frac{d(x^3 \operatorname{Exp}[x])}{dx} = 3x^2 \operatorname{Exp}[x] + x^3 \operatorname{Exp}[x]$$

Example 6.14 *A Half-General Rule*

If we choose $f[x] = x^5$ but do not make a specific choice for $g[x]$, plugging into the Product Rule will yield

$$\boxed{\frac{d(f[x]\cdot g[x])}{dx} = \frac{df[x]}{dx} \cdot g[x] + f[x] \cdot \frac{dg[x]}{dx}}$$

$$\frac{d(x^5 g[x])}{dx} = 5x^4 g[x] + x^5 \frac{dg[x]}{dx}$$

You need to know the Product Rule as a functional identity, not just learn shortcut computation methods that use it. Look at the **DiffRules** program to see a "practical" use of the identity as a computer program command.

Exercise Set 6.3

1. Drill on Products
   Break each of the following expressions into a product of two terms and apply the Product Rule to find $\frac{dy}{dx}$ for each of the following functions $y = y[x]$. (The letters a, b, c, and h denote constants or parameters, and e is the natural base for logs and exponentials.)

   a) $y = (2\,x^3 - 4)(3\,x + 5)$

   b) $y = (x^3 - 2\,x^2 + 4)(5\,x + 6)$

   c) $y = (\frac{2}{x^3} - 4)(3\sqrt{x} + 5)$

   d) $y = (x^3 - 2\,x^2 + 3\,x + 4)\sqrt{x}$

   e) $y = (\frac{3}{x^2} - \frac{4}{\sqrt{x}})(5\sqrt{x} + \frac{3}{x^5})$

   f) $y = (x + \frac{1}{\sqrt{x}})(\sqrt{x} + \frac{1}{x})$

   g) $y = \frac{\operatorname{Log}[x]}{\sqrt[5]{x}}$

   h) $y = (e^x - 3)(\frac{1}{x^4})$

   i) $y = x^2\,e^x$

   j) $y = 7\,e^x\,\operatorname{Cos}[x]$

   k) $y = (a + b\,x)(c\,x^2 + h\,x^3)$

   l) $y = 3\operatorname{Sin}[x] \cdot \operatorname{Cos}[7e]$

   m) $y = \left(x^{1/2} + x^{1/3} + x^{1/4}\right)\left(x^{-2} + x^{-3} + x^{-4}\right)$

   n) $y = \left(x^2 + x^3 + x^4\right)\left(x^{-1/2} + x^{-1/3} + x^{-1/4}\right)$

2. Which parts of Problem superdrill can you do now using the Product Rule that you could not do without it? (Again, you can check your work with the **DiffRules** program by entering the rules up to the Product Rule but not entering the Chain Rule.)

3. Let $f[x]$ and $g[x]$ be unknown functions that satisfy $f[1] = 2$, $\frac{df}{dx}[1] = 3$, $g[1] = -3$, $\frac{dg}{dx}[1] = 4$. Let $h[x] = f[x]g[x]$. Compute $\frac{dh}{dx}[1]$.

4. Show that the derivative of $\mathrm{Sin}[\theta] \times \mathrm{Cos}[\theta]$ is $\mathrm{Cos}[2\theta]$ using the addition formula for cosine. (You can look that formula up in Chapter 28 on the CD.)

   Combine this fact with the previous example to show that

$$\frac{d(\mathrm{Sin}[2\,x])}{dx} = \frac{d(2\ \mathrm{Sin}[x]\ \mathrm{Cos}[x])}{dx} = 2\ \mathrm{Cos}[2\,x]$$

   without using the Chain Rule! (We will be able to verify this more simply once we have the Chain Rule. The point is that there are often several ways to apply the rules of algebra and trig in combination with the rules of differentiation.)

   It is important to be able to apply the differentiation rules to functions defined in terms of letters other than x.

5. Other Variables

   (a) $v = e^u\ \mathrm{Cos}[u] \quad \Rightarrow \quad \frac{dv}{du} = ?$

   (b) $v = (u^3 + 3\,u + 3)(\mathrm{Sin}[u] + \mathrm{Cos}[u]) \quad \Rightarrow \quad \frac{dv}{du} = ?$

   (c) $u = (4\sqrt{v} - \pi\,\frac{1}{v^2})(v^{1/4} - \frac{1}{v^{1/3}}) \quad \Rightarrow \quad \frac{du}{dv} =?$

   (d) $y = (\mathrm{Cos}[v] - \mathrm{Sin}[v])(\mathrm{Log}[v] - e^v) \quad \Rightarrow \quad \frac{dy}{dv} = ?$

Problem 6.5

1. Verify that $\varepsilon_3 \approx 0$ and $\varepsilon_4 \approx 0$ in the proof of the product rule above.

2. Let $f[x] = x^2$ and $g[x] = x^3$ and write out the steps of the proof of the product rule given above for these specific functions.

Problem 6.6

The "rule" that says the derivative of a product is the product of the derivatives might make things simpler, but it is false. Show this by finding a counterexample from among functions you know how to differentiate directly from increment computations (such as simple powers of x.) For example, let $f[x] = x^2$ *and* $g[x] = x$*, so* $\frac{df}{dx} = 2x$ *and* $\frac{dg}{dx} = 1$*. How much is* $\frac{df}{dx} \times \frac{dg}{dx}$*? How much is* $\frac{d(f \cdot g)}{dx} = \frac{d(x^3)}{dx}$*? How much is* $\frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$*?*

Explain why the derivative of a product is not necessarily the product of the derivatives.

The function notation in the general proof of the Product Rule may seem obscure, but the idea is not. The next problem shows you why.

Problem 6.7 A CONCRETE INSTANCE OF THE PRODUCT RULE

1. (a) A contractor's crew is making forms to lay a rectangular concrete floor. One member of the crew measures the length l (feet) and makes an error Δl (feet), and another measures the width w (feet) and makes a separate error Δw (feet). The contracted area is $A = lw$, but the actual area is $A + \Delta A$ where the error in area is $\Delta A = \_$? (Write a formula in terms of l, w, Δl and Δw. See the Figure 6.3:2 for help.)

Figure 6.3:2: Three error rectangles

   (b) Check your formula with a numerical example. Suppose that the floor has design dimensions of 20 by 30 feet, the error in length is 2 inches, and the error in width is 1 inch - both too long. How much too large is the floor? Is the error larger or smaller than a desktop?

   (c) With the same dimensions as in the previous part and same error in length, suppose the error in width is 2 inches too short. Use $\Delta w = -2/12$ in your formula and verify that it gives the correct result. Is the floor too large or too small? What is the error in area?

(d) Suppose that the length and width are changing with time (instead of a measurement error), so $\frac{\Delta l}{\Delta t}$ and $\frac{\Delta w}{\Delta t}$ are the rates of change of length and width during the time increment Δt. (Imagine the floor expanding with a change in temperature.) Show that

$$\frac{\Delta A}{\Delta t} = \frac{\Delta w}{\Delta t} \cdot l + w \cdot \frac{\Delta l}{\Delta t} + \frac{\Delta w}{\Delta t} \cdot \frac{\Delta l}{\Delta t} \cdot \Delta t$$

What is the Product Rule for $A = l[t]\, w[t]$? How do these expressions differ when Δt is very small?

The Scientific Project on the Expanding Economy shows another way that the Product Rule arises in everyday discussions.

6.4 The Chain Rule

The Chain Rule for derivatives shows how to differentiate functions that are "hooked together in a chain." Mathematically this occurs in expressions like $y = \mathrm{Sin}[x^2 + 1]$, with "chain" $u = x^2 + 1$ "linked to" $y = \mathrm{Sin}[u]$.

If we let $u = x^2 + 1$ and let $y = \mathrm{Sin}[u]$, then the original formula, $y = \mathrm{Sin}[x^2 + 1]$ is what we would get if we start with x, compute u, and then use that answer for u to compute y.

$$u = x^2 + 1 \quad \to \quad y = \mathrm{Sin}[u]$$

The functional notation for this "chaining" if $y = f[u]$ and $u = g[x]$ is

$$y = f[g[x]] = \mathrm{Sin}[x^2 + 1]$$

Of course, we could have broken the final formula down in other ways. For example,

$$v = x^2 \quad \to \quad y = \mathrm{Sin}[v + 1]$$

or $y = f[g[x]]$, where $f[v] = \mathrm{Sin}[v+1]$ and $v = g[x] = x^2$. The advantage of the first decomposition is that we can differentiate each of the component pieces with rules already at our disposal,

$$\frac{du}{dx} = 2x \;\;\&\;\; \frac{dy}{du} = \mathrm{Cos}[u]$$

Notice the importance of being able to differentiate with respect to a letter other than x. This is one reason that we emphasized "other letters" in the early sections of the chapter.

The Chain Rule given next tells us how to use the derivatives of the components to find the derivative of the whole composition. In this case,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \mathrm{Cos}[u] \cdot 2x \\ &= \mathrm{Cos}[x^2+1] \cdot 2x \\ &= 2x\,\mathrm{Cos}[x^2+1]\end{aligned}$$

We removed the "link" variable u in our final expression for $\frac{dy}{dx}$ because we introduced it only to help solve the problem. (In some applications, the link variables actually have important separate meanings.)

Theorem 6.6 *The Chain Rule*
If $y = f[u]$ is smooth on the range of $u = g[x]$ and g is smooth for $\alpha < x < \beta$, then the chained composition $y = h[x] = f[g[x]]$ is smooth for $\alpha < x < \beta$ and $h'[x] = f'[g[x]] \cdot g'[x]$ or

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Proof of the Chain Rule

The general proof is only a little more complicated use of the increment approximation.

$$\begin{aligned}f[g[x+\delta x]] - f[g[x]] &= f[g[x] + (g'[x]\delta x + \varepsilon_1 \cdot \delta x)] - f[g[x]] \\ &= f[g[x] + L_1\delta x] - f[g[x]] \\ &= f'[g[x]]\, L_1\delta x + \varepsilon_2 \cdot L_1\delta x \\ &= f'[g[x]]\,(g'[x] + \varepsilon_1)\delta x + \varepsilon_2 \cdot L_1\delta x \\ &= f'[g[x]]\, g'[x]\,\delta x + (f'[g[x]]\,\varepsilon_1 + \varepsilon_2 \cdot L_1)\delta x \\ &= f'[g[x]]\, g'[x]\delta x + \varepsilon_3\delta x\end{aligned}$$

where $L_1 = (g'[x] + \varepsilon_1)$ is finite so that $L_1\delta x \approx 0$ and we may apply the increment approximation to f at $g[x]$ with change $L_1\delta x$. Also, $\varepsilon_3 = (f'[g[x]]\varepsilon_1) + \varepsilon_2 L_1 \approx 0$ since $f[u]$ is smooth on the range of $g[x]$.

Procedure 6.1

To differentiate an expression like $v = \sqrt{x^2+1}$ with the Chain Rule, you need to find a decomposition of the formula satisfying the following

(a) Each piece of the decomposition can be differentiated by known rules.

(b) When chained back together, the pieces "compose" the original formula.

You can view the 'links' in function notation or by using a new variable.

Example 6.15 $\frac{d(\sqrt{x^2+1})}{dx}$

In this case, we let $u = x^2 + 1$ and $v = \sqrt{u}$ because substituting this value for u into the u-expression for v makes $v = \sqrt{x^2+1}$. The Power Rule and the Superposition Rule tell us

so the Chain Rule above says

$$\boxed{\frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx}}$$

$$\begin{aligned} \frac{dv}{dx} &= \frac{1}{2\sqrt{u}} \cdot 2x \\ &= \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

In function notation, this example is solved as follows: The functions $v = f[u] = u^{1/2}$ and $u = g[x] = x^2 + 1$ have $v = f[g[x]] = \sqrt{x^2+1}$. The derivatives are

so

$$\begin{aligned} (f[g[x]])' &= f'[g[x]] \cdot g'[x] \\ &= \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

Example 6.16 $\frac{d(\mathrm{Sin}[x^2+2x+1])}{dx}$

In this case, we let $y = \mathrm{Sin}[u]$ and $u = x^2+2x+1$, because substituting this value for u into the u-expression for y makes $y = \mathrm{Sin}[x^2+2x+1]$. The Sine Rule, the Power Rule and the Superposition Rule tell us

so the Chain Rule says

$$\boxed{\frac{dv}{dx} = \frac{dv}{du} \cdot \frac{du}{dx}}$$

$$\begin{aligned}\frac{dy}{dx} &= \mathrm{Cos}[u](2x+2)\\ &= 2(x+1)\,\mathrm{Cos}[x^2+2x+1]\end{aligned}$$

Example 6.17 *More Links* $y = \mathrm{Log}[\mathrm{Cos}[e^x + x^6]]$

We decompose $y = \mathrm{Log}[\mathrm{Cos}[e^x+x^6]]$ into three pieces below because resubstituting these yields the original expression.

The simple two-link Chain Rule says

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$

and

$$\boxed{\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx}}$$

$$\frac{du}{dx} = -\,\mathrm{Sin}[v] \cdot (e^x + 6x^5)$$

so

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \\
\frac{dy}{dx} &= \frac{1}{u} \cdot (-\operatorname{Sin}[v]) \cdot (e^x + 6x^5) \\
&= \frac{1}{\operatorname{Cos}[v]}(-\operatorname{Sin}[e^x + x^6])(e^x + 6x^5) \\
&= -(e^x + 6x^5)\frac{\operatorname{Sin}[e^x + x^6]}{\operatorname{Cos}[e^x + x^6]} \\
&= -(e^x + 6x^5)\operatorname{Tan}[e^x + x^6]
\end{aligned}$$

It is easy to generalize this example to see that if we decompose an expression into three links,

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}}$$

Example 6.18 *"Generalized" Differentiation Rules*

Some folks like to remember special cases of the Chain Rule such as

$$\frac{d(e^u)}{dx} = e^u \cdot \frac{du}{dx}$$

which they apply as follows. If we want to differentiate $y = e^{x^3}$, let $u = x^3$, so $\frac{du}{dx} = 3x^2$ and use the "Generalized Derivative" above,

$$\frac{d(e^u)}{dx} = e^u \cdot \frac{du}{dx} = e^u \cdot 3x^2 = 3x^2\, e^{x^3}$$

Of course, this is just the Chain Rule with $y = e^u$ and $u = x^3$. There is no need to remember a "generalized" formula, but you may if you wish.

6.4.1 The Everyday Meaning of the Chain Rule

The Scientific Projects contain a chapter on the "Expanding House." The wood in your house expands during the course of a normal day's warming. A 40-foot long house only expands about 0.03 inches on a normal Fall day, but there is a substantial increase in the volume of the house. Do you think it is about a thimble, a bucket, or a bathtub full? The numerical calculation might surprise you, but the ideas of this calculation are similar to the way that the rules of calculus are built. The project starts with simple but surprising arithmetic and progresses to a symbolic formulation of volume expansion. A final section uses the Chain Rule from the next section to give a direct solution. We recommend that you at least skim through the Expanding House project now.

Chapter 7 has several applications of the Chain Rule.

You may want to review chaining or function composition in Exercise 28.7 before doing the Chain Rule exercises that follow.

Exercise Set 6.4

1. Drill on Chains

   Break each of the following expressions into a composition of two functions and apply the Chain Rule to find $\frac{dy}{dx}$. (The letters a, b, c denote constants or parameters, and e is the natural base for logs and exponentials.)

a) $y = (1+x)^{33}$	*b*) $y = (a+b\,x)^{33}$	*c*) $y = (a+b\,x)^{c}$
d) $y = e^{a\,x}$	*e*) $y = e^{x^2+3}$	*f*) $y = e^{a\,x^2+b}$
g) $y = e^{\mathrm{Cos}[x]}$	*h*) $y = e^{\mathrm{Sin}[a\,x]+b}$	*i*) $y = e^{\mathrm{Log}[x]}$
j) $y = 3(\mathrm{Sin}[x])^3$	*k*) $y = 3(\mathrm{Sin}[x^3])$	*l*) $y = 3(\mathrm{Sin}[x^3])^3$
m) $y = \mathrm{Log}[x\,\sqrt[5]{x}]$	*n*) $y = \frac{6}{5}\,\mathrm{Log}[x]$	*o*) $y = \mathrm{Log}[x^{\frac{6}{5}}]$
p) $y = \mathrm{Log}[x^c]$	*q*) $y = c\,\mathrm{Log}[x]$	*r*) $y = \mathrm{Cos}[7\,e^x\,]$

2. The product $\mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta] = \frac{1}{2}\,\mathrm{Sin}[2\theta]$. Show this using the addition formula for the sine. (HINT: $\mathrm{Sin}[\theta+\theta] = ?$) Use the Chain Rule to show that the derivative of the product is $\mathrm{Cos}[2\theta]$ by differentiating the right side of the equality. Check your work using the Product Rule on the left side of the equality.

3. The sine function in degrees can be thought of this way

   $$\mathrm{Sin}[D] = \mathrm{Sin}[\frac{\pi}{180}\,D]$$

   where $\mathrm{Sin}[u]$ denotes the radian measure sine function. Use the Chain Rule to show that the derivative of the sine in degrees is $\frac{\pi}{180}$ times the cosine in degrees,

   $$y = \mathrm{Sin}[D] \Rightarrow \frac{dy}{dD} = \frac{\pi}{180}\,\mathrm{Cos}[D]$$

4. Which parts of Exercise 6.2 can you now do using the Chain Rule (in addition to the other rules) that you could not do without it?

6.5 General Exponentials

This section uses the Chain Rule to find $\frac{d(a^x)}{dx}$ and the derivatives of more general exponential expressions.

The project on Numerical Differentiation of Exponentials shows you a direct way to approximate the derivative of a general exponential function. The next topic shows you an exact symbolic method. This is important in the theory of calculus because it reduces the calculus of other bases to the natural base.

Suppose we have $y = e^{ct}$ for some constant c. The Chain Rule says

$$\boxed{\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}}$$

$$y = e^u \qquad u = ct$$

$$\frac{dy}{du} = e^u \qquad \frac{du}{dt} = c$$

$$\frac{d\big(e^{ct}\big)}{dt} = \frac{dy}{du} \times \frac{du}{dt} = c\, e^{ct}$$

If we want to differentiate $y = 2^t$, we can use the preceding Chain Rule computation and two other important facts about exponentials. We know the general exponential law

$$(a^c)^x = a^{cx}$$

so we try to find a constant c that satisfies

$$2 = e^c$$

Once we find this c we have $2^t = e^{ct}$ for all t because of the law of exponents. We know how to differentiate $y = e^{ct}$, and with this value of c, we learn how to differentiate 2^t,

$$\frac{d(2^t)}{dt} = \frac{d(e^{ct})}{dt} = c\, e^{ct} = c\, 2^t$$

Now we solve for c using natural log. Natural log and exponential are inverse functions. This simply means

$$\text{Log}[e^t] = t \qquad \text{and} \qquad e^{\text{Log}[s]} = s, \quad s > 0$$

We apply this to our problem by taking logs of both sides of

$$2 = e^c \, \text{Log}[2] = \text{Log}[e^c] = c$$

Thus $c = \text{Log}[2]$ and we see that

$$\frac{d(2^t)}{dt} = \text{Log}[2] \times 2^t$$

In general,

$$\boxed{\frac{d(a^x)}{dx} = \text{Log}[a] \, a^x}$$

but the procedure used above to find the derivative of 2^t is what you should learn. That procedure also applies to formulas like $y = x^x$.

Example 6.19 $\frac{d(\left(\text{Sin}[x]^{\text{Cos}[x]}\right)}{dx}$

$$\text{Sin}[x] = e^{\text{Log}[\text{Sin}[x]]} \qquad \text{for } \text{Sin}[x] > 0$$

so

$$\text{Sin}[x]^{\text{Cos}[x]} = \left(e^{\text{Log}[\text{Sin}[x]]}\right)^{\text{Cos}[x]} = e^{\text{Cos}[x] \; \text{Log}[\text{Sin}[x]]}$$

and we want to differentiate the chain

Use of the Product Rule

$$\frac{du}{dx} = \frac{d(\text{Cos}[x] \; \text{Log}[\text{Sin}[x]])}{dx} = \frac{d(\text{Cos}[x])}{dx} \, \text{Log}[\text{Sin}[x]] + \text{Cos}[x] \, \frac{d(\text{Log}[\text{Sin}[x]])}{dx}$$
$$= -\,\text{Sin}[x] \; \text{Log}[\text{Sin}[x]] + \text{Cos}[x] \, \frac{\text{Cos}[x]}{\text{Sin}[x]}$$

since another application of the Chain Rule gives

$$\frac{d(\text{Log}[\text{Sin}[x]])}{dx} = \frac{dw}{dv} \times \frac{dv}{dx} = \frac{\text{Cos}[x]}{\text{Sin}[x]}$$

This shows that

$$\frac{du}{dx} = \frac{\text{Cos}[x]^2}{\text{Sin}[x]} - \text{Sin}[x] \, \text{Log}[\text{Sin}[x]]$$

So finally,

$$\frac{d(\left(\text{Sin}[x]^{\text{Cos}[x]}\right)}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left(\frac{\text{Cos}[x]^2}{\text{Sin}[x]} - \text{Sin}[x] \, \text{Log}[\text{Sin}[x]]\right) \text{Sin}[x]^{\text{Cos}[x]}$$

Exercise Set 6.5

1. *For* $y[t]$ *as follows, find* $\frac{dy}{dt}[t]$*:*

$$a)\quad y = 3^t \qquad b)\quad y = 10^t \qquad c)\quad y = a^t$$

$$d)\quad y = t^t \qquad e)\quad y = 2^{\mathrm{Cos}[t]} \qquad f)\quad y = te^t$$

$$g)\quad y = e^{\frac{1}{t}} \qquad h)\quad y = 3^{\sqrt{t}} \qquad i)\quad y = \sqrt{3^t - 2^t}$$

(*Check your work with the computer.*)

2. *Find* $\frac{dy}{dx}[x]$

$$a)\quad y = x^x \qquad b)\quad y = x^{\mathrm{Sin}[x]} \qquad c)\quad y = (\mathrm{Cos}[x])^x$$

Problem 6.8

If the number of algae cells is $N[t] = N_0 2^{t/6}$*, how long does it take to double the number of cells? How long does it take to triple the number of cells? What is the instantaneous rate of growth of algae at* $t = 0$*? At* $t = 6$*? What is the instantaneous rate of growth of algae as a percentage of* N *at* $t = 0$*? At* $t = 6$*?*

Problem 6.9

Let r *be a constant. Use rules of calculus to prove that* $y = e^{r\,x}$ *grows at the instantaneous rate of* $r \times 100\%$*. (Compare this with Exercise NaturalPercentEx.)*

Problem 6.10

What is wrong with the following nonsensical differentiation?

$$\frac{dx^x}{dx} = xx^{x-1} = x^1x^{x-1} = x^{1+x-1} = x^x$$

(HINT: Differentiate with the computer, and try $y = x^x = (e^{Log[x]})^x = e^{xLog[x]}$ *yourself.)*

6.6 Derivative of the Natural Log

$\frac{d\,\mathrm{Log}[u]}{du} = \frac{1}{u}$

The inverse of $x = \mathrm{Log}[y]$ is $y = e^x$ and has derivative $\frac{dy}{dx} = e^x = y$; therefore

$$dy = e^x \; dx$$

or

$$dy = y \; dx$$

and

$$\frac{dx}{dy} = \frac{1}{y}$$

This computation is explored in more detail in the Project on Inverse Functions. The point is that the derivative of the inverse function is the reciprocal of the derivative of the function.

Example 6.20 $\frac{d\ \mathrm{Log}[\mathrm{Tan}[x]]}{dx}$

Use the chain

$$y = \mathrm{Log}\,[y] \qquad u = \mathrm{Tan}\,[x] = \tfrac{\mathrm{Sin}[x]}{\mathrm{Cos}[x]}$$

$$\tfrac{dy}{du} = \tfrac{1}{u} \qquad \tfrac{du}{dx} = \tfrac{1}{(\mathrm{Cos}[x])^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{\mathrm{Cos}[x]}{\mathrm{Sin}[x]} \times \frac{1}{(\mathrm{Cos}[x])^2} = \frac{1}{\mathrm{Cos}[x]\ \mathrm{Sin}[x]}$$

Compare this with the answer to

$$\begin{aligned}\frac{d\ \mathrm{Log}[\mathrm{Sin}[x]]}{dx} - \frac{d\ \mathrm{Log}[\mathrm{Cos}[x]]}{dx} &= \frac{\mathrm{Cos}[x]}{\mathrm{Sin}[x]} + \frac{\mathrm{Sin}[x]}{\mathrm{Cos}[x]} \\ &= \frac{(\mathrm{Cos}[x])^2 + \mathrm{Sin}[x])^2}{\mathrm{Sin}[x]\ \mathrm{Cos}[x]}\end{aligned}$$

since $\mathrm{Log}[\mathrm{Tan}[x]] = \mathrm{Log}[\mathrm{Sin}[x](\mathrm{Cos}[x])^{-1}] = \mathrm{Log}[\mathrm{Sin}[x] - \mathrm{Log}[\mathrm{Cos}[x]]$

Exercise Set 6.6

1. *For $y[t]$ as follows, find $\frac{dy}{dt}[t]$*

$$a)\quad y = (\mathrm{Log}[t])^3 \qquad b)\quad y = \mathrm{Log}[\mathrm{Cos}[x]] \qquad c)\quad y = t\,\mathrm{Log}[t] - t$$

$$d)\quad y = \mathrm{Log}[\mathrm{Log}[x]] \qquad e)\quad y = \mathrm{Log}[t^{1/t}] \qquad f)\quad y = \mathrm{Log}[t^2 + 2x]$$

   Assume first that x is independent of t. Second, if x is a function of t but we forgot to give you a formula for $x = x[t]$. Express your answers in terms of x and $\frac{dx}{dt}$. What is $\frac{dx}{dt}$ if x is independent of t?

2. *Differentiate $y = \mathrm{Log}[x^3]$ using $u = x^3$ and $y = \mathrm{Log}[u]$. Also use the log identity, $\mathrm{Log}[x^p] = p\ \mathrm{Log}[x]$ and differentiate $y = \mathrm{Log}[x^3] = 3\ \mathrm{Log}[x]$ without the Chain Rule. Compare the two answers.*

6.7 Combined Symbolic Rules

You can make some additional rules of differentiation for general cases.

Custom Rules

Suppose you often need to differentiate a cube of a product of functions, $y = (f[x] \cdot g[x])^3$, for various smooth functions $f[x]$ and $g[x]$. We use the Chain Rule with unknown functions:

$$y = u^3 \qquad u = f[x] \cdot g[x]$$

$$\frac{dy}{du} = 3u^2 \qquad \frac{du}{dx} = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = 3\,u^2 \left(\frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx} \right) \\ &= 3(f[x] \cdot g[x])^2 \left(\frac{df}{dx}[x] \cdot g[x] + f[x] \cdot \frac{dg}{dx}[x] \right) \end{aligned}$$

Exercise Set 6.7

1. Use all the rules of differentiation to find the following:

   a) $y = \frac{1}{ax+b}, \frac{dy}{dx} = ?$

   b) $y = \frac{1}{2x-1}, \frac{dy}{dx} = ? \left({u=ax+b \atop y=\frac{1}{u}=u^{-1}} \right)$

   c) $y = \mathrm{Cos}[x^2], \frac{dy}{dx} = ?$

   d) $y = \mathrm{Cos}^2[x], \frac{dy}{dx} = ?$ (unchain two ways)

   e) $y = \sqrt{1-x^2}, \frac{dy}{dx} = ?$

   f) $y = \frac{1}{\sqrt{1-x^2}}, \frac{dy}{dx} = ?$

   g) $y = (12-3x^7)^8, \frac{dy}{dx} = ?$

   h) $y = \mathrm{Sin}[x^2+x^3], \frac{dy}{dx} = ?$

   i) $y = (2x^3+4)(x^2-\sqrt{x}), \frac{dy}{dx} = ?$

   j) $y = [(ax+b)^{-1}+c]^{-1}, \frac{dy}{dx} = ?$

   k) $y = \mathrm{Sin}[x]\,\mathrm{Cos}[x], \frac{dy}{dx} = ?$

   l) $y = \mathrm{Sin}[2x], \frac{dy}{dx} = ?$

   m) $y = \mathrm{Sin}[3x], \frac{dy}{dx} = ?$

   n) $y = \frac{1}{\mathrm{Cos}[3x]}, \frac{dy}{dx} = ?$

   o) $y = \mathrm{Sin}[x]\cdot[\mathrm{Cos}[x]]^{-1} = \frac{\mathrm{Sin}[x]}{\mathrm{Cos}[x]} = \mathrm{Tan}[x], \ \frac{dy}{dx} = ?$

   p) $y = \frac{2-3x}{1+2x} = (2-3x)[(1+2x)^{-1}], \ \frac{dy}{dx} = ?$

   Check your work with the computer.

2. Differentiate $y = \sqrt{x^2+2x+1}$ using the Chain Rule, the Power Rule and the Superposition Rule. The graph of this function is rather simple, as we saw in Exercise kinkex. Where does your symbolic answer not make sense? Can you sketch the graph? Why does the symbolic answer not work at the bad point?

3. Use the Superposition Rule and the Product Rule repeatedly to show in general that if $f[x]$, $g[x]$ and $h[x]$ are smooth on an interval, then so are their sum and product and

$$\frac{d(f[x]+g[x]+h[x])}{dx} = \frac{df}{dx}[x] + \frac{dg}{dx}[x] + \frac{dh}{dx}[x]$$
$$\frac{d[f[x]g[x]h[x]]}{dx} = \frac{df}{dx}[x]\cdot g[x]\cdot h[x] + f[x]\cdot\frac{dg}{dx}[x]\cdot h[x] + f[x]\cdot g[x]\cdot\frac{dh}{dx}[x]$$

   (HINT: Let $G[x] = (g[x]\cdot h[x])$ and apply the Product Rule to $f[x]\cdot G[x]$.)

4. The Second Derivative Product Rule

(a) Differentiate the general Product Rule identity to get a formula for

$$\frac{d^2(f[x] \cdot g[x])}{dx^2}$$

(b) Let $h[x] = f[x] \cdot g[x]$ and use your rule to compute $\frac{d^2(h)}{dx^2}[1]$ if $f[1] = 2$, $\frac{df}{dx}[1] = 3$, $\frac{d^2(f)}{dx^2}[1] = 5$, $g[1] = -3$, $\frac{dg}{dx}[1] = 4$, and $\frac{d^2(g)}{dx^2}[1] = -2$.

6.7.1 The Quotient Rule

5. The Quotient Rule
Derive the quotient rule: If $q[x] = \frac{f[x]}{g[x]}$, where f and g are smooth and $g[x] \neq 0$ for $\alpha < x < \beta$, then $q[x]$ is also smooth for $\alpha < x < \beta$ and

$$\frac{d(\frac{f[x]}{g[x]})}{dx} = \frac{\frac{df[x]}{dx} g[x] - f[x] \frac{dg[x]}{dx}}{[g[x]]^2}$$

Use the Chain Rule and Product Rule on the formula

$$\frac{f[x]}{g[x]} = f[x] \times (g[x])^{-1} = f[x] \times h[x]$$

You will have to put your answer on a common denominator to get the formula above.

6.7.2 The Relative Growth Rule

We often make relative measurements stating the error (or accuracy) as a fraction of the amount (or stating a percentage). A similar notion is the relative rate of change given by

$$f^*[x] = \frac{1}{f[x]} \frac{df}{dx}[x]$$

6.

(a) Let $f[x] = e^{r\,x}$ for a constant r. Compute $f^*[x]$.

(b) Let $f[x] = b^x$ for a constant base b. Compute $f^*[x]$.

7. Give a general symbolic rule for the relative rate of change of a product in terms of $f^*[x]$ and $g^*[x]$.

$$(f[x]\ g[x])^* = ?$$

(HINT: Substitute a product into the rule for * and rewrite using ordinary rules.)

6.8 Review - Inside the Microscope

Calculus lets us "see" inside a powerful microscope without actually magnifying the nonlinear graph. We know that the curve looks like its tangent line at high magnification. The "rules"of differentiation are the way we "see." This section combines the rules with "looking."

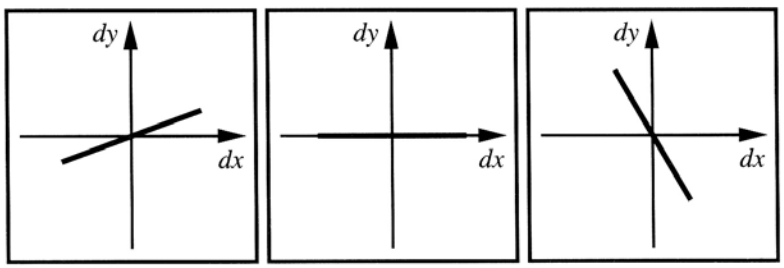

Figure 6.8:3: Possible microscopic views

6.8.1 Review - Numerical Increments

When a function is smooth, we summarize the local linear approximation by

$$y = f[x] \qquad \Rightarrow \qquad dy = f'[x]\ dx$$

The differential $dy = f'[x]\ dx$ is a linear function of the local variable dx, with dependent variable dy. This is the linear equation in microscope variables. The variable x in $f'[x]$ is considered fixed until we move the point where we focus our microscope. The quantity dy is an approximation to the change $f[x + dx] - f[x]$ in the actual function.

You should memorize the microscope approximation or Definition 5.2 and strive to understand its algebraic and geometric consequences. Functions given by formulas are important in science and mathematics, but they are not the only kind of functions.

The rules of calculus are theorems that guarantee that the local linear approximation is valid. These rules are remarkably easy to use compared with the direct verification of the approximations as in Chapter 5. You simply compute and look at the answers. We used the symbolic approximation in Chapter 5 to estimate Sin[46°].

Contrast what we learn about a function from the approximation with the simplicity of the computation that guarantees that the approximation holds. If

$$y = x^{-3}, \quad \text{then} \quad dy = -\frac{3}{x^4}\ dx$$

according to the rules. Both of these formulas are valid when $x \neq 0$, so the increment approximation defining the derivative holds and the change in f is approximated by dy with $dx = \delta x$. In general,

$$f[x+dx] - f[x] = f'[x] \cdot dx + \varepsilon \cdot dx$$
$$f[x+dx] - f[x] = dy + \varepsilon \cdot dx \approx dy$$

In this case,

$$\begin{aligned}
&\tfrac{1}{(x+dx)^3} - \tfrac{1}{x^3} = \tfrac{-3}{x^4} \cdot dx + \varepsilon \cdot dx \approx \tfrac{-3}{x^4} \cdot dx \\
&\tfrac{1}{(3+.01)^3} - \tfrac{1}{27} \approx -\tfrac{3}{81} \times 0.01 \\
&\tfrac{1}{(3.01)^3} - \tfrac{1}{27} \approx -\tfrac{.01}{27}
\end{aligned}$$

so,

$$\frac{1}{(3.01)^3} \approx .99 \times \frac{1}{27} \approx 0.0366667$$

the computer gives $1/(3.01)^3 = 0.0366691$, so the increment approximation is quite close even though the x increment $\delta x = 0.01$ is not infinitesimal.

To make numerical differential approximations, do the following steps.

1. Compute $f'[x]$ by rules. The rules guarantee the approximation when $dx \approx 0$.
2. Substitute the fixed x to find the numerical value of the derivative
$$m = f'[x]$$
3. Compute $df = m\ dx$ when $dx =$*your perturbation* using the *number* m.
4. Compute $f[x+dx]$ using this approximate change and the value of $f[x]$,
$$f[x+\delta x] \approx f[x] + df[x]$$

In short, $f[x+dx] - f[x] \approx df[x]$, with an error small compared to dx when dx is small.

6.8.2 Differentials and the (x, y)-Equation of the Tangent Line

The equation of the tangent line to $y = f[x]$ in local coordinates is simply the differential $dy = f'[x]\, dx$, but there is possible confusion when we try to convert back to regular coordinates because we are treating x as fixed in the local dx-dy-equation. Here is a way to find the equation of the tangent line to $y = x^2$ when $x = -1/3$ as shown in Figure 6.8:4. We know $dy = 2x\, dx$ and $x = -1/3$, so the slope is $2 \cdot (-1/3) = -2/3$. When $x = -1/3$, $y = x^2 = 1/9$, so the line goes through $(-1/3, 1/9)$ and has slope $-2/3$. Using the change form of a line (or the point-slope formula),

---

To find the (x, y) equation of the tangent line, do the following steps.

1. Compute $f'[x]$ by rules.
2. Substitute the fixed $x = a$ ($x = -1/3$, in this case) to find the numerical value of the slope $m = f'[x]$.
3. Compute the specific y value, $y = f[x]$ at the point of tangency. This gives you a specific point $(x, y) = (a, b)$ that lies on both the curve and tangent line.
4. Change the local equation of the line $dy = m\ dx$ to the point-slope (x, y) form of a line $\frac{y-b}{x-a} = m$ and simplify to the slope-intercept form $y = m\,x + i$.

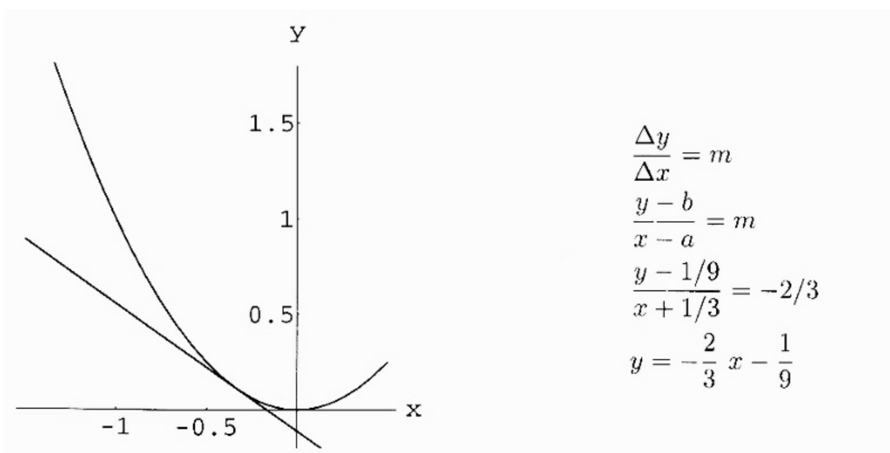

Figure 6.8:4: $y = x^2$ & $y = -\frac{2}{3}x - \frac{1}{9}$

Exercise Set 6.8

1. A Partial View of the "Bell Shaped" Curve
   The derivative of the function $f[x] = e^{-x^2}$ is $f'[x] = -2x \cdot e^{-x^2}$ (as you may verify using rules of differentiation.) This question asks, So what? (or what does this tell us mathematically?) You answer it as follows: Draw microscopic views of the graph $y = e^{-x^2}$ when the microscope is focused on the graph over the x-points, $x = 0, \pm 0.1, \pm 1, \pm 10$. Give the numerical values of the derivatives and sketch the slopes to scale on equal axes.

   The next problem asks you to do all the steps involved in "looking" in an infinitesimal microscope. This is a question that requires you to summarize the steps in writing. This should help you combine the facts you have learned.

2. Find the (x, y)-equation of the tangent to $y = x^3$ at $x = -2$.

   Find the (x, y)-equation of the tangent to $y = \mathrm{Sin}[x]$ at $x = \pi/3$.

   Find the (x, y)-equation of the tangent to $y = \mathrm{Log}[x]$ at $x = 1$.

   Plot these pairs of curves with the computer program **Tangents**.

3. Differential Approximation
   Approximate $\sqrt[3]{1,000,000,000,000,002} - \sqrt[3]{1,000,000,000,000,000}$ using the differential increment of the function $f[x] = \sqrt[3]{x}$, $x = 1,000,000,000,000,000$ and $\delta x = 2$. (Two is not infinitesimal, but it is small compared to 10^{15}. Computers have a very hard time with this kind of computation because they work in fixed length decimal approximations.) We only need a simple way to estimate $f[x + \delta x] - f[x]$, since we know $f[x] = 100,000$ when $x = 10^{15}$.

   How many decimals of your approximation are accurate in this case? Try it with your calculator or the computer; you'll get the wrong answer unless you work with very very high-precision arithmetic. The differential is very accurate.

Problem 6.11

You are interested in the accuracy of your speedometer and perform the following experiment on a stretch of flat, straight, deserted Interstate highway. You drive at constant speed with your speedometer reading 60 mph, crossing between two consecutive mile markers in 57 seconds as measured by your quartz watch. For constant speed, we know the formula, "distance equals rate times time," $D = R \times T$, *so* $R = D/T$ *when the units are correct.*

1. (a) *Express the rate of speed* R *in miles per hour as a function of the distance* D *in miles and the time* t *in* ***seconds****.*

   (b) *Compute the differential* $dR = ? \times dt$ *using the appropriate rules, assuming that* $D = 1$ *is measured exactly.*

   (c) *Approximate the speed of your car in the above experiment using the differential to approximate the increment* ΔR. *(See Exercise 6.8.)*

   (d) *Use the computer to compare the actual rate with the differential approximation using your formula for* r *and its differential*:

Summarize the idea of this problem in a few sentences.

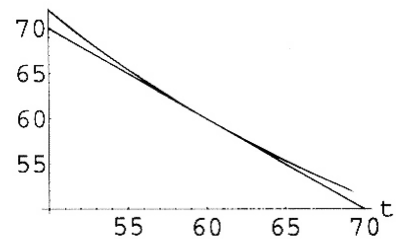

Figure 6.8:5: Rate and Approximation

Problem 6.12

1. *Sketch a pair of (x, y)-axes and plot the point $(1, -1)$. Let x run from 0 to 3, and y run from -2 to 1.*
2. *The point $(x, y) = (1, -1)$ lies on the explicit curve $y = x^2 \operatorname{Cos}[\pi/x]$. Verify this.*
3. *Add a pair of (dx, dy)-axes at the (x, y)-point $(1, -1)$. How are these axes related to the (x, y) axes?*
4. *Use rules of calculus to show that*

$$y = x^2 \operatorname{Cos}[\pi/x] \qquad \Rightarrow \qquad dy = (2x\ \operatorname{Cos}[\pi/x] + \pi\ \operatorname{Sin}[\pi/x])\ dx$$

5. *Substitute $x = 1$ into your differential to show that*

$$dy = -2\ dx$$

at the (x, y)-point $(1, -1)$ or the (dx, dy)-point $(0, 0)$.

6. *Plot the line $dy = -2\ dx$ on your (dx, dy)-axes.*
7. *What would you see if you looked at the graph of $y = x^2 \operatorname{Cos}[\pi/x]$ under a very powerful microscope?*
8. *Use the computer NoteBook* ***Micro1D*** *to plot the function and its differential and to make an animation of a microscope zooming in on the graph at the (x, y)-point $(1, -1)$.*
9. *Explain how the Differential cell of the* ***Micro1D*** *NoteBook is actually solving parts (2) - (7) of this exercise. How does calculus let us "see" a graph in a powerful microscope?*

Problem 6.13

What is wrong with the following "general formula" for the tangent to $y = x^2$ at the point $x = a$? We have $dy = 2x\,dx$ and we know that $dx = x - a$, while $dy = y - a^2$. So (false conclusion), the equation of the tangent line is $\frac{\Delta y}{\Delta x} = \frac{y-a^2}{x-a} = 2x$ and we can simplify to the form $y = m\,x + b$.

The (x, y) equation of the tangent is often not needed (if you plot in (dx, dy) coordinates), but the idea of working in the correct coordinates is important. Here is another kind of example of tangency and keeping track of all the variables. A circle of radius 1 centered on the y axis is moved down the axis until it touches the parabola $y = x^2$, as shown in the next figure. Since the circle just touches the parabola, both curves are tangent at the point of contact.

Problem 6.14 Tangent Curves

Write the equation of a circle of radius 1 centered on the y axis in terms of a parameter c for the unknown y coordinate of the center.

Calculate the differential of the equation of your circle, using the unknown parameter. Solve for $\frac{dy}{dx}$, and write the equation that says

"the slope of the circle at (x, y) equals the slope of the parabola at (x, y)"

Write the system of three equations that say "(x, y) is the point of tangency":

$y = x^2$ *"(x, y) lies on the parabola"*

$? = ?$ *"(x, y) lies on the circle through $(0, c)$"*

$? = ?$ *"the circle and parabola have the same slope at (x, y)"*

Finally, solve the three equations in three unknowns, using the computer if you like.

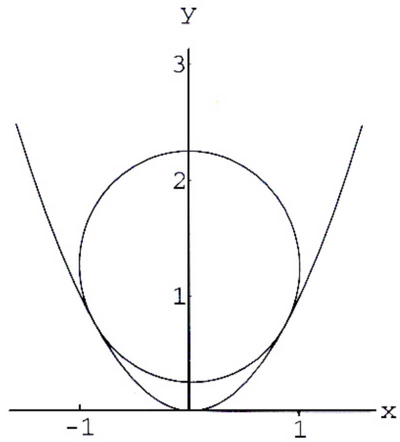

Figure 6.8:6: A circle tangent to a parabola

Here is a sample of the use of the computer to solve a set of equations:

equns = { y == x∧2 , x∧2 + (y - c)∧2 == 1 , x == 2 x (c - y) }
Solve[**equns , { x , y , c }**]

Problem 6.15 MORE TANGENT CURVES

A circle with center $(0, 1)$ *is expanded until it just touches the parabola* $y = x^2$*. What is it's radius? Where does it make contact?*

A line passes through the origin and is tangent to $y = x^2 + 1$*. What is the point of tangency?*

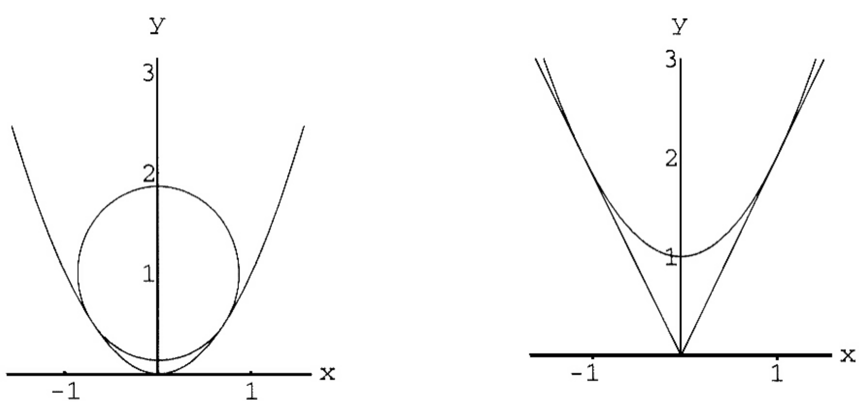

Figure 6.8:7: Tangent curves

6.9 Projects

6.9.1 Functional Identities

The Mathematical Background chapter on Functional Identities (on the CD) may help you understand the role of unknown functions in mathematics. It is important for you to see the Product Rule and the Chain Rule as identities in unknown functions.