22m:033 Notes:
1.1 Systems of Linear Equations

Dennis Roseman
University of Iowa
Iowa City, IA

http://www.math.uiowa.edu/~roseman

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1 Linear Equations

Definition 1.1 *A **linear equation in the variables** $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form*

$$a_1x_1 + a_2x_2, \cdots a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers.

*The numbers $a_1, a_2, \ldots, a_n$ are called **coefficients**.*

Remark 1.2 We could also allow $a_1, a_2, \ldots, a_n$ and b to be complex numbers and almost everything we do will be unchanged, however in this class we will use real numbers.

It is possible that some of the $a_1, a_2, \ldots, a_n$ and b are zero. Also there is nothing special about using subscripts or these particular letters. So that:

$x + 2y - 13z = 11$ is a linear equation in x, y, z.

$x - 13z = 11$ is a linear equation in x, y, z, if we view this as $x + 0y - 13z = 11$.

$3a_1 + 4a_2 - 16a_3 = 1066$ is a linear equation in a_1, a_2, a_3

Remark 1.3 Notice that the definition says *"can be written"* not "is written".

So $x - y = z$ is a linear equation in x, y, z since we can rewrite this as $x - y - z = 0$

Remark 1.4 The graph of a linear equation in two variables such as $y = 5x - 3$ is a line. That is why we use the term "linear".

So $y = x^2 + 1$ is not a linear function since the graph is a parabola and not a line.

Definition 1.5 *A* ***system of linear equations*** *is a set of one or more linear equations in the same variables.*

Definition 1.6 *A* ***solution to a system of linear equations*** *in variables $x_1, x_2, \ldots, x_n$ is a sequence of numbers $s_1, s_2, \ldots, s_n$ which when substituted for the respective variables satisfies each of the equations of the system. The* ***solution set to a system of linear equations*** *is the set of all possible solutions to the system.*

Two systems of linear equations in variables $x_1, x_2, \ldots, x_n$ ***are called equivalent*** *if they have the same solution set.*

Definition 1.7 *If a set of linear equations has no solution we say the system of linear equations is* ***inconsistent****; otherwise (if there is one or infinitely many solutions) we say the system is* ***consistent****.*

Remark 1.8 If we have two linear equations in two variables, this corresponds to two lines L_1 and L_2 in the plane. The solution set of the the system corresponds to the intersection of these two lines.

We note that the solution set could be an empty set (that is no solutions), a point (a unique solution) or even infinitely many points (when the lines are the same):

The system

$$\begin{aligned} x + y &= 1 \\ x - y &= 0 \end{aligned}$$

has exactly one solution, $x = \frac{1}{2}$ and $y = \frac{1}{2}$.

The system:

$$\begin{aligned} x+y &= 1 \\ x+y &= 0 \end{aligned}$$

has no solutions, so the solution set is the empty set and we say the system is inconsistent.

The system

$$\begin{aligned} x+y &= 1 \\ 2x+2y &= 2 \end{aligned}$$

has infinitely many solutions. For any number t, $x=t$ and $y=1-t$ is a solution.

Remark 1.9 When we have more that two variables, the situation is more complex as we later analyze in this chapter.

Remark 1.10 For a set of linear equations it is sometimes clear from context what the variables are. But sometimes we need to specify.

For the system of linear equations

$$\begin{aligned} 2x-3z &= -2 \\ -x+14z &= 14 \end{aligned}$$

we would most reasonably assume that the variables are x and z unless it is stated otherwise.

However if we state that the variables are x, y and z then we would understand that there are three variables:

$$\begin{aligned} 2x + 0y - 3z &= -2 \\ -x + 0y + 14z &= 14 \end{aligned}$$

An equation in three variables corresponds to a plane in space. The number of variables is critical in understanding the solution set since, generally two lines intersect in a point but two planes intersect in a line.

2 Matrtix notation

Definition 2.1 *A **matrix** is a rectangular array of either numbers or algebraic expressions.*

If we have an ordered set of equations and ordered set of variables, we get matrix of coefficients:

Example 2.2 For the system of linear equations in x, y, z:

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= 4 \\ x - z &= 0 \end{aligned}$$

the matrix of coefficients is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix}$$

The matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

is called the **augmented matrix** of the system.

Definition 2.3 *If a matrix has m rows and n columns it is called an $m \times n$* ***matrix*** *and we say that "$m \times n$" is the* ***size of the matrix****.*

Remark 2.4 It is important to remember that if we have an $m \times n$ matrix that m is the number of rows (*not* the number of columns).

So

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

is a 3×4 matrix.

Remark 2.5 If A is a matrix, the entries of the matrix are often referenced using a double subscript notation a_{ij}.

The first subscript is the *position* in the row and the second is the *position* in the columns. This will take some getting used to since this is the same as saying that the first index is the column number and the second index is the row number.

For example if A is a 4×5 matrix we might write this as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{pmatrix}$$

Alternatively we could use notation as follows: A is the matrix (a_{ij}) where $1 \leq i \leq m$ and $1 \leq j \leq n$.

3 Solving a system of linear equations

We can solve a linear equations using the following three rules:

1. Add a multiple of one equation to another equation
2. Change the order of equations
3. multiply both sides of an equation by a non-zero number

Example 3.1 To solve

$$\begin{aligned} 2x+2y+2z &= 12 \\ x-y+z &= 2 \\ 2x+y &= 4 \end{aligned}$$

we multiply equation 1 by $\frac{1}{2}$ and get:

$$\begin{aligned} x+y+z &= 6 \\ x-y+z &= 2 \\ 2x+y &= 4 \end{aligned}$$

We multiply the equation 1 by -1 and add to equation 2 and get

$$\begin{aligned} x+y+z &= 6 \\ -2y &= -4 \\ 2x+y &= 4 \end{aligned}$$

We multiply equation 2 by $-\frac{1}{2}$ and get

$$\begin{aligned} x+y+z &= 6 \\ y &= 2 \\ 2x+y &= 4 \end{aligned}$$

We multiply equation 1 by -2 and add to equation 3 and get

$$\begin{aligned} x+y+z &= 6 \\ y &= 2 \\ -y-2z &= -8 \end{aligned}$$

We multiply the equation 2 by 1 and add to equation 3 and get (just add equation 2 to equation 3) and get

$$\begin{aligned} x+y+z &= 6 \\ y &= 2 \\ -2z &= -6 \end{aligned}$$

We multiply equation 3 by $-\frac{1}{2}$ and get

$$\begin{aligned} x + y + z &= 6 \\ y &= 2 \\ z &= 3 \end{aligned}$$

Multiply equation 2 by -1 and add to equation 1 and get

$$\begin{aligned} x + z &= 4 \\ y &= 2 \\ z &= 3 \end{aligned}$$

Multiply equation 3 by -1 and add to equation 1 and get

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

At this point we have found that this system of equations had a unique solution $x = 1, y = 2$ and $z = 3$.

4 Row operations and matrices

Two observations on Example 3.1. In order to solve the equations we need only concentrate on the coefficients. Also the order of the equations was artificial because we have a *set* of equations, not an ordered set of equations—we could switch any two equations and not alter the system.

We can now streamline (or abstract) solving equations by considering the augmented coefficient matrix. Then each equation corresponds to a row of the matrix and each column (except the last) corresponds to a variable. Then our allowable algebraic operations will correspond to changes called operations for the rows.

Here is the definition of these **row operations**:

Definition 4.1 *(Row Operations) Let M be a matrix. The following changes on M are called row operations:*

1. *(Replacement) Replace any row by the sum of that row and any multiple of another row*
2. *(Interchange)Interchange two rows of M*

3. *(Scaling) Multiply any row of M by a non-zero number*

Definition 4.2 *Two matrices are* ***row equivalent*** *if there is a sequence of row operations that takes one to the other.*

Example 4.3 We now solve the equations of Example 3.1 using this method, using the same steps To solve

$$\begin{aligned} 2x + 2y + 2z &= 12 \\ x - y + z &= 2 \\ 2x + y &= 4 \end{aligned}$$

we first write the augmented matrix:

$$\begin{pmatrix} 2 & 2 & 2 & 12 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & 0 & 4 \end{pmatrix}$$

Next, we take our description and replace the word "equation" by "row".

We multiply row 1 by $\frac{1}{2}$ and get:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & 0 & 4 \end{pmatrix}$$

We multiply the equation 1 by -1 and add to equation 2 and get We multiply row 1 by $\frac{1}{2}$ and get:

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 2 & 1 & 0 & 4 \end{pmatrix}$$

We multiply equation 2 by $-\frac{1}{2}$ and get

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 2 & 1 & 0 & 4 \end{pmatrix}$$

We multiply equation 1 by -2 and add to equation 3 and get

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & -2 & -8 \end{pmatrix}$$

We multiply the equation 2 by 1 and add to equation 3 and get (just add equation 2 to equation 3) and get

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -6 \end{pmatrix}$$

We multiply equation 3 by $-\frac{1}{2}$ and get

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Multiply equation 2 by -1 and add to equation 1 and get

$$\begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Multiply equation 3 by -1 and add to equation 1 and get

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

At this point we reinterpret this matrix as a linear system and have found that this system of equations had a unique solution $x = 1, y = 2$ and $z = 3$.