On Restrictions of Representations for GSpin

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Melissa Emory
joint with Shuichiro Takeda

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University of Toronto

Multiplicity at most one theorems

- F: a non-archimedean local field of characteristic zero
- G : a reductive group over F
- $H \subseteq G$: a reductive subgroup
- π, σ irreducible admissible representations of G and H, resp. Question - "How many times σ" appears as a quotient of π, when $\pi|_H$? namely, to know the dimension

$$\dim_{\mathbb{C}} \operatorname{Hom}_H(\pi, \sigma)$$

- When is this space at most one? This can be formulated as

$$\dim_{\mathbb{C}} \operatorname{Hom}_H(\pi, \sigma) \leq 1$$

This assertion is referred to as a "multiplicity at most one theorem".

An Example: Spherical Harmonics

Spherical harmonics give a decomposition of the functions of the 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

as a representation of SO(3).

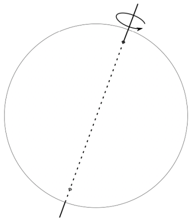
Let W_ℓ be the vector space of homogeneous polynomials $f(x, y, z)$ of degree ℓ which are harmonic on $\mathbb{R}^3$

$$\Delta(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

The W_ℓ is an irreducible representation of SO(3) of dimension $2\ell + 1$ and $\mathcal{F}(S^2) = \hat{\bigoplus}_{\ell \geq 0} W_\ell$

An Example: Spherical Harmonics

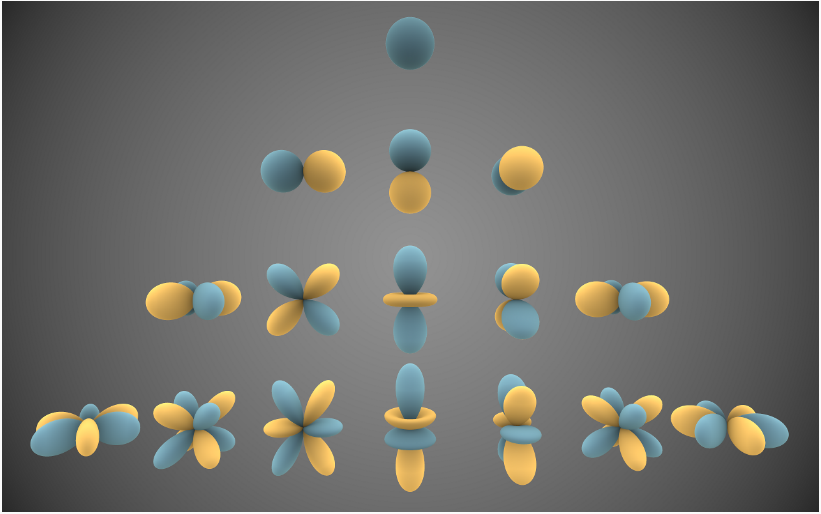
The subgroup pf SO(3) which fixes a point on the 2-sphere is isomorphic to the rotation group SO(2).



The restriction of W_ℓ decomposes as a sum of one-dimensional representations

$$\mathrm{Res}_{\mathrm{SO}(2)} W_\ell = \bigoplus_{-\ell \leq m \leq \ell} \chi_m, \qquad \chi_m(z) = z^m.$$

An example: Spherical Harmonics



What is GSpin?

Let F be a field with characteristic not equal to 2 and (V, q) be a quadratic space.

The tensor algebra is denoted

$$T(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k} = F \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

Denote

$$I(q) = \langle v \otimes v - q(v) \cdot 1 \rangle \in T(V), \quad v \in V.$$

The Clifford algebra: $C(V) = T(V)/I(q)$.

The definition of GSpin

$C(V) = T(V)/I(q)$ is a graded algebra - inherited by

$$T(V)_+ = \coprod_{n \text{ even}} M^{\otimes n}, T(V)_- = \coprod_{n \text{ odd}} M^{\otimes n}$$

Then

$$C(V) = T(V)_+/I(q)_+ \bigoplus T(V)_-/I(q)_-$$

and

$$C(V) = C^+(V) \oplus C^-(V)$$

which we call the even Clifford algebra and the odd part of the Clifford algebra, respectively.

Note that

$$\dim C^+(V) = \dim C^-(V) = 2^{n-1}.$$

Definition

$\mathrm{GSpin}(V) = \{g \in C^+(V)^\times : gVg^{-1} = V\}$

A group (we call) GPin

Definition

$\mathrm{GSpin}(V) = \{g \in C^+(V)^\times : gVg^{-1} = V\}$

Let α be the automorphism $\alpha : C(V) \to C(V)$ where

$$\alpha|_{C+(V)} = 1 \text{ and } \alpha|_{C-(V)} = -1$$

Definition

$\mathrm{GPin}(V) = \{g \in C(V)^\times : \alpha(g)Vg^{-1} = V\}$

$$1 \longrightarrow F^\times \longrightarrow \mathrm{GPin}(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1$$

$$1 \longrightarrow F^\times \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1$$

The Clifford norm

$C(V)$ is equipped with a natural involution

$$(v_1 \dots v_k)^* = v_k \dots v_1.$$

For all $x \in C(V)$, the Clifford conjugation is

$$\bar{x} := \alpha(x)^* = \alpha(x^*)$$

giving rise to the Clifford norm

$$N : C(V) \to C(V), \quad N(x) = x\bar{x}.$$

The Clifford norm descends to $\mathrm{O}(V) \to F^\times / F^{\times 2}$ because $N(z) \in F^{\times 2}$ for $z \in Z^0 = F^\times$, which is called the spinor norm

$$\mathrm{Pin}(V) := \ker(N : \mathrm{GPin}(V) \to F^\times)$$

Why we call it GPin

We hope that you like commutative diagrams of group schemes!!

$$1 \longrightarrow \mathrm{GL}_1 \longrightarrow \mathrm{GPin}(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1$$

$$\mathrm{Spin}(V) := \mathrm{GSpin}(V) \cap \mathrm{Pin}(V)$$

$$1 \longrightarrow \mathrm{GL}_1 \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1$$

Why GSpin?

- A step beyond classical groups
- Representation theory of GSpin subsumes that of SO
- Shimura varieties - work around using work for SO
- Relationship with the Langlands dual group of similitude groups
  - Let $\widehat{G}$ be the dual group of the group G.
  - The L-group of G is denoted ${}^L G$ and ${}^L G \cong \widehat{G} \rtimes W_F$

G	$\widehat{G}$	G	$\widehat{G}$
GL_m	$GL_m(\mathbb{C})$	SL_m	$PGL_m(\mathbb{C})$
SO_{2m+1}	$Sp_{2m}(\mathbb{C})$	GSp_{2m}	$GSpin_{2m+1}(\mathbb{C})$
SO_{2m}	$SO_{2m}(\mathbb{C})$	GSO_{2m}	$GSpin_{2m}(\mathbb{C})$

Multiplicity at most one theorems

- (2010) -(non-archimedean) Aizenbud, Gourevitch, Rallis, Schiffman proved a multiplicity at most one theorem for the pairs

$$(G,H) = (\mathrm{GL}_{n+1}, \mathrm{GL}_n), (\mathrm{U}_{n+1}, \mathrm{U}_n), \text{ and } (\mathrm{O}_{n+1}, \mathrm{O}_n)$$

- (2012) -(non-archimedean) Waldspurger proved for

$$(G,H) = (\mathrm{SO}_{n+1}, \mathrm{SO}_n)$$

- (2012) -(archimedean) Sun and Zhu

Applications of multiplicity at most one theorems

Multiplicity at most one theorems have many applications, some of them are

- Local and global liftings of automorphic representations
- Automorphic descent
- Determination of L-functions
- The relative trace formula
- A first step in proving the local Gan-Gross-Prasad conjecture

Statement of the main theorem

- Let F be a nonarchimedean local field of characterstic zero
- Assume $W \subseteq V$ is a nondegenerate subspace of dimension $n-1$

Main Theorem (E., Takeda)

Let

$$(G, H) = (\mathrm{GPin}(V), \mathrm{GPin}(W)) \text{ or } (\mathrm{GSpin}(V), \mathrm{GSpin}(W))$$

. For all $\pi \in \mathrm{Irr}(G)$ and $\tau \in \mathrm{Irr}(H)$, we have

$$\dim_{\mathbb{C}}(\mathrm{Hom}_H(\pi, \tau) \leq 1$$

Note that $\omega_\pi \neq \omega_\tau$ on Z^0, then the Hom space is automatically zero.

The involution

Suppose that we are not in the case where $\dim_F V = 2k$ with k odd.

First, let

$$sign : \mathrm{GPin}(V) \to \{\pm 1\}$$

be the homomorphism which sends the nonidentity component to -1, so that its kernel is $\mathrm{GSpin}(V)$.

We define our involution

$$\sigma_n(g) = \begin{cases} g^* & \text{if } n = 2k \\ sign(g)^{k+1} g^* & \text{if } n = 2k-1. \end{cases}$$

Two essential properties of σ_n

1. σ_n fixes $Z_{\mathrm{GPin}(V)}$ pointwise
2. For each semisimple $g \in \mathrm{GSpin}(V), \sigma_n(g)$ and g are conjugate in $\mathrm{GPin}(V)$

Vanishing Theorem

Recall, we have an orthogonal basis

$$e_1, \dots e_{n-1} e_n$$

and

$$W = \mathrm{Span}\{e_1, \dots, e_{n-1}\}.$$

Let E be a quadratic extension over F. We define

$$\beta : V \to V$$

by sending $a_1 e_1 + \cdots + a_n e_n$ to $\overline{a_1} e_1 + \cdots + \overline{a_n} e_n$ so that $\beta^2 = 1$. We define

$$\widetilde{\mathrm{GSpin}}(V) = \langle g, e_n^k \beta \quad : \quad g \in \mathrm{GSpin}(V) \rangle$$

with the relations $g\beta = \beta g$ for all $g \in \mathrm{GSpin}(V)$ and $\beta^2 = 1$.

Let χ be the character

$$\chi : \widetilde{\mathrm{GSpin}}(V) \longrightarrow \{\pm 1\}$$

that sends β to -1.

The Theorems

Recall that we want to show

Main Theorem (E., Takeda)

For all $\pi \in \mathrm{Irr}(\mathrm{GSpin}(V))$ and $\tau \in \mathrm{Irr}(\mathrm{GSpin}(W))$, we have

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{GSpin}(W)}(\pi, \tau) \leq 1.$$

Let $\mathcal{S}'(\mathrm{GSpin}(V))^{\widetilde{\mathrm{GSpin}}(W),\chi}$ be the space of distributions on which $\widetilde{\mathrm{GSpin}}(W)$ acts via χ. The main technical part is to prove

Vanishing Theorem (E., Takeda)

$$\mathcal{S}'(\mathrm{GSpin}(V))^{\widetilde{\mathrm{GSpin}}(W),\chi} = 0$$

Vanishing Theorem implies Main Theorem

- Uses Cor 1.1 from AGRS. Let G be an lctd group and $H \subseteq G$ a closed subgroup, both unimodular. Assume there exits an involution $\sigma : G \to G$ such that $\sigma(H) = H$ and every distribution on G invariant under the conjugation action of H is also fixed by σ; namely if $T \in \mathcal{S}'(G)^H$, then $\sigma \cdot T = T$, where the action of σ on T is defined in the obvious way. Then for all $\pi \in \mathrm{Irr}(G)$ and $\tau \in \mathrm{Irr}(H)$, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, \tau^\vee) \cdot \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi^\vee, \tau) \leq 1,$$

- For $\pi \in \mathrm{Irr}(\mathrm{GSpin}(V))$, we have

$$\pi^\vee \simeq \begin{cases} \omega_\pi^{-1} \otimes \pi, & \text{if } n = 2k \text{ with } k \text{ even, or } n = 2k+1; \\ \omega_\pi^{-1} \otimes \pi^\delta, & \text{if } n = 2k \text{ with } k \text{ odd,} \end{cases}$$

  where π^δ is the representation obtained by twisting π by any $\delta \in \mathrm{GPin}(V) \setminus \mathrm{GSpin}(V)$.

Proof of Vanishing Theorem

Vanishing Theorem (E., Takeda)

$$\mathcal{S}'(\mathrm{GSpin}(V))^{\widetilde{\mathrm{GSpin}}(W),\chi} = 0$$

- Classical groups (AGRS, Waldspurger) Induction argument which boils down to the centralizer of a semisimple element. For $\mathrm{GL}(n), \mathrm{U}(n), \mathrm{O}(n), \mathrm{SO}(n)$, it is a direct product.
- For GSpin the centralizer of a semisimple element has a part which is not a direct product, i.e.

$$\{(g_1, \ldots, g_m) \in \mathrm{GSpin}(V_1) \times \cdots \times \mathrm{GSpin}(V_m) : N(g_1) = N(g_2) \cdots = N(g_m)\}$$

Proof of Vanishing Theorem

Vanishing Theorem (E., Takeda)

$$\mathcal{S}'(\mathrm{GSpin}(V))^{\widetilde{\mathrm{GSpin}}(W),\chi} = 0$$

- Reduces to showing

$$\mathcal{S}'(\mathrm{GSpin}(V) \times V)^{\mathrm{GSpin}(V),\chi} = 0$$

  using Frobenius descent, and then Harish-Chandra's descent
- We use that the involution σ_n and g are conjugate in $\mathrm{GSpin}(V)$ and prove some isomorphisms to reduce to AGRS and Waldspurger

The local Gan-Gross-Prasad conjecture

The local Gan-Gross-Prasad conjecture asserts when

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, \sigma) = 1$$

Several ingredients to the proof

- Multiplicity at most one theorem
- Endoscopic Classification of Representations
- local GGP proof

Endoscopic Classification of Representations of GSpin

As with Arthur, we assume our group is quasi-split. From BIRS conference, it appears there is a way to handle inner forms using a method of Kaletha or Gan using a theta correspondence.

Gee-Taibi: Arthur's multiplicity formula for GSp_4 and restriction to Sp_4

- GSpin_{2n+1}: Split group
- $\mathsf{GSpin}_{2n}^{\alpha}$: Quasi-split group, where $\alpha \in F^{\times}/(F^{\times})^2$.
- We have the spin double cover

$$0 \longrightarrow \mu_2 \longrightarrow \mathsf{Spin}_{2n}^{\alpha} \longrightarrow \mathsf{SO}_{2n}^{\alpha} \longrightarrow 0$$

- Set

$$\mathsf{GSpin}_{2n}^{\alpha} := (\mathsf{Spin}_{2n}^{\alpha} \times GL_1)/\mu_2$$

Endoscopic Classification of Representations of GSpin

- Let $\mu : \mathrm{GL}_1 \to Z(G)$ be dual to the similitude factor $\hat{\mu} : \hat{G} \to \mathbb{C}^\times$.
- Recall that ${}^L G = \hat{G} \rtimes W_F$
- When $\mathrm{GSpin}_{2n}^{\alpha}, \alpha \neq 1$, then the action of W_F factors through $Gal(F(\sqrt{\alpha}/F) = \{1, \sigma\}$
- σ acts by outer conjugation on GSO_{2n} and $1 \rtimes \sigma$ is the element of $\mathrm{O}_{2n}(\mathbb{C})$ that switches e_n and e_{n+1} and fixes the other e_i.
- We have the standard representation

$$Std_G :^L G \to \mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$$

Endoscopic datum

Endoscopic datum for a connected reductive group G over a local field F is a tuple $(H, \mathcal{H}, s, \xi)$

- H is a quasi-split connected group over F
- $\xi : \hat{H} \to \hat{G}$ is a continuous embedding
- $\mathcal{H}$ is a closed subgroup of ${}^L G$ that surjects onto W_F with kernel $\xi(\hat{H})$ such that the induced outer action of W_F on $\xi(\hat{H})$ coincides with the usual one on $\hat{H}$ transported by ξ and such that there exists a continuous splitting $W_F \to \mathcal{H}$
- $s \in \hat{G}$ is a semisimple element whose connected centralizer in $\hat{G}$ is $\xi(\hat{H})$ and such that the map $W_F \to \hat{G}$ induced by $h \in \mathcal{H} \mapsto shs^{-1}h^{-1}$ takes values in $Z(\hat{G})$ and is trivial in $H^1(W_F, Z(G))$

Endoscopic groups for GSpin

If $G = \mathrm{GSpin}_{2n+1}$, then

$$H = (\mathrm{GSpin}_{2a+1} \times \mathrm{GSpin}_{2b+1}) / \mathrm{GL}_1$$

with $a + b = n, ab \neq 0$

If $G = \mathrm{GSpin}^{\alpha}_{2n}$, then

$$H = (\mathrm{GSpin}^{\beta}_{2a} \times \mathrm{GSpin}^{\gamma}_{2b}) / \mathrm{GL}_1$$

where $a + b = n, \beta\gamma = \alpha, \beta \neq 1$ if $a = 1$ and $\gamma = 1$ if $b = 1$.

Let

$$\theta : GL_N \times GL_1 \to GL_N \times GL_1$$
$$(g, x) \mapsto (J^t g^{-1} J^{-1}, x \det g)$$

- θ fixes the pinning of G
- $GL_N \times \mathrm{GL}_1 \rtimes \{\theta\}$ is the connected component of $GL_N \times GL_1 \rtimes \langle\theta\rangle$

Thank you!!