Alcove Walks in Affine Flags & Matrix Coefficients

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Based on joint work with
Yusra Naqvi, Petra Schwer, Anne Thomas;
and Ben Brubaker

Affine Flag Varieties

Local Fields:

- $F = \mathbb{F}_q((t))$
- $\mathcal{O} = \mathbb{F}_q[[t]]$
- project $\mathcal{O} \to \mathbb{F}_q$

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The *affine flag variety* is the quotient $G(F)/I$.

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The *affine flag variety* is the quotient $G(F)/I$.

Theorem (Affine Bruhat Decomposition)

$$G(F)/I = \bigsqcup_{w \in \widetilde{W}} IwI/I$$

Example ($G = \mathrm{SL}_3$)

B is upper-triangular matrices and T is the diagonal matrices:

$$T = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \subset B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \subset G$$

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Given these choices, the Iwahori subgroup I is then

$$I = \left\{ \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \mathcal{O}^\times & \mathcal{O} \\ t\mathcal{O} & t\mathcal{O} & \mathcal{O}^\times \end{pmatrix} \right\} \subset G(F)$$

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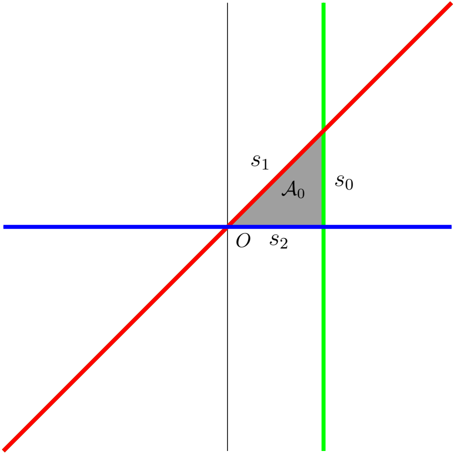
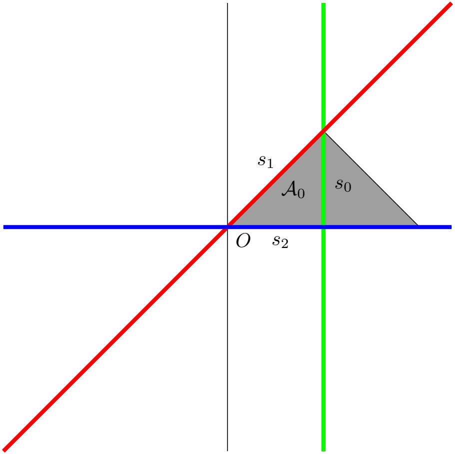
$\widetilde{W} = \widetilde{S_3}$ is the affine symmetric group

Example: $G = Sp_4$



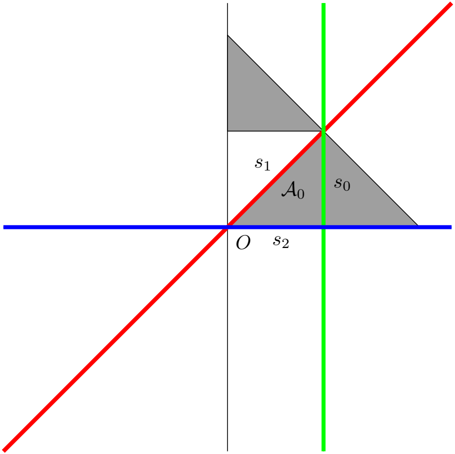
Generated by three reflections s_0, s_1, and s_2.

Affine Flag Varieties

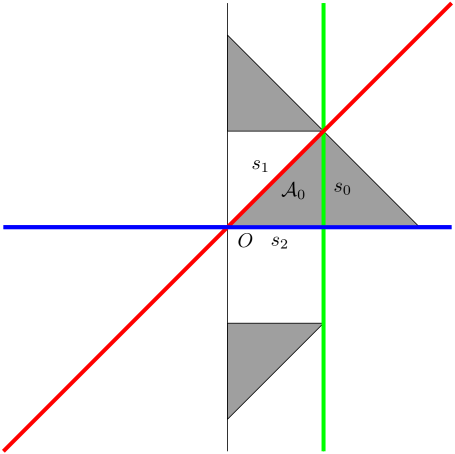


The result of applying s_0 to the base alcove $\mathcal{A}_\circ$.

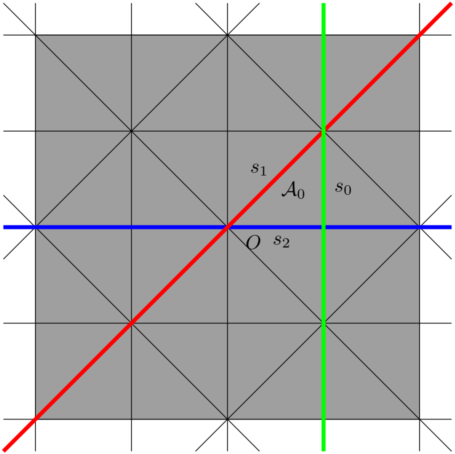
Affine Flag Varieties



The result of applying s_1 to $s_0(\mathcal{A}_\circ)$ is $s_1 s_0(\mathcal{A}_\circ)$.



The result of applying s_2 to $s_1 s_0(\mathcal{A}_\circ)$ is $s_2 s_1 s_0(\mathcal{A}_\circ)$.



Elements in $\widetilde{W} = N_G(T(F))/T(\mathcal{O})$ correspond to alcoves in $\mathbb{R}^r$.

Groups over Local Fields:

- Fix $G \supset B \supset T$ a Borel containing a split maximal torus
- The unipotent subgroups satisfy $B = TU$ and $B^- = TU^-$
- The maximal compact subgroup is $K = G(\mathcal{O})$

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Example ($G = \mathrm{SL}_3$)

$$U = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} = B$$

$$U^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\} = B^-$$

$$K = \left\{ \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O}^\times & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O}^\times \end{pmatrix} \right\} \subset G(F)$$

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Matrix Coefficients

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Definition

For λ an antidominant weight, the Whittaker coefficient is

$$W(t^{\lambda}) = \int_{U^-} v_K(ut^{\lambda})\psi(u)du,$$

where here

- $v_K \in \mathrm{Ind}_B^G(\chi)^K$ for χ a character of B trivial on K, and
- ψ a character of U^-.

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To evaluate, we integrate over the double Iwasawa cells

$$C_{\lambda\mu} := U^- t^\lambda K \cap U^+ t^\mu K,$$

inside the affine Grassmannian $G(F)/K$.

Matrix Coefficients

Using $K = \bigcup_{w \in W} IwI$ and that λ is antidominant, we can write

$$U^- t^\lambda K = \bigcup_{w \in W} U^- t^\lambda wI = \bigcup_{\substack{w \in W \\ v \in \widetilde{W}}} U^- t^\lambda wI \cap IvI.$$

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$$U^- t^\lambda K = \bigcup_{w \in W} U^- t^\lambda w I = \bigcup_{\substack{w \in W \\ v \in \widetilde{W}}} U^- t^\lambda w I \cap IvI.$$

Therefore, we get a decomposition of the double Iwasawa cells

$$\begin{aligned} C_{\lambda\mu} &= U^- t^\lambda K \cap U^+ t^\mu K \\ &= \bigcup_{\substack{w, w' \in W \\ v \in \widetilde{W}}} U^- t^\lambda w I \cap IvI \cap U^+ t^\mu w' I. \end{aligned}$$

Matrix Coefficients

We can then rewrite the Whittaker coefficient as

$$W(t^\lambda) = \frac{1}{\sum\limits_{w \in W} q^{\ell(w)}} \sum_{\substack{w,w' \in W \\ v \in \widetilde{W}}} \chi(t^\mu) \left(\int\limits_{U^- t^\lambda w I \cap I v I \cap U^+ t^\mu w' I} \psi(u) du \right).$$

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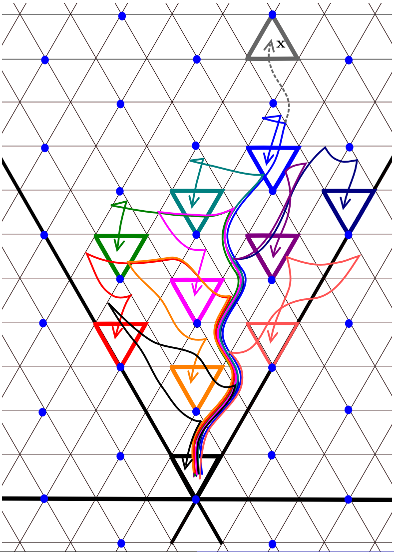
Theorem (Brubaker-M)

For $SL_2(\mathbb{F}_q((t)))$, we recover Tokuyama's formula bijectively. Each Gelfand-Tsetlin pattern corresponds to a stratum in $C_{\lambda\mu}$; the statistics are recording its weighted volume.

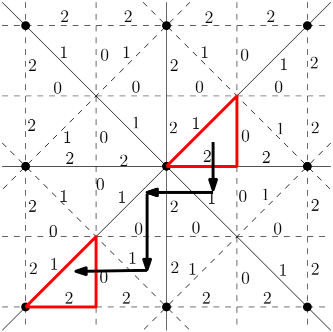
Labeled Folded Alcove Walks

Proof = labeled folded alcove walks



Definition

An *alcove walk* is a sequence of moves from an alcove to an adjacent alcove obtained by crossing an affine hyperplane.

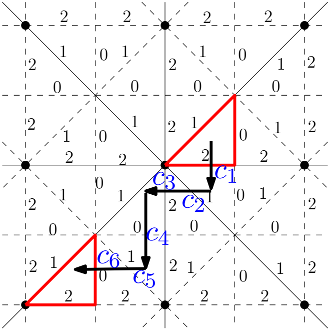


An alcove walk corresponding to the word $s_2 s_1 s_2 s_0 s_1 s_0$.

Labeled Folded Alcove Walks

Theorem (Steinberg 1967, Parkinson-Ram-C. Schwer 2009)

$$\{\textit{labeled alcove walks}\} \longleftrightarrow \{\textit{double cosets } IwI/I\}$$

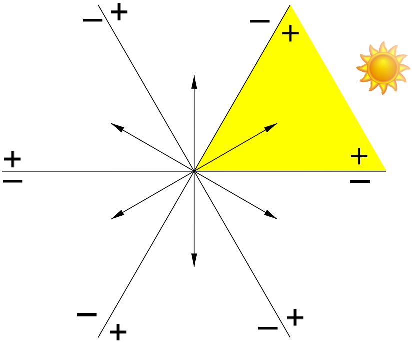


All points in $Sp_4(F)/I$ which lie in $Is_{212010}I$, varying $c_i \in \mathbb{F}_q$.

Labeled Folded Alcove Walks

For each $w \in W$, the periodic orientation on hyperplanes induced by w is defined such that:

1. alcove w is on the positive side of H_α and
2. hyperplanes parallel to H_α have the same orientation.



Standard orientation on hyperplanes induced by $w = 1$

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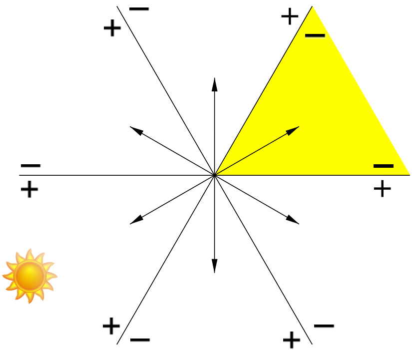


Opposite orientation on hyperplanes induced by $w = w_0$

Labeled Folded Alcove Walks

Definition

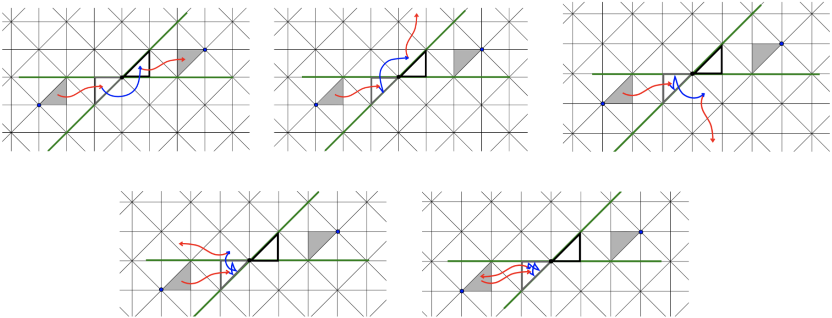
A fold is positive if the fold occurs on the positive side of the hyperplane, with respect to a fixed periodic orientation.

Rules for creating folded alcove walks:

1. Can only do positive folds.
2. Must fold tail-to-tip.

Labeled Folded Alcove Walks

Positively folding an alcove walk with the opposite orientation.



Labeled Folded Alcove Walks

Theorem (Parkinson-Ram-C. Schwer 2009)

For any $x, y \in \widetilde{W}$,

$$\left\{ \begin{array}{c} \textit{positively folded labeled alcove walks} \\ \textit{folded from } x \textit{ ending at } y = t^{\lambda}w \end{array} \right\} \longleftrightarrow UyI \cap IxI.$$

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Remark: These theorems have also been substantially generalized in joint work with Naqvi, P. Schwer, and Thomas.

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Example

In $SL_2(\mathbb{F}_q((t)))$,

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Example

In $SL_2(\mathbb{F}_q((t)))$, the elements of $U^-(F)$ such that

$$U^- t^{(1,-1)} I \cap I t^{(3,-3)} s I \neq \emptyset$$

Alcove Walks & Matrix Coefficients

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ψ is trivial on $\mathcal{O}$, and so this path contributes 0 to $W(t^{(1,-1)})$.

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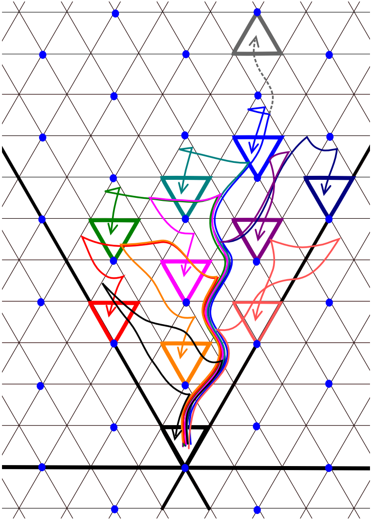
These combinatorial tools provide a flexible (and fun!) framework for many vast generalizations of this mini theorem.



Thank you!