Local Gross-Prasad conjecture over archimedean local fields

Cheng Chen

University of Minnesota

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Setting of the Conjecture

- F a local field;
- S is an even-dimensional and split quadratic space;
- D is an anisotropic line;
- V, W non-deg quadratic spaces $/F$ s.t. $V = W \oplus S \oplus D$.

Gross-Prasad triple (G, H, ξ)

- $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$, $H = \Delta\mathrm{SO}(W) \ltimes N$;
- ξ a unitary character on $H(F)$ induced from a generic unitary character ξ_N on $N(F)$.

Example

- **Codimension-one case**: $S = 0$. *In this case,* $\dim V = \dim W + 1$, $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$, $H = \Delta\mathrm{SO}(W)$ *and* ξ *is the trivial character of* $H(F)$.
- **Whittaker Case**: $\dim W = 0, 1$. *In this case,* $G = \mathrm{SO}(V)$ *is quasi-split,* $H = N$ *maximal unipotent and* ξ *is a Whittaker character.*

Multiplicity For an irreducible admissible(nonarchimedean)/ Casselman-Wallach(archimedean) representation π of $G(F)$

$$m(\pi) = \dim \mathrm{Hom}_{H(F)}(\pi, \xi)$$

Theorem (AGRS10, GGP12; SZ12, JSZ11, SZ10)

$$m(\pi) \leqslant 1$$

Pure inner form of spherical pairs

For every $\alpha \in H^1(F, H) \to H^1(F, G)$, we have pure inner forms

- $G_\alpha = \mathrm{SO}(V_\alpha) \times \mathrm{SO}(W_\alpha)$;
- $H_\alpha = \mathrm{SO}(W_\alpha) \rtimes N$;

where $V_\alpha = W_\alpha \oplus H \oplus D$. Together with ξ_α induced by ξ_N, we have a Gross-Prasad triple

$$(G_\alpha, H_\alpha, \xi_\alpha)$$

Vogan packet

Given a generic L-parameter $\varphi : \mathcal{W}_F \to {}^L G (= {}^L G_\alpha)$, for every $\alpha \in H^1(F, G)$, the LLC gives L-packets $\Pi_\varphi(G_\alpha)$,

$$\Pi_\varphi^{\mathrm{Vogan}} = \coprod_{\alpha \in H^1(F,G)} \Pi_\varphi(G_\alpha)$$

In was proved by Shelstad over archimedean fields and conjectured by Vogan over nonarchimedean fields that, fixing a Whittaker datum, there exists a non-degenerate $\mathbb{Z}/2\mathbb{Z}$-bilinear pairing

$$\Pi_\varphi^{\mathrm{Vogan}} \times \mathcal{S}_\varphi \to \{\pm 1\}$$

where

$$\mathcal{S}_\varphi = \pi_0(\mathrm{Cent}_{\widehat{G}}(\mathrm{Im}(\varphi)))$$

Local Gross-Prasad conjecture

Conjecture (GP92,GP94)

Given a generic parameter φ

- **Multiplicity-one for Vogan packets** *There exists exactly one representation in the Vogan packet $\Pi_{\varphi}^{\mathrm{Vogan}}$ with multiplicity equal to one.*
- **Epsilon-Dichotomy** *Gross and Prasad defined a character $\chi_{\varphi,H}$ of $\mathcal{S}_{\varphi}$ using local epsilon factors. They also specified a choice of Whittaker datum. Under this Whittaker datum, the distinguished representation in the Vogan packet corresponds to $\chi_{\varphi,H}$.*

Example: $\mathrm{SO}(3) \times \mathrm{SO}(2)$, discrete series case

- $G = \mathrm{SO}(3,0) \times \mathrm{SO}(2,0)$, $H = \Delta\mathrm{SO}(2,0)$, $\xi = 1_{H(\mathbb{R})}$
- $G_\alpha = \mathrm{SO}(3,0) \times \mathrm{SO}(2,0)$ or $\mathrm{SO}(1,2) \times \mathrm{SO}(0,2)$
- ${}^L G_\alpha = \mathrm{Sp}(2,\mathbb{C}) \times \mathrm{O}(2,\mathbb{C}) \subset \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})$
- For non-negative integer l, $\varphi_l : \mathcal{W}_\mathbb{R} = \mathbb{C}^\times \coprod \mathbb{C}^\times j \to \mathrm{GL}(2,\mathbb{C})$

$$z \mapsto \begin{pmatrix} |z|^{2t}(\frac{z}{|z|})^l & \\ & |z|^{2t}(\frac{z}{|z|})^{-l} \end{pmatrix} \quad j \mapsto \begin{pmatrix} 0 & 1 \\ (-1)^l & 0 \end{pmatrix}$$

- When l is even, $\varphi_l : \mathcal{W}_\mathbb{R} \to \mathrm{O}(2,\mathbb{C})$, LLC gives

$$\text{One-dim } \mathrm{SO}(2,0)(\mathbb{R})\text{-repn } \pi_l : \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix} \mapsto e^{i\theta l\pi}$$

- When l is odd, $\varphi_l : \mathcal{W}_\mathbb{R} \to \mathrm{Sp}(2,\mathbb{C})$, LLC gives

$$l\text{-dim } \mathrm{SO}(3,0)(\mathbb{R})\text{-repn } F_l \text{ with highest weight } \frac{l-1}{2}$$

$$\text{A discrete series repn } D_l \text{ of } \mathrm{SO}(1,2)(\mathbb{R}) = \mathrm{PGL}(2,\mathbb{R})$$

Example: $\mathrm{SO}(3) \times \mathrm{SO}(2)$, discrete series case

Given an odd integer l and an even integer n

$$m(F_l \otimes \pi_n) = \mathrm{Hom}_{H(F)}(F_l \otimes \pi_n, 1) = 1 \text{ iff } l > n$$

$$m(D_l \otimes \pi_n) = \mathrm{Hom}_{H(F)}(D_l \otimes \pi_n, 1) = 1 \text{ iff } l < n$$

The component group $\mathcal{S}_{\varphi_l} = \{\pm 1\}$

$$\begin{aligned}\chi_{\varphi_l \times \varphi_n, H}(-1, 1) &= (-1)^{\frac{4}{4}} \varepsilon(\varphi_l \otimes \varphi_n) = -\varepsilon(\varphi_{l+n} \oplus \varphi_{|l-n|}) \\ &= -i^{l+n+1} i^{|l-n|+1} = i^{2max\{l,n\}} \\ &= \begin{cases} 1 & \text{if } l < n \\ -1 & \text{if } l > n \end{cases}\end{aligned}$$

Related results (local Gross-Prasad conjecture)

Over nonarchimedean local fields

- Waldspurger proved the **full conjecture** for **tempered parameters**
- Mœglin and Waldspurger proved the **full conjecture** for **generic parameters**

Over archimedean local fields

- Möllers proved the **full conjecture** for **codimension-one case** over $\mathbb{C}$
- Luo proved the **multiplicity-one part** of the conjecture for **tempered parameters** over $\mathbb{R}$
- C.-Luo-Wan proved the **epsilon-dichotomy part** of the conjecture for **tempered parameters** over $\mathbb{R}$
- C. proved the **full conjecture** for **generic parameters** over $\mathbb{R}$ and $\mathbb{C}$

Related results (local Gan-Gross-Prasad conjecture for unitary groups)

Over nonarchimedean local fields

- Beuzart-Plessis proved the **full conjecture** for **tempered parameters**
- Gan and Ichino proved the **full conjecture** for **generic parameters**

Over archimedean local fields

- Beuzart-Plessis proved the **multiplicity-one part** of the conjecture for **tempered parameters**
- Xue proved the **full conjecture** for **tempered parameters** using theta correspondence
- Xue proved the **full conjecture** for **generic parameters**

(These results are before our proof of Gross-Prasad conjecture.)

Waldspurger's proof for Epsilon-dichotomy (tempered, nonarchimedean)

Step 1: Local trace formula (on $\mathrm{SO}(V)$)

$$J_{\mathrm{spec}}(f) = J_{\mathrm{geom}}(f) \Longrightarrow m(\pi) = m_{\mathrm{geom}}(\pi)$$

Step 2: Express the epsilon factor in terms of the Harish-Chandra characters $\Theta_{\Pi_\varphi(G)}$ using twisted endoscopy and twisted local trace formula (on $\mathrm{GL}(n) \rtimes \theta$, θ =transpose inverse)

$$J_{\mathrm{spec}}(\widetilde{f}) = J_{\mathrm{geom}}(\widetilde{f}) \Longrightarrow m(\widetilde{\pi}) = m_{\mathrm{geom}}(\widetilde{\pi})$$

Step 3: Study $m_{\mathrm{geom}}(\pi)$ under endoscopy with a multiplicity formula.

Our proof (tempered, $F = \mathbb{R}$)

Step 1: Local trace formula (proved by Luo)

Step 2: Easier over archimedean fields. When $\dim V > 3$, the parameter φ_V is either be parabolic type or endoscopic type. In both case, we are able to reduce the question to smaller cases. So the question can be reduced to the Waldspurger's model.

Step 3: Instead of considering the geometric multiplicity for one pure inner form, we sum over the geometric multiplicity of all pure inner forms with the same Kottwitz sign.
(We computed the Fourier transform of orbital integrals of regular nilpotent conjugacy classes at some special reg. s.s conj. classes)

Proof for generic case

Mœglin and Waldspurger's framework

Step 1: A structure theorem showing every representation in generic packets can be expressed as a parabolic induction;
(The proof for the structure theorem uses **Casselman-Shahidi's standard module conjecture** (proved by Muić) and an **irreducibility criterion**)

Step 2: Reduction from co-dimension one cases to tempered cases with a **mathematical induction**;
(The induction steps were proved with a **multiplicity formula**.)

Step 3: Reduction from general cases to co-dimension one cases using tshe **multiplicity formula** in **Step 2**.

Proof for generic case: $F = \mathbb{R}$

Step 1: **standard module conjecture** was proved by Vogan; **irreducibility criterion** was proved by Speh and Vogan.

Step 2: With **a multiplicity formula** as in Mœglin and Waldspurger ($s_{\pi_W,\sigma} = s_{\pi_W} + s_\sigma$), the **mathematical induction** won't work.
I proved **a refined multiplicity formula** ($s_{\pi_W,\sigma} = s_{\pi_W} - s_\sigma$) so that the mathematical induction works.
(Multiplicity formula are proved using Schwartz homologies)

Step 3: Reduction from general cases to co-dimension one cases using a **multiplicity formula**.

The **multiplicity formula** are in the form of

$$m(I_P^G(|\det|^s \sigma \otimes \pi_V)\widehat{\otimes}\pi_W) = m(\pi_V \widehat{\otimes} \pi_W) \quad \text{for } \mathrm{Re}(s) \geqslant s_{\pi_W,\sigma}$$

Proof for generic case: $F = \mathbb{C}$

(Codimension-one case proved by Möllers.) Notice that $|\Pi_\varphi^{\mathrm{Vogan}}| = 1$.

- **Step 1** We can find a Borel subgroup $B = B_V \times B_W$ of G such that $H \cap B = 1$ and BH is Zariski-open in G.
- **Step 2** A $(B(F) \times H(F), \delta_{B(F)}^{1/2}\sigma \times \xi)$-equivalent tempered measure

$$\mu = \delta_{B(F)}^{-1/2}\sigma^{-1}(b)\xi(h)dbdh.$$

  can be constructed on $B(F)H(F)$
- **Step 3** From [GSS 16], this measure can be "extended" to a nonzero $(B(F) \times H(F), \delta_{B(F)}^{1/2}\sigma \times \xi)$-equivalent tempered distribution on $G(F)$.
- **Step 4** We can construct a nonzero element in

$$\mathrm{Hom}_H(I_B^G(\sigma), \xi)$$

  with this distribution.

Thank you!