A Converse Theorem without Root Numbers

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What is a Converse Theorem?

"A converse theorem characterizes automorphic forms in terms of analytic properties of their L-functions."

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- Can associate to f the completed L-function

$$\Lambda(s; f) = (2\pi)^{-s}\,\Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s}$$

A classical result

Theorem (Hecke '36)

f is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight k if and only if $\Lambda(s; f)$

(i) *has an analytic continuation to the whole s-plane*

(ii) *is bounded in vertical strips*

(iii) *satisfies the functional equation*

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The if part of this statement is a prototypical example of a Converse theorem.

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- A single functional equation does not suffice in this case.
- Weil (1967) proved a converse theorem requiring a family of ‘twisted’ L-functions.

Weil's setup

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- Associate to them a pair of functions f, $\tilde{f}$

$$f(z) = \sum_{n=1}^{\infty} \lambda_n n^{\frac{k-1}{2}} e^{2\pi i n z} \quad \text{and} \quad \tilde{f}(z) = \sum_{n=1}^{\infty} \tilde{\lambda}_n n^{\frac{k-1}{2}} e^{2\pi i n z}.$$

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- Define the L-function twisted by the Dirichlet character χ

$$\Lambda(s; \lambda, \chi) := \Gamma_{\mathbb{C}}\left(s + \frac{k-1}{2}\right) \sum_{n=1}^{\infty} \lambda_n \chi(n) n^{-s}.$$

Weil's Converse theorem

Weil showed that if the L-functions defined above are 'nice' for every Dirichlet character χ with conductor q relatively prime to N and satisfy the functional equation

$$\Lambda(s;\lambda,\chi) = C_\chi (q^2N)^{\frac{1}{2}-s}\Lambda(1-s;\tilde{\lambda},\bar{\chi}),$$

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The complex number $C_\chi = i^k\xi(q)\chi(-N)\tau(\chi)/\tau(\bar{\chi})$, with $\tau(\chi)$ the Gauss sum for χ and ξ the nebentypus character of f, is called the *root number* of the functional equation.

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Assume

- the central character χ of π is an idele class character, and
- the L-function $L(s;\pi) = \prod_v L(s;\pi_v)$ converges in some right half plane.

Representation theoretic statement

Theorem (Jacquet and Langlands '70)

Suppose, for each idele class character ω, the twisted L-functions $L(s; \pi \otimes \omega)$ and $L(s; \check{\pi} \otimes \omega^{-1})$ can be continued to entire functions of s, are bounded in vertical strips and satisfy the functional equation

$$L(s; \pi \otimes \omega) = \varepsilon(s; \pi \otimes \omega) L(1 - s; \check{\pi} \otimes \omega^{-1}).$$

Then π is a cuspidal automorphic representation.

Jacquet-Langlands proof (idea)

- For each $\xi = \otimes_v \xi_v \in V_\pi$ let $W_\xi = \prod_v W_{\xi_v} \in \mathcal{W}(\pi, \psi)$ and set

$$\varphi_\xi(g) = \sum_{\gamma \in k^\times} W_\xi \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right).$$

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  This gives another embedding of π in a space of functions on $G(\mathbb{A})$.
- Show, for all g

$$\varphi_\xi(wg) = \varphi_\xi(g),$$

  where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This shows φ_ξ, and hence π, is automorphic.

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Theorem (Booker '19)

Let π be an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with automorphic central character and conductor N. Suppose each π_v is unitary and that π_∞ is a discrete series or limit of discrete series representation. For each unitary idele class character ω of conductor q coprime to N, suppose the completed L-functions $\Lambda(s, \pi \otimes \omega)$ and $\Lambda(s, \check{\pi} \otimes \omega^{-1})$ continue to entire functions on $\mathbb{C}$, are bounded in vertical strips and satisfy a functional equation of the form

$$\Lambda(s, \pi \otimes \omega) = \epsilon_\omega (Nq^2)^{\frac{1}{2} - s} \Lambda(1 - s, \check{\pi} \otimes \omega^{-1})$$

for some complex number ϵ_ω. Then there is a cuspidal automorphic representation $\Pi = \otimes_v \Pi_v$ such that $\Pi_\infty \cong \pi_\infty$ and $\Pi_v \cong \pi_v$ at every finite v at which π_v is unramified.

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- The values for ϵ_ω require some additional (natural) constraints

The case of a rational function field

- $F = \mathbb{F}_q(t)$
- $\mathbb{A}$ the adele ring of F
- Fix a place ∞ of F
- π an irreducible admissible generic representation of $GL_2(\mathbb{A})$ with conductor $\mathfrak{a}$, and automorphic central character χ

The case of a rational function field

Theorem (A)

For each unitary idele class character ω whose conductor $\mathfrak{f}$ is disjoint from $\mathfrak{a}$, assume the L-function $L(s, \pi \otimes \omega)$ continues to a holomorphic function on $\mathbb{C}$ and satisfies the functional equation

$$L(s, \pi \otimes \omega) = \epsilon_\omega |\mathfrak{a}\mathfrak{f}^2|^{s-\frac{1}{2}} L(1-s, \check{\pi} \otimes \omega^{-1}),$$

where the complex number ϵ_ω is such that

(i) *if ω is unramified or ramified only at ∞, then $\epsilon_\omega = 1$, and*

(ii) *for any unramified unitary idele class character ω', we have $\epsilon_{\omega'\omega} = \epsilon_\omega$.*

Then there is a cuspidal automorphic representation Π so that $\Pi_v \cong \pi_v$ at all places v away from the support of the divisor $\mathfrak{a}$.

Key ingredients in the proof

- Basic idea of showing $\varphi_\xi(wg) = \varphi_\xi(g)$ remains the same

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- Average the subsequent equality we get for the twisted φ and its dual over all unitary characters mod a fixed divisor
- Primes in arithmetic progression in a rational function field

Twists mod a conductor

- Let $\xi^0 = \otimes_v \xi_v^0 \in V_\pi$, where ξ_v^0 is the new vector in V_{π_v}. Like before, set

$$\varphi_{\xi^0}(g) = \sum_{\gamma \in k^\times} W_{\xi^0}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

- For ω an idele class character, define

$$I(s; \varphi_{\xi^0}, \omega) = \int_{\mathbb{A}^\times} W_{\xi^0}\left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right) \omega(u)|u|^{s-\frac{1}{2}}\, du.$$

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- If ω is ramified at any place π is unramified, this integral becomes zero.

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$$\varphi_{\tau,\mathfrak{f}_0}(x, y) = \int_{\prod_v \mathcal{O}_v^\times} \tau(u)\varphi\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & wu \\ 0 & 1 \end{pmatrix}\right) du,$$

where w is an adele given in terms of $\mathfrak{f}_0$.

Twists mod a conductor

Working with the integral $I(s; \varphi_{\omega,\mathfrak{f}_0}, \omega)$ instead, we can pick out local L-factors of $L(s, \pi \otimes \omega)$ even at places where ω is ramified. By varying $\mathfrak{f}_0$, we get finer control on what terms in the Dirichlet series corresponding to $L(s, \pi \otimes \omega)$ we pick up.

Applications?

We can explore the role of root numbers in functional equations in the context converse theorems. The Langlands-Shahidi method gives a well developed theory of ε-factors, so I don't see any direct application. However, if we had a method of constructing L-functions that did not give precise ε-factors, converse theorems not requiring root numbers could be useful.

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Thank You!