The Cohomology of GU_n local Shimura varieties

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(with Kieu Hieu Nguyen)

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Motivation

Langlands Conjectures:

$$\left\{ \begin{array}{c} \text{automorphic} \\ \text{representations of } G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \widehat{G} - \text{valued} \\ \text{Galois representations} \end{array} \right\}$$

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- ***p*-adic fields**: Lubin–Tate spaces (GL_n), Rapoport–Zink spaces, local Shimura varieties, moduli of shtuka

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   - GSp_4 (Hamann): Gan–Takeda $\leftrightarrow$ Fargues–Scholze

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- The datum (GU_n, μ) determines a *basic* local Shimura variety whose cohomology $H_c^*(\mathbb{M}_\mu)$ has an action of $GU_n(\mathbb{Q}_p) \times GU_n(\mathbb{Q}_p) \times W_{E_\mu}$
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$$\mathrm{Mant}_\mu : \mathrm{Groth}(GU_n(\mathbb{Q}_p)) \to \mathrm{Groth}(GU_n(\mathbb{Q}_p) \times W_{E_\mu})$$

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- We study $\rho \in \mathrm{Irr}(GU_n(\mathbb{Q}_p))$ such that $\varphi_\rho : W_{\mathbb{Q}_p} \to {}^L GU_n$ is *supercuspidal*.

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Let φ be supercuspidal and $\rho \in \Pi_{\varphi}$. Then $\mathrm{Mant}_{\mu}(\rho)$ has an explicit description in terms of LLC:

$$\mathrm{Mant}_{\mu}(\rho) = \sum_{\rho' \in \Pi_{\varphi}} \rho' \boxtimes \mathrm{Hom}_{S_{\varphi}}(\tau_{\rho',\rho}, r_{\mu} \circ \varphi).$$

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Application (work in progress with Hamman and Nguyen)

Mok LLC and Fargues–Scholze LLC are compatible for GU_n as above.

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with associated "highest weight rep"

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   - Size 4: $r_\mu \circ \varphi|_{W_E}$ is a sum of three characters.

Size 1 Case

- Recall that in LLC, L-packets are controlled by rep theory of the centralizer group $S_{\varphi} = Z_{\widehat{GU_n}}(\varphi)$

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The decomposition of $\mathrm{Mant}_{\mu}(\rho)$ is also determined by rep theory of S_{φ}. In particular: the action of S_{φ} on V via r_{μ}.

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The decomposition of $\mathrm{Mant}_\mu(\rho)$ is also determined by rep theory of S_φ. In particular: the action of S_φ on V via r_μ.

- In the Size 1 case, S_φ acts trivially on V and we simply get

$$\mathrm{Mant}_\mu(\rho) = \rho \boxtimes r_\mu \circ \varphi|_{W_E}$$

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$$V \cong \chi_{\mathrm{triv}} \boxtimes \eta_{\mathrm{dim}\,1} \oplus \chi_{\mathrm{sgn}} \boxtimes \eta_{\mathrm{dim}\,2}$$

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- In other words, the Galois rep attached to ρ_j in $\mathrm{Mant}_\mu(\rho_i)$ is the $\iota(\rho_i) \otimes \iota(\rho_j)$-isotypic part of V.

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- In other words, Mant_μ knows about A-packets!