Study of parity sheaves arising from graded Lie algebras

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Overview

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- This defines a grading on $\mathfrak{g}$,

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n.$$

Clearly, $\mathfrak{g}_0 = \text{Lie}(G_0)$ and G_0 acts on $\mathfrak{g}_n$.

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- **In this talk we want to study $D^b_{G_0}(\mathfrak{g}_n, \Bbbk)$, where $\Bbbk$ has positive characteristic.**
- We can think of Local systems(locally constant sheaves) on a space X as the representations of the fundamental group, $\pi_1(X)$.
- Intersection cohomology complexes are some important objects in $D^b_H(X)$, the equivariant derived category on X.
- For example, intersection cohomology complexes in $D^b_G(\mathcal{N}_G)$($\mathcal{N}_G$ is the nilpotent cone) are in bijection with the pairs $(C, \mathcal{F})$, where C is a G-orbit and $\mathcal{F}$, a local system on C.

Example of $\mathcal{IC}$'s on Sp_4

Table: Orbits in $\mathfrak{sp}_4$

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orbits:	$\mathcal{O}[4]$	$\mathcal{O}[2^2]$	$\mathcal{O}[2,1^2]$	$\mathcal{O}[1^4]$
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This means we have 7 simple $\mathcal{IC}$'s in $D^b_{Sp_4}(\mathcal{N}_{Sp_4})$.

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  This means we have 7 simple $\mathcal{IC}$'s in $D^b_{Sp_4}(\mathcal{N}_{Sp_4})$.

- Intersection cohomology complexes on $D^b_{G_0}(\mathfrak{g}_n)$ are $\mathcal{IC}(\mathcal{O},\mathcal{L})$, where $\mathcal{O}\subset\mathfrak{g}_n$ is G_0-orbit and $\mathcal{L}$ is a local system on $\mathcal{O}$. Denote the set of all collection of these pairs by $\mathscr{I}(\mathfrak{g}_n)$.

$\mathcal{IC}$'s in $D^b_{G_0}(\mathfrak{g}_n)$ for $G = GL_4$.

Let $G = GL_4$ and $\chi : \mathbb{C}^\times \to G$ be defined as $t \to (t, 1, 1, t^{-1})$.

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Including the zero orbit and what we have in the table, we have 5 simple $\mathcal{IC}$'s in $D^b_{G_0}(\mathfrak{g}_{-1})$.

Parabolic induction and restriction on nilpotent cone

- We consider a diagram for a Levi subgroup L contained in a parabolic P with U_P, the unipotent radical of P.

$$\mathcal{N}_L \xleftarrow{\pi_P} \mathcal{N}_L + \mathfrak{u}_P \xrightarrow{e_P} G \times^P (\mathcal{N}_L + \mathfrak{u}_P) \xrightarrow{\mu_P} \mathcal{N}_G$$

$$i_P = \mu_P \circ e_P : \mathcal{N}_L + \mathfrak{u}_P \to \mathcal{N}_G.$$

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- Res_P^G is left adjoint to Ind_P^G.

Cuspidal pairs

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Definition

A pair $(C, \mathcal{E}) \in \mathscr{I}(G)$, is called cuspidal if $\mathrm{Res}_P^G \mathcal{IC}(C, \mathcal{E}) = 0$ for any proper parabolic P and a Levi factor $L \subset P$.

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- For nilpotent cones, how to get a $\mathcal{IC}(C, \mathcal{F})$ from the induction of some cuspidal pair have been studied in "Springer theory".

Induction and restriction in the graded setting.

- The diagram below defines induction and restriction.

$$\mathfrak{l}_n \xleftarrow{\pi} \mathfrak{p}_n \xrightarrow{e} G_0 \times^{P_0} \mathfrak{p}_n \xrightarrow{\mu} \mathfrak{g}_n$$

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} : D^b_{L_0}(\mathfrak{l}_n) \to D^b_{G_0}(\mathfrak{g}_n),$$

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- Here we need L to contain $\chi(\mathbb{C}^\times)$.
- $\mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}}$ is left adjoint to $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$.

Lusztig's work in characteristic 0.

Definition (Cuspidal on $\mathfrak{g}_n$)

$(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n)$ will be called cuspidal if there exists a cuspidal pair $(C, \mathcal{E}) \in \mathscr{I}(G)$, such that $C \cap \mathfrak{g}_n = \mathcal{O}$ and $\mathcal{L} = \mathcal{E}|_{\mathcal{O}}$.

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Theorem (Lusztig)

In characteristic 0*, for any* $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n)$*, there exists a Levi subgroup* L *contained in a parabolic subgroup* P *with a cuspidal pair* $(\mathcal{O}_L, \mathcal{L}') \in \mathscr{I}(\mathfrak{l}_n)$ *so that,* $\mathcal{IC}(\mathcal{O}, \mathcal{L})$ *appears as direct summand of* $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{IC}(\mathcal{O}_L, \mathcal{L}')$.

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- This theorem is not true when the characteristic of the field of sheaf coefficients is positive.
- Following the pattern from other works in modular representation theory, often the appropriate replacement for "semisimple complex" or $\mathcal{IC}$'s is "parity complex".

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- We denote the parity sheaf associated to the pair $(\mathcal{O}, \mathcal{L})$ by $\mathcal{E}(\mathcal{O}, \mathcal{L})$, where $\mathcal{E}(\mathcal{O}, \mathcal{L})|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}]$.

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- **Unlike for $\mathcal{IC}$'s, $\mathcal{E}(\mathcal{O}, \mathcal{L})$ does not exist automatically.**

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Assumption

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Let C be a nilpotent orbit in $\mathcal{N}_G$ and $\mathcal{L} \in \mathrm{Loc}_{\mathrm{f},G}(C, \Bbbk)$, then $\mathrm{H}^*_G(\mathcal{L})$ vanishes in odd degrees.

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Proposition (C)

Let $\mathcal{O}$ be a G_0-orbit in $\mathfrak{g}_n$ and $\mathcal{L} \in \mathrm{Loc}_{\mathrm{f},G_0}(\mathcal{O}, \Bbbk)$, then $\mathrm{H}^*_{G_0}(\mathcal{L})$ vanishes in odd degrees.

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Cohomology vanishing needed for parity sheaves to make sense.

Mautner's cleanness conjecture.

Definition (Clean)

A pair $(C, \mathcal{E}) \in \mathscr{I}(G, \mathbb{k})$ is called l-clean if the corresponding $\mathcal{IC}(C, \mathcal{E})$ has vanishing stalks on $\bar{C} - C$. Similarly, a pair $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n, \mathbb{k})$ is called l-clean if the corresponding $\mathcal{IC}(\mathcal{O}, \mathcal{L})$ has vanishing stalks on $\bar{\mathcal{O}} - \mathcal{O}$.

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Conjecture (1)

If the characteristic l of $\mathbb{k}$ is a "pretty good" prime for G, then every 0*-cuspidal pair $(C, \mathcal{E}) \in \mathscr{I}(G)$ is l-clean.*

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Conjecture (1)

If the characteristic l of $\Bbbk$ is a "pretty good" prime for G, then every 0-cuspidal pair $(C, \mathcal{E}) \in \mathscr{I}(G)$ is l-clean.

- This already holds when the characteristic does not divide the order of the Weyl group of the group G.

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If the characteristic l of $\Bbbk$ is a "pretty good" prime for G, then every 0-cuspidal pair $(C, \mathcal{E}) \in \mathscr{I}(G)$ is l-clean.

- This already holds when the characteristic does not divide the order of the Weyl group of the group G.
- If every irreducible factor of the root system of G is either of type A, B_4, C_3, D_5 or of exceptional types then also the conjecture holds.

Existence of parity for cuspidal pairs

Assuming the conjecture is true.

Theorem (C)

Under the assumption on the characteristic of $\mathbb{k}$, *any cuspidal pair* $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n, \mathbb{k})$ *is clean.*

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Under the assumption on the characteristic of $\mathbb{k}$, any cuspidal pair $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n, \mathbb{k})$ is clean.

Corollary

For any cuspidal pair $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n, \mathbb{k})$, $\mathcal{IC}(\mathcal{O}, \mathcal{L}) = \mathcal{E}(\mathcal{O}, \mathcal{L})$. Therefore parity sheaf exists for cuspidal pairs.

Conjecture 2

Conjecture (2)

Let P be a parabolic subgroup of G and L be its Levi subgroup. For a pair $(C, \mathcal{E}) \in \mathscr{I}(L)^{\mathrm{cusp}}$, $\mathrm{Ind}_P^G \mathcal{IC}(C, \mathcal{E})$ is a parity complex.

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- In the last section of our paper we have calculated $\mathrm{Ind}_P^G \mathcal{IC}(C, \mathcal{E})$ for SL_4 and Sp_4, where the conjecture is true.
- In our next paper we are trying to prove this conjecture for classical groups.

Main results

Assuming both the conjectures are true.

Theorem (C)

For any pair $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n)$ *, there exists a parabolic subgroup P with L, its Levi subgroup L and* $(\mathcal{O}_L, \mathcal{L}') \in \mathscr{I}(\mathfrak{l}_\mathrm{n})^{\mathrm{cusp}}$ *such that* $\mathcal{E}(\mathcal{O}, \mathcal{L})$ *occurs as direct summand of* $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{E}(\mathcal{O}_L, \mathcal{L}'))$.

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For any pair $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n)$, $\mathcal{E}(\mathcal{O}, \mathcal{L})$ exists.

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For any pair $(\mathcal{O}, \mathcal{L}) \in \mathscr{I}(\mathfrak{g}_n)$, there exists a parabolic subgroup P with L, its Levi subgroup L and $(\mathcal{O}_L, \mathcal{L}') \in \mathscr{I}(\mathfrak{l}_n)^{\mathrm{cusp}}$ such that $\mathcal{E}(\mathcal{O}, \mathcal{L})$ occurs as direct summand of $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{E}(\mathcal{O}_L, \mathcal{L}'))$.

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Theorem (C)

Let P be a parabolic subgroup of G with a Levi factor L, the induction functor sends parity complexes to parity complexes.

Calculation of Ind_P^G for $G = Sp_4$.

- $L = GL_1 \times Sp_2$ is a Levi subgroup. $(\mathcal{O}_{prin}, \mathcal{L}) \in \mathscr{I}(L)^{\mathrm{cusp}}$.

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Table: Stalks of $\mathrm{Ind}_P^G \mathcal{IC}(\mathcal{O}_{prin}, \mathcal{L})$

-

dim	$\mathcal{O}[4]$	$\mathcal{O}[2^2]$	$\mathcal{O}[2,1^2]$	$\mathcal{O}[1^4]$
-2			rank 1	
-3				
-4			rank 1	
-5				
-6				
-7				
-8				
-9				
-10	rank 1			

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Hence the parity condition is satisfied.

Thank you for your attention!