Diving into the Shallow End

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Supercuspidal Representations

Notation. $G = \mathrm{SL}_n(\mathbb{Q}_p),\ \mathrm{SO}_n(\mathbb{Q}_p),\ \mathrm{Sp}_n(\mathbb{Q}_p),$ etc. (connected, semisimple, split).

A **supercuspidal representation** is a smooth homomorphism

$$\pi : G \to \mathrm{GL}_k(\mathbb{C})$$

whose matrix coefficients are compactly supported modulo the center $Z(G) \subseteq G$.

Theorem (Yu 2001 and Kim 2007). *Suppose that p is large. Given a supercuspidal representation π of G, there exists*

- *H a compact-open subgroup of G*
- *V a finite-dimensional complex representation of H*

such that π is isomorphic to the compactly-induced representation

$$\mathrm{ind}_H^G(V) := \left\{ \phi : G \to V \;\middle|\; \begin{array}{c} \phi(hx) = h \cdot \phi(x) \\ \phi \text{ compactly supported} \end{array} \right\}.$$

Remarks: **1.** Yu's construction is dependent on the prime p.

2. Can be a relatively complicated construction.

3. $\mathrm{ind}_H^G(V)$ is irreducible only for very special pairs (H, V).

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An Example for $\mathrm{Sp}_4(\mathbb{Q}_2)$

Let $G = \mathrm{Sp}_4(\mathbb{Q}_2)$ with **Iwahori subgroup**

$$I := \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ (2) & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ (2) & (2) & \mathbb{Z}_2 & \mathbb{Z}_2 \\ (2) & (2) & (2) & \mathbb{Z}_2 \end{bmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}_2)$$

Then we consider **Moy-Prasad filtration** by open compact subgroups:

$$I > I_+ > I_{++} > \cdots > I_r > I_{r+} > \cdots$$

In this talk, we are interested in the following Moy-Prasad subgroups:

$$I_+ := \left(1 + \begin{bmatrix} (2) & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ (2) & (2) & \mathbb{Z}_2 & \mathbb{Z}_2 \\ (2) & (2) & (2) & \mathbb{Z}_2 \\ (2) & (2) & (2) & (2) \end{bmatrix}\right) \cap \mathrm{Sp}_4(\mathbb{Z}_2) \subseteq I$$

$$I_{++} := \left(1 + \begin{bmatrix} (2) & (2) & \mathbb{Z}_2 & \mathbb{Z}_2 \\ (2) & (2) & (2) & \mathbb{Z}_2 \\ (2) & (2) & (2) & (2) \\ (4) & (2) & (2) & (2) \end{bmatrix}\right) \cap \mathrm{Sp}_4(\mathbb{Q}_2) \subseteq I_+$$

$$I_1 := \left(1 + \begin{bmatrix} (2) & (2) & (2) & (2) \\ (4) & (2) & (2) & (2) \\ (4) & (4) & (2) & (2) \\ (4) & (4) & (4) & (2) \end{bmatrix}\right) \cap \mathrm{Sp}_4(\mathbb{Z}_2) \subseteq I_{++}$$

Goal: Classify all complex characters

$$\chi : I_+ \longrightarrow \mathbb{C}^\times,$$

and the supercuspidal representations which arise via compact induction.

Problem: The full commutator subgroup $[I_+, I_+]$ is complicated.

Method: Find less complicated, intermediate subgroups

$$[I_+, I_+] \subseteq Q \subseteq I_+$$

and classify the complex characters

$$\chi : I_+/Q \to \mathbb{C}^\times$$

and the supercuspidal representations which arise via compact induction.

Examples: 1. $Q = I_{++} \rightsquigarrow$ **simple supercuspidal repr** (Gross-Reeder 2010).

2. $Q = P_{++}$ for general parahoric subgroups $P \rightsquigarrow$ **epipelagic repr** (Reeder-Yu 2013).

3. $Q = P_1$ for general parahoric subgroups P (SSG current).

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Constructing Supercuspidals

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Epipelagic Characters

A character $\chi : I_+ \to \mathbb{C}^\times$ is called **epipelagic** if it is trivial on $Q = I_{++}$. The quotient

$$I_+/I_{++} \cong \bigoplus_{\alpha \text{ simple}} (\mathbb{F}_p)_\alpha$$

is an elementary abelian p-group, and so an epipelagic character is uniquely determined by giving any additive characters

$$\chi_\alpha : (\mathbb{F}_p)_\alpha \to \mathbb{C}^\times$$

and setting $\chi(x_\alpha) := \chi_\alpha(x_\alpha)$.

The group I/I_+ acts on characters of I_+. An epipelagic character is **stable** if over $\mathbb{Q}_p^{\mathrm{un}}$

- it belongs to a closed orbit.
- it has a finite stabilizer.

Theorem (Gross-Reeder 2013). *Let χ be an extension to ZI_+ of a stable epipelagic character with trivial stabilizer in I/I_+. Then the compactly-induced representation $\mathrm{ind}_{ZI_+}^G(\chi)$ is an irreducible supercuspidal representation of G.*

Remarks. **1.** Simple construction, independent of p.
2. Can make predictions about the Langlands parameter.
3. Connects LLC with geometric invariant theory (GIT).
4. Relatively few supercuspidal representations come in this form.

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Shallow Characters

A character $\chi : I_+ \to \mathbb{C}^\times$ is called **shallow** if it is trivial on $Q = I_1$. The quotient

$$I_+/I_1 = \langle (\mathbb{F}_2)_\alpha \mid \underbrace{0 < \alpha(\text{alcove}) < 1}_{\text{shallow affine root}} \rangle$$

is a p-group, not necessarily abelian. Then shallow characters of I_+ are uniquely determined by giving additive characters

$$\chi_\alpha : (\mathbb{F}_2)_\alpha \to \mathbb{C}^\times$$

for each shallow affine root and setting $\chi(x_\alpha) := \chi_\alpha(x_\alpha)$.

Theorem (SSG). *Given additive characters $\chi_\alpha : \mathbb{F}_2 \to \mathbb{C}^\times$ for each shallow affine root α, there exists a shallow character $\chi : I_+/I_1 \to \mathbb{C}^\times$ such that*

$$\chi(x_\alpha) = \chi_\alpha(x_\alpha)$$

for all $x_\alpha \in (\mathbb{F}_p)_\alpha$ if and only if for each shallow α, β the following relation holds:

$$1 = \prod \chi_{i\alpha + j\beta}(C_{\alpha\beta ij} x^i y^j)$$

for all $x, y \in \mathbb{F}_p$. Here the $C_{\alpha\beta ij}$ are constants from the Chevalley Commutator Formula.

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A New Supercuspidal Representation

With respect to the simple affine roots

$$\alpha_0 \Rightarrow \alpha_1 \Leftarrow \alpha_2,$$

a shallow character of I_+ is uniquely determined by specifying the additive characters $\epsilon_i : \mathbb{F}_2 \to \mathbb{C}^\times$ for $i = 1, \dots, 5$ and setting

shallow affine root α	additive character χ_α
α_0	ϵ_1
α_1	ϵ_2
α_2	ϵ_3
$\alpha_0 + \alpha_1$	ϵ_4
$\alpha_1 + \alpha_2$	ϵ_5
$\alpha_0 + 2\alpha_1$	ϵ_4
$\alpha_0 + \alpha_1 + \alpha_2$	1
$2\alpha_1 + \alpha_2$	ϵ_5

Example: If ϵ_1, ϵ_2 are trivial while $\epsilon_3, \epsilon_4, \epsilon_5$ are non-trivial, then the corresponding shallow character gives rise to a new supercuspidal representation of $\mathrm{Sp}_4(\mathbb{Q}_2)$.

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(1) In order to define a group homomorphism

$$\chi : P_+/P_1 \to \mathbb{C}^\times,$$

for parahoric subgroups P of a split group G, it is necessary and sufficient that χ be trivial on commutators

$$[U_\alpha, U_\beta] \subseteq \prod U_{i\alpha+j\beta}$$

for "shallow" affine roots α, β.

(2) We can use Mackey theory to determine which shallow characters give rise to supercuspidal representations G. Currently, no clear pattern has presented itself, but we have

- $\exists$ supercupsidals representations coming from characters not defined over $\mathbb{Q}_p^{\mathsf{un}}$.
- $\exists$ non-epipelagic supercuspidals coming from shallow characters that are defined over $\mathbb{Q}_p^{\mathsf{un}}$.

(3) Using methods similar to those of Gross-Reeder 2010 and Reeder-Yu 2013 we can make predictions for Langlands Parameters for supercuspidal representations coming from shallow characters defined over $\mathbb{Q}_p^{\mathsf{un}}$