The Formal Degree of a Regular Supercuspidal

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Outline

Let $\underline{G}$ be a split semisimple algebraic group over $\mathbb{Q}_p$, e.g.,

$$\underline{G} = \mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_n, \mathrm{Aut}(\text{octonions}), \ldots$$

$G := \underline{G}(\mathbb{Q}_p)$ is a locally compact group.

1. Formal degree
2. Yu's supercuspidals
3. Connection to Langlands correspondence

Part 1. Formal degree

The *unitary dual* of G is

$$\mathrm{Spec}_{\mathrm{u}}\, G := \frac{\{\text{irreducible unitary representations of } G\}}{\text{iso.}}$$

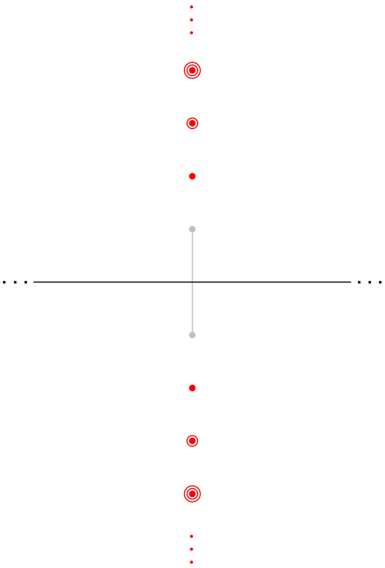
It carries natural structures of

- a topological space (Fell topology) and
- a Borel measure space (Plancherel measure).

Natural questions:

1. What is $\mathrm{Spec}_{\mathrm{u}}\, G$ as a set?
2. What is the topology and measure on $\mathrm{Spec}_{\mathrm{u}}\, G$?

Example: $SL_2(\mathbb{R})$



$\operatorname{Spec}_u SL_2(\mathbb{R})$ has both discrete parts and continuous parts.

- discrete series
- principal series
- complementary series
- (three more reps)

The nth discrete series has Plancherel measure n.

Formal degree

Definition

1. A unirrep π of G is discrete series if it is isolated in $\operatorname{Spec_u} G$.
2. The formal degree of π is its Plancherel measure.

Like the Plancherel measure, the formal degree depends (inversely) on a choice of Haar measure on G.

If $\dim \pi < \infty$, the formal degree of an irrep "equals" its dimension.

If $\dim \pi = \infty$, the formal degree can still be finite, and is thus a good replacement for dimension.

Part 2. Yu's supercuspidal representations

Parabolic induction reduces the study of $\mathrm{Spec}_\mathrm{u}\, G$ to the study of the supercuspidal representations of G.

(supercuspidal $\subsetneq$ discrete series)

In 2001, Yu constructed many supercuspidal representations.

In 2007, Kim proved that for a given G and for p large enough, Yu's construction yields all supercuspidals.

In 2018, Fintzen extended Kim's exhaustion theorem to an expected optimal bound on p (e.g., need $p > n$ if $G = \mathrm{SL}_n$).

Input data for the construction

Yu's construction takes as input a 5-tuple $\Psi = (\vec{\underline{G}}, x, \rho, \vec{r}, \vec{\phi})$ and outputs a supercuspidal π_Ψ.

- $\vec{\underline{G}} = (\underline{G}^0 \subsetneq \underline{G}^1 \subsetneq \cdots \subsetneq \underline{G}^d = \underline{G})$, twisted Levi subgroups.
- $x \in \mathcal{B}(\underline{G})$, the Bruhat-Tits building of G.
- ρ is a finite-dimensional irrep of G^0_x (stabilizer).
- $\vec{r} = (0 \leq r_0 < r_1 < \cdots < r_d)$.
- $\vec{\phi} = (\phi_0, \phi_1, \ldots \phi_d)$ with $\phi_i : G^i \to \mathbb{C}^\times$ a character.

There are various additional requirements, e.g.:

- $\text{c-Ind}_{G^0_x}^{G^0} \rho$ is supercuspidal;
- $r_i = \text{depth}\, \phi_i$.

Yu's construction mixes earlier constructions in depth zero (Moy-Prasad) and in positive depth (Adler).

Formal degree of a Yu supercuspidal: formula

Theorem (S)

The formal degree of π_Ψ *is*

$$\frac{\dim \rho}{[G_x^0 : G_{x,0+}^0]} \exp_p\left(\tfrac{1}{2}\dim \underline{G} + \tfrac{1}{2}\dim \underline{G}_{x,0:0+}^0 + \tfrac{1}{2}\sum_{i=0}^{d-1} r_i(|R_{i+1}| - |R_i|)\right).$$

- $R_i = R(G_i, S)$.
- $\exp_p t := p^t$.
- G_x is the stabilizer of x in G.
- $\underline{G}_{x,0}^0$ is a $\mathbb{Z}_p$-group, its special fiber is an $\mathbb{F}_p$-group, and $\underline{G}_{x,0:0+}^0$ is the maximal reductive quotient of the special fiber.

Formal degree of a Yu supercuspidal: proof sketch

Yu's representations are of the form $\text{c-Ind}_K^G \kappa$.

If H is a finite group then

$$\dim \text{Ind}_K^H \kappa = [H : K] \dim \kappa.$$

Similarly, if K is a compact-open subgroup of G then

$$\deg \text{c-Ind}_K^G \kappa = \frac{\dim \kappa}{\text{vol}\, K}.$$

- Computing $\dim \kappa$ is "simple".
- Computing $\text{vol}\, K$ is complicated and uses Moy-Prasad Theory.

Computation of vol K: proof sketch

1. Reduce to computing the index of K in a subgroup of known volume:
$$\operatorname{vol} K = \frac{\operatorname{vol} G_{x,0}}{[G_{x,0} : K]}.$$

2. $K \approx G_{x,r}$. Use the Moy-Prasad isomorphism to reduce the index to the length of a finite Lie algebra:
$$[G_{x,0} : G_{x,r}] = \operatorname{len}(\mathfrak{g}_{x,0}/\mathfrak{g}_{x,r}).$$

3. Decompose $\mathfrak{g}$ into root lines and compute the length by studying the breaks in the Moy-Prasad filtration.

Part 3. Langlands correspondence

Conjecture (Langlands)

1. *There is a surjective map*

$$\frac{\{\text{Smooth irreps of } G\}}{\text{iso.}} \to \frac{\{L\text{-parameters } W' \to {}^L G\}}{\text{equiv.}}$$

   satisfying many nice properties.

2. *The fibers of this map, called L-packets, are finite.*
3. *For each L-parameter $\varphi : W' \to {}^L G$, there is a bijection*

$$L\text{-packet of } \varphi \longleftrightarrow \mathrm{Spec_u}\, S_\varphi$$

   where S_φ is a certain finite group canonically constructed from φ.

(${}^L G = \widehat{G} \rtimes W$, the Weil form of the Langlands dual $\widehat{G}/\mathbb{C}$.)

Formal degree conjecture: statement

Conjecture (Hiraga-Ichino-Ikeda)

Let π be a discrete series representation of G with extended parameter $(\varphi : W' \to {}^L G, \rho \in \mathrm{Spec_u}\, S_\varphi)$. Then

$$\deg \pi = \frac{\dim \rho}{|S_\varphi|} \cdot |\gamma(\mathrm{Ad} \circ \varphi)|.$$

- $\mathrm{Ad} : {}^L G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation.
- γ is the γ-factor, a product of an ε-factor and two L-factors.
- $|S_\varphi|$ is the cardinality of S_φ.
- (This ρ is unrelated to the one from Yu's construction.)

The formal degree conjecture is a "2-conjecture": it depends on the conjectural LLC.

Formal degree conjecture: known cases

The conjecture is known for the following G, where the LLC has been constructed in entirety.

- real Lie groups [Harish-Chandra]
- (inner forms of) GL_n [Silberger-Zink]
- (inner forms of) SL_n [Harris-Taylor, Henniart]

Even if the full LLC is unavailable, we can still test the conjecture as long as we have some L-packets.

The conjecture is known for the following L-packets.

- unipotent discrete series [Lusztig, Reeder]
- depth-zero supercuspidals [DeBacker-Reeder]

Kaletha's L-packets

Kaletha has organized most of Yu's representations, the "regular representations", into L-packets.

The construction of each L-packet is organized around a pair $(\underline{S}, \theta)$ consisting of a maximal torus $\underline{S} \subseteq \underline{G}$ and a character $\theta : S \to \mathbb{C}^\times$.

On the automorphic side, one can unspool $(\underline{S}, \theta)$ into an input for Yu's construction.

$$(\underline{S}, \theta) \mapsto (\underline{\vec{G}}, x, \rho, \vec{r}, \vec{\phi})$$

On the Galois side, one "uses" functoriality with the help of χ-data [Langlands-Shelstad].

$$\begin{array}{ccccc} W & \xrightarrow{{}^L\theta} & {}^LS & \overset{{}^Lj_\chi}{\dashrightarrow} & {}^LG \\ & \searrow^{\hat{\theta}} & \uparrow & & \uparrow \\ & & \widehat{S} & \xrightarrow{\hat{\jmath}} & \widehat{G} \end{array}$$

Formal degree conjecture for regular supercuspidals

Theorem (S)

Kaletha's L-packets satisfy the formal degree conjecture:

$$\deg \pi = \frac{\dim \rho}{|S_\varphi|} \cdot |\gamma(\mathrm{Ad} \circ \varphi)|.$$

Proof sketch:

- On the automorphic side, we use our formula for $\deg \pi$.
- On the Galois side, for regular representations, S_φ is abelian and $|S_\varphi|$ is known. So we need only compute $|\gamma(\mathrm{Ad} \circ \varphi)|$.

Proof sketch: Galois side

We start by computing the adjoint representation of φ. It decomposes as

$$\mathrm{Ad} \circ \varphi = V_{\text{toral}} \oplus V_{\text{root}},$$

where V_{toral} comes from $\underline{S}$ and V_{root} comes from $R(G, S)$. Since γ is additive, we can handle each factor separately.

$V_{\text{toral}} \simeq X^*(S) \otimes \mathbb{C}$, so we can compute $\gamma(V_{\text{toral}})$ by understanding the Galois action on $X^*(S)$.

V_{root} is a sum of monomial representations, so we can compute $\gamma(V_{\text{root}})$ by understanding how γ behaves under (tame) induction.

- Key technical result: base change for χ-data.

Key references

- Kaoru Hiraga, Atsushi Ichino, and Tamotsu Ikeda, *Formal degrees and adjoint γ-factors*, Journal of the American Mathematical Society **21** (2008), no. 1, 283–304. MR 2350057
- Tasho Kaletha, *Regular supercuspidal representations*, Journal of the American Mathematical Society **32** (2019), no. 4, 1071–1170. MR 4013740
- Jiu-Kang Yu, *Construction of tame supercuspidal representations*, Journal of the American Mathematical Society **14** (2001), no. 3, 579–622. MR 1824988