Singular modular forms on quaternionic E_8

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Goal

This talk is about: The construction of two very nice automorphic forms on quaternionic E_8

- $E_{8,4}$: real reductive group of type E_8 with split rank four; this is quaternionic E_8
- The symmetric space $E_{8,4}/K$ does not have Hermitian structure, but still possesses automorphic forms that behave **similarly** to classical holomorphic modular forms
- **Similarly**: They have a 'robust' Fourier expansion; called 'modular' forms
- There are two modular forms on $E_{8,4}$ that can write down explicitly
- **Theorem**: These modular forms have all Fourier coefficients in $\mathbf{Q}$

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A very nice exceptional group

$E_{7,3}$: has a symmetric space with Hermitian tube structure

- Θ: octonions with positive-definite norm form. This is an 8-dimensional, non-associative $\mathbf{R}$-algebra that comes equipped with a quadratic form $\Theta \to \mathbf{R}$ and an $\mathbf{R}$-linear conjugation $* : \Theta \to \Theta$.
- $J = H_3(\Theta)$: Hermitian 3×3 matrices with elements in Θ.

$$J = \left\{ \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix} : c_i \in \mathbf{R}, x_j \in \Theta \right\}.$$

$E_{7,3}$ acts on

$$\mathcal{H}_J = \{Z = X + iY : X, Y \in J, Y > 0\}$$

by "fractional linear" transformations.

For an integer $\ell > 0$, $f : \mathcal{H}_J \to \mathbf{C}$ is a holomorphic modular form of weight ℓ if

- f is holomorphic, moderate growth
- $f(\gamma Z) = j(\gamma, Z)^{\ell} f(Z)$ for all $\gamma \in \Gamma \subseteq E_{7,3}$ a congruence subgroup

These holomorphic modular forms on $E_{7,3}$ have a Fourier expansion:

$$f(Z) = \sum_{T \in J_{\mathbf{Q}}, T \geq 0} a_f(T) e^{2\pi i \,\mathrm{tr}(TZ)}$$

with the $a_f(T) \in \mathbf{C}$.

Kim's modular forms on $E_{7,3}$

Rank

Note that $J \supseteq S_3$ the symmetric 3×3 matrices. There is a function $rank : J \to \{0, 1, 2, 3\}$ extending the rank of symmetric matrices on S_3.

Theorem 1 (H. Kim)

There exists holomorphic modular forms $\Theta_{Kim,4}$ and $\Theta_{Kim,8}$ for $E_{7,3}$ with the following properties:

1. *$\Theta_{Kim,4}$ is a weight 4, level 1 modular form with Fourier coefficients in $\mathbf{Z}$. Moreover, the Fourier coefficients $a_{\Theta_{Kim,4}}(T)$ are 0 unless $rank(T) \in \{0, 1\}$.*
2. *$\Theta_{Kim,8}$ is a weight 8, level 1 modular form with Fourier coefficients in $\mathbf{Z}$. Moreover, the Fourier coefficients $a_{\Theta_{Kim,8}}(T)$ are 0 unless $rank(T) \in \{0, 1, 2\}$.*

The modular forms $\Theta_{Kim,4}, \Theta_{Kim,8}$ are said to be **singular**.

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Exceptional groups have 'modular forms'

The groups

$G : G_2 \subseteq D_4 \subseteq F_4 \subseteq E_{6,4} \subseteq E_{7,4} \subseteq E_{8,4}$

- $K \subseteq G$ the maximal compact. $K \twoheadrightarrow \mathrm{SU}(2)/\mu_2$.
- G/K: no Hermitian structure

Definition of modular forms on G

Let $\ell \geq 1$ be an integer. A modular form on G of weight ℓ is

- an automorphic form $\varphi : \Gamma\backslash G \to Sym^{2\ell}(\mathbf{C}^2)$
- satisfying $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $g \in G$, $k \in K$
- and $\mathcal{D}_\ell \varphi = 0$ for a certain special linear differential operator $\mathcal{D}_\ell$

- Definition due to Gross-Wallach, Gan-Gross-Savin

These modular forms have nice properties

Theorem 2

The modular forms of weight $\ell \geq 1$ on G have a robust Fourier expansion, normalized over the integers, that is compatible with pullbacks between groups G above.

The theorem means:

- Given a modular φ form of weight ℓ, one can ask the question "Are all of φ's Fourier coefficients in some ring $R \subseteq \mathbf{C}$?"
- If $\iota : G_1 \subseteq G_2$ in the above sequence of groups, and if φ is modular form on G_2 of weight ℓ, then the pullback $\iota^*(\varphi)$ on G_1 is a modular form of weight ℓ.
- Moreover, the Fourier coefficients of $\iota^*\varphi$ are **finite sums** of the Fourier coefficients of φ

Motivating question

Fix G and $\ell \geq 1$. Does there exist a basis of the modular forms on G of weight ℓ, all of whose Fourier coefficients are in $\overline{\mathbf{Q}}$?

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Let $P = MN \subseteq E_{8,4}$ be the Heisenberg parabolic subgroup, $M = GE_{7,3}$.

Theorem 3 (Gan,P,Savin)

There exists square integrable automorphic forms Θ_{min} and Θ_{ntm} on $E_{8,4}$ with the following properties.

1. *Θ_{min} is a weight 4 modular form with all Fourier coefficients in $\mathbf{Z}$. Its constant term along N, $\Theta_{min,N}$ is essentially $\Theta_{Kim,4}$.*
2. *Θ_{ntm} is a weight 8 modular form with all Fourier coefficients in $\mathbf{Q}$. Its constant term along N, $\Theta_{ntm,N}$ is essentially $\Theta_{Kim,8}$.*

These modular forms are **singular** *in the sense that many of their Fourier coefficients are 0.*

The Fourier coefficients are parametrized by elements in a lattice in $W = (N/[N,N])^\vee$. There is a function $\mathrm{rank} : W \to \{0,1,2,3,4\}$.

- The Fourier coefficients $a_{\Theta_{min}}(w)$ of Θ_{min} are 0 unless $\mathrm{rank}(w) \in \{0,1\}$
- The Fourier coefficients $a_{\Theta_{ntm}}(w)$ of Θ_{ntm} are 0 unless $\mathrm{rank}(w) \in \{0,1,2\}$

Remarks

1. Gross-Wallach constructed unitary representations π_4 and π_8 of the real group $E_{8,4}$ that are small in the sense of GK dimension. The automorphic forms Θ_{min}, Θ_{ntm} should be[1] thought of as globalizations of these representations.
2. On **split** E_8 there are analogues of Θ_{min} and Θ_{ntm}. These are completely spherical automorphic forms
   - constructed by Ginzburg-Rallis-Soudry, in the case of the minimal;
   - constructed by Green-Miller-Vanhove, Ciubotaru-Trapa in the case of next-to-minimal;
   - next-to-minimal recently studied by Gourevitch-Gustafsson-Kleinschmidt-Persson-Sahi.
3. Gan constructed Θ_{min} as a special value of an Eisenstein series associated to $Ind_P^{E_{8,4}}(\delta_P^{s_{min}})$, proved it's square integrable.

[1] Proved by Gan-Savin for Θ_{min} and π_4. Should be true but not proved for Θ_{ntm} and π_8.

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Heisenberg Eisenstein series

Suppose $G = E_{8,4}$, P Heisenberg parabolic.

$$\nu : P \to \mathrm{GL}_1$$

generating the character group of P. On $G = E_{8,4}$,

$$|\nu(p)|^{29} = \delta_P(p)$$

for $p \in P$. Suppose

- $\ell \geq 1$ even
- $f(g, \ell; s) \in Ind_{P(\mathbf{A})}^{G(\mathbf{A})}(|\nu|^s)$, certain $Sym^{2\ell}(\mathbf{C}^2)$-valued section.
- $E(g, \ell; s) = \sum_{\gamma \in P(\mathbf{Q}) \backslash G(\mathbf{Q})} f(\gamma g, \ell; s)$ absolutely convergent for $Re(s) > 29$.
- If $s = \ell + 1$ in range of absolute convergence, $E(g, \ell; s = \ell + 1)$ a **modular form of weight** ℓ for G

Question

Does $E(g, \ell; s = \ell + 1)$ have rational Fourier coefficients?

Next to minimal

Motivated by work of Gross-Wallach on continuation of quaternionic discrete series, take $\ell = 8$ and $G = E_{8,4}$.

Proposition

The Eisenstein series $E(g, \ell = 8; s)$ is regular at $s = 9$ (even though outside the range of absolute convergence), and defines square integrable weight 8 modular form at this point.

Set

$$\theta_{ntm}(g) = E(g, \ell = 8; s = 9)$$

Theorem 4 (Savin)

The spherical constituent of the degenerate principal series $Ind_{P(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}(|\nu|^9)$ is "small", i.e., many twisted Jacquet modules are 0. Consequently, the rank three and rank four Fourier coefficients of θ_{ntm} are 0.

More on next-to-minimal modular form

Theorem 5

The weight 8 modular form θ_{ntm} has rational Fourier coefficients.

Proof.

1. Savin's result gives vanishing of rank three and four Fourier coefficients
2. Explicit computation (outside range of abs. convergence) gives rationality of rank 1 and rank 2 Fourier coefficients
3. Constant term analyzed using work of H. Kim on weight 8 singular modular form on $GE_{7,3}$

□

Explicit computation of θ_{ntm}

1. Define special $Sym^{2\ell}(\mathbf{C}^2)$-valued Eisenstein series $E_\ell(g)$ on $SO(3, 4k+3)$
2. Prove that the constant term θ_{ntm} from $E_{8,4}$ down to $SO(3, 11)$ is $E_8(g)$
3. **Theorem**: the $E_\ell(g)$ have rational Fourier coefficients (in a precise sense)
4. The Fourier coefficients of $E_8(g)$ can be identified with rank 1 and rank 2 Fourier coefficients of θ_{ntm}.

To prove the $E_\ell(g)$ have rational Fourier coefficients:

Jacquet integral

Explicit computation of certain Archimedean Jacquet integral

$$\int_{V_{2,4k+2}(\mathbf{R})} e^{2\pi i (v,x)} f_\ell(wn(x))\, dx.$$

Thank you

Thank you for your attention!