# Interpreting the Harish-Chandra—Howe local character expansion via branching rules

Monica Nevins

Department of Mathematics and Statistics
University of Ottawa

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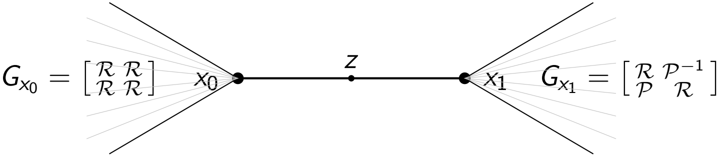

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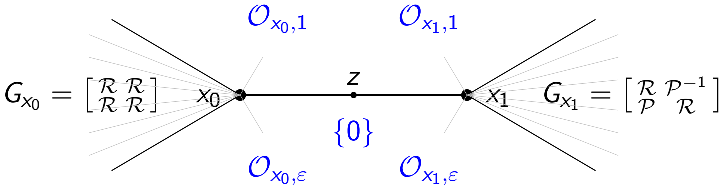


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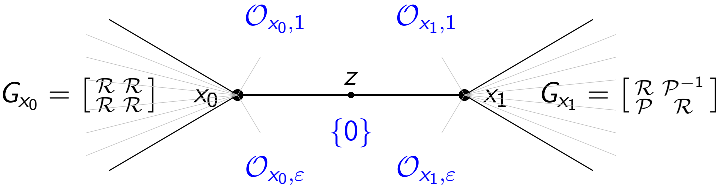


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$\pi$ : an irreducible admissible representation of $G$, of depth $r \geq 0$, with character $\Theta_\pi$

# Three perspectives

1. Harish-Chandra–Howe local character expansion:

$$\Theta_\pi(\varphi(X)) = \sum_{\mathcal{O}\in\mathcal{N}} c_{\mathcal{O}}\widehat{\mu_{\mathcal{O}}}(X)$$

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3. Orbit method philosophy: construct key representations of $G$ from its admissible nilpotent coadjoint orbits.

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**Definition**

*We call Shalika's representation*

$$\mathcal{S}_d(\theta, X) := \mathrm{Ind}_{ZNG_{u,e}}^{G_x} \theta \otimes \psi(X)$$

*a basic irreducible representation of $G_x$, of depth $d$ and central character $\theta$. It depends only on the $G_x$-orbit of $X$.*

# Representations of $G_x$ attached to nilpotent $G$-orbits

Each nilpotent $G$-orbit $\mathcal{O}$ decomposes as $G_x$-orbits:

$$\mathcal{O} = G \cdot X_a = \bigsqcup_{t \in \mathbb{Z}} G_x \cdot X_{\varpi^{2t} a}$$

## Definition

*Let $\tau(0) = 1$. For $\mathcal{O} \in \mathcal{N} \setminus \{0\}$ set*

$$\tau_x(\mathcal{O})_\theta = \bigoplus_{X_d} \mathcal{S}_d(\theta, X_d) \qquad \text{(a representation of } G_x\text{)}$$

*where $X_d$ runs over a set of representatives of*

$$G_x\text{-orbits in } \mathcal{O} \setminus \mathfrak{g}_{x,0}.$$

# Back to branching rules for $SL(2, k)$

For any $\pi$ of depth $r \geq 0$, we have a complete description of $\mathrm{Res}_{G_x}\pi$ [N05, N13].
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- In between

$$\mathrm{Res}_{G_x}\pi = \pi^{G_{x,r+}} \oplus \pi_{r<d\leq 2r} \oplus \pi_{>2r}$$

are many (non-basic) irreducible representations of intermediate depth that are types for increasingly large families of representations (bigger than one Bernstein block).

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## Proposition

*If $\pi$ has depth $r$, with branching rules*

$$\mathrm{Res}_{G_x}\pi = \pi^{G_{x,r+}} \oplus \pi_{>r},$$

*then there is a subset $\mathcal{N}_\pi$ of $\mathcal{N}$ such that*

$$\mathrm{Res}_{G_{x,r+}}(\pi_{>r}) = \bigoplus_{\mathcal{O}\in\mathcal{N}_\pi} \tau_x(\mathcal{O})_{>r}.$$

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## Corollary

*For each $\pi$ of depth $r$, there is an integer $c$ and a subset $\mathcal{N}_\pi \subset \mathcal{N}$ such that on $G_{x,r+}$ we have*

$$\pi = c1 \oplus \bigoplus_{\mathcal{O}\in\mathcal{N}_\pi} \tau_x(\mathcal{O}).$$

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$\mathcal{O}_{u,a}$ : for each vertex $x \in \mathcal{B}$ set

$$\Theta_{u,a}|_{G_{x,0+}^{rss}} = \begin{cases} \frac{q}{2} + \chi_x(\mathcal{O}_{u,a}) & \text{if } u \sim x; \\ \frac{1}{2} + \chi_x(\mathcal{O}_{u,a}) & \text{if } u \not\sim x \end{cases}$$

$\Theta_{u,a}$ is well-defined (as a consequence of branching rules).

# Branching rules and the LCE

**Theorem**

*Let $\pi$ be an irreducible admissible representation of $SL(2,k)$ of depth $r$. Then there exist $t_0 \in \mathbb{Q}$ and $t_{u,a} \in \{0,1\}$ such that on $G^{rss}_{x,r+}$*

$$\Theta_\pi = t_0\Theta_0 + \sum_{\mathcal{O}_{u,a}\in\mathcal{N}} t_{u,a}\,\Theta_{u,a}.$$

*Moreover, these coefficients agree with the local character expansion, in the sense that*

$$\Theta_\pi \circ \varphi = t_0\widehat{\mu_0} + \sum_{\mathcal{O}_{u,a}\in\mathcal{N}} t_{u,a}\,\widehat{\mu_{\mathcal{O}_{u,a}}}.$$

The coefficients (and much more) have been calculated for $SL(2,k)$ in an abundance of ways: Sally–Shalika 1968, Assem 1994, Barbasch–Moy 1997, Cunningham–Gordon 2000, DeBacker–Sally 2000, Spice 2005, $\cdots$.

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- More test cases?
  - Campbell-N (2010) + Onn-Singla (2014) give the complete explicit branching rules for unramified principal series of $GL(3, k)$