The local symmetric square L-function for $GL(2)$

Yeongseong Jo

The University of Iowa

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Rankin-Selberg Integrals

- F : a non-archimedean local field
- π, σ : irreducible admissible generic (complex) representations of $GL_n(F)$
- $\mathcal{W}(\pi, \psi), \mathcal{W}(\sigma, \psi^{-1})$: Whittaker models for π and σ
- $\mathcal{S}(F^n)$: Bruhat-Schwartz functions on F^n
- $\omega_\pi, \omega_\sigma$: central characters of $F^\times$

$W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\sigma, \psi^{-1})$ and $\Phi \in \mathcal{S}(F^n)$

$$\Psi(s, W, W', \Phi) = \int_{N_n(F)\backslash GL_n(F)} W(g)W'(g)\Phi(e_n g)|\det(g)|^s \, dg$$

$\Psi(s, W, W', \Phi)$ converges absolutely for $\mathrm{Re}(s) \gg 0$.

Rankin-Selberg Integrals

Theorem (Jacquet, Piatetski-Shapiro, and Shalika)

1. *For* $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\sigma, \psi^{-1})$ *and* $\Phi \in \mathcal{S}(F^n)$, $\Psi(s, W, W', \Phi) \in \mathbb{C}(q^{-s})$. *Hence we have a meromorphic continuation.*
2. $\langle \Psi(s, W, W', \Phi) \rangle$ *is a* $\mathbb{C}[q^{\pm s}]$*-fractional ideal in* $\mathbb{C}(q^{-s})$.
3. $\langle \Psi(s, W, W', \Phi) \rangle = \left\langle \dfrac{1}{P(q^{-s})} \right\rangle$ *such that* $P(X) \in \mathbb{C}[X]$ *and* $P(0) = 1$.

Definition

$$L(s, \pi \times \sigma) = \frac{1}{P(q^{-s})}.$$

Classification of Bernstein and Zelevinsky

Theorem (Bernstein and Zelevinsky)

1. π : *irreducible admissible generic representations of* $GL_n(F)$
$\Longrightarrow \pi \simeq \mathrm{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$
    - *Induction is normalized from standard parabolic subgroup of type* $(n_1, n_2, \cdots, n_t)$ *with* $\sum n_i = n$.
    - Δ_i : *irreducible quasi-square integrable representations of* $GL_{n_i}(F)$.
2. $\Delta_i \simeq [\rho_i, \rho_i\nu, \cdots, \rho_i\nu^{\ell_i - 1}], \quad \ell_i r_i = n_i$
    - ρ_i : *irreducible supercuspidal representations of* GL_{r_i}
    - *Determinantal unramified character* $\nu(g) := |\det(g)|, \quad g \in GL_{r_i}$
    - $[\rho_i, \rho_i\nu, \cdots, \rho_i\nu^{\ell_i - 1}]$ *is the unique irreducible quotient of* $\mathrm{Ind}(\rho_i \otimes \rho_i\nu \otimes \cdots \otimes \rho_i\nu^{\ell_i - 1})$.

Inductivity relations I

Theorem (Jacquet, Piatetski-Shapiro and Shalika)

1. $\pi = \mathrm{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_i \otimes \cdots \otimes \Delta_t), \sigma = \mathrm{Ind}(\Delta'_1 \otimes \cdots \otimes \Delta'_j \otimes \cdots \otimes \Delta'_r)$: *irreducible admissible generic representations of* $GL_n(F)$
   - $L(s, \pi \times \sigma) = \prod_{i,j} L(s, \Delta_i \times \Delta'_j)$
2. $\Delta = [\rho\nu^{-\frac{\ell-1}{2}}, \cdots, \rho\nu^{\frac{\ell-1}{2}}], \Delta' = [\rho'\nu^{-\frac{\ell'-1}{2}}, \cdots, \rho'\nu^{\frac{\ell'-1}{2}}]$: *irreducible square integrable representation with* ρ, ρ' *irreducible unitary supercuspidal representations of* $GL_r(F)$ *and* $GL_{r'}(F)$
   - $L(s, \Delta \times \Delta') = \prod_{j=0}^{\ell'-1} L\left(s + \frac{\ell - \ell'}{2} + j, \rho \times \rho'\right)$

Functional equations

- $\widetilde{\pi}$: contragredient representation of π
- $w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$: the long Weyl element
- $\mathcal{W}(\widetilde{\pi}, \psi^{-1}) = \{\widetilde{W}(g) = W(w_n {}^t g^{-1}) \mid W \in \mathcal{W}(\pi, \psi)\}$
- Fourier transformation : $\hat{\Phi}(y) = \int_{F^n} \Phi(x)\psi(x\ {}^t y)dx$.

Theorem (Jacquet, Piatetski-Shapiro and Shalika)

- $\Psi(1-s, \widetilde{W}, \widetilde{W}', \hat{\Phi}) = \omega_\sigma(-1)^{n-1}\gamma(s, \pi \times \sigma, \psi)\Psi(s, W, W', \Phi)$.

- $\varepsilon(s, \pi \times \sigma, \psi) := \gamma(s, \pi \times \sigma, \psi)\dfrac{L(s, \pi \times \sigma)}{L(1-s, \widetilde{\pi} \times \widetilde{\sigma})}$
- $\dfrac{\Psi(1-s, \widetilde{W}, \widetilde{W}', \hat{\Phi})}{L(1-s, \widetilde{\pi} \times \widetilde{\sigma})} = \omega_\sigma(-1)^{n-1}\varepsilon(s, \pi \times \sigma, \psi)\dfrac{\Psi(s, W, W', \Phi)}{L(s, \pi \times \sigma)}$.

Supercuspidal representations

- Theory of types and covers - Bushnell, Henniart and Kutzko
- Inductive formula II
  - Paskunas and Stevens ($GL_n \times GL_n$), J. Kim ($GL_n \times GL_m$)
  $\rightsquigarrow$ The Langlands-Shahidi method (?)

Theorem (J.-Krishnamurthy)

$\rho_i \simeq \mathrm{c}-\mathrm{Ind}_{F^\times K_n}^{GL_n}(\widetilde{\lambda}_i)$, $i = 1, 2$: *depth (level) zero supercuspidal representations.*

$$\varepsilon_{LS}(s, \rho_1 \times \rho_2, \psi) = \omega_{\rho_2}(-1)^n \varepsilon(s, \rho_1 \times \widetilde{\rho}_2, \psi)$$

- Shahidi
- Siegel Levi subgroups inside classical groups - Asai (U(n,n)) and exterior square cases (SP(2n))
- Different endo-classes $\rightsquigarrow$ Stability of $\gamma(s, \rho_1 \times \rho_2, \psi)$

Integral representations

- π : an irreducible admissible generic representation of $GL_n(F)$

$$L(s, \pi \times \pi) = L_{LS}(s, \pi, \wedge^2) L_{LS}(s, \pi, \mathrm{Sym}^2)$$

- F : a characteristic zero - Shahidi
- F : a positive characteristic - Ganapathy, Henniart, and Lomeli

Assumption : F a non-archmidean local field of the characteristic zero
- Integral representations (1990's)

$$L(s, \pi, \wedge^2)$$

- Bump-Friedberg integrals
  - $L_{BF}(s, \pi, \wedge^2) = L_{LS}(s, \pi, \wedge^2)$: Matringe $\leftsquigarrow$ Kewat-Raghunathan
- Jacquet-Shalika integrals
  - $L_{JS}(s, \pi, \wedge^2) = L_{LS}(s, \pi, \wedge^2)$: J. $\leftsquigarrow$ Kewat-Raghunathan

Integral representations

- $L(s, \pi, \mathrm{Sym}^2)$
- $n = 2$: Gelbart-Jacquet $L(s, \pi \times \pi) = L(s, \omega_\pi) L(s, \pi, \mathrm{Sym}^2)$
- $n = 3$: Patterson and Piatetski-Shapiro
- any n : (non-twisiting) Bump-Ginzburg (twisiting) Takeda, Yamana

Theorem (J.)

$\pi = \mathrm{Ind}_B^{GL_2}(\mu_1 \boxtimes \mu_2)$ *: irreducible principal series representation of* $GL_2(F)$.

$$L(s, \pi, \mathrm{Sym}^2) = \prod_{1 \leqslant i \leqslant j \leqslant 2} \frac{1}{1 - \mu_i(\varpi)\mu_j(\varpi) q^{-s}}.$$

Corollary (+Yamana)

π *: irreducible admissible generic representation of* $GL_2(F)$

$$L(s, \pi, \mathrm{Sym}^2) = L_{LS}(s, \pi, \mathrm{Sym}^2).$$

Sections

- The metaplectic group: $1 \to \{\pm 1\} \longrightarrow \widetilde{GL}_2 \xrightarrow{pr} GL_2 \longrightarrow 1$
- The mirabolic subgroup: $P = \left\{ \begin{pmatrix} a & x \\ & 1 \end{pmatrix} \middle| a \in F^\times, x \in F \right\} \simeq A \ltimes N$
- $Z^2 = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \middle| a \in (F^\times)^2 \right\}$
- η : a character of $F^\times$
- $\widetilde{\eta}((1,\xi)\begin{pmatrix} a & \\ & a \end{pmatrix}) = \xi\eta(a)$ on $\widetilde{Z^2}$, $\quad \widetilde{1}_{GL_1}((1,\xi)\begin{pmatrix} a & \\ & 1 \end{pmatrix}) = \xi$ on $\widetilde{A}$,

Sections

- $V(s,\eta) := \mathrm{Ind}_{\widetilde{Z^2}\widetilde{P}}^{\widetilde{GL}_2}(\delta_B^{s/4}(\widetilde{\eta} \boxtimes \widetilde{1}_{GL_1}))$
- $V_{std}(s,\eta)$ is called a standard section or a flat section if for any $k \in \widetilde{K}$, $f_s|_{\widetilde{K}}$ is independent of s.
- $V_{hol}(s,\eta) = \mathbb{C}[q^{s/4}, q^{-s/4}] \otimes_{\mathbb{C}} V_{std}(s,\eta)$: holomorphic sections
- $V_{rat}(s,\eta) = \mathbb{C}(q^{-s/4}) \otimes_{\mathbb{C}} V_{std}(s,\eta)$: rational sections

Intertwining operators

- $M(s,\eta) : \mathrm{Ind}_{\widetilde{Z^2}\widetilde{P}}^{\widetilde{GL_2}}(\delta_B^{s/4}(\widetilde{\eta}\boxtimes\widetilde{1}_{GL_1})) \to \mathrm{Ind}_{\widetilde{Z^2}\,{}^{w_2}\widetilde{A}N*}^{\widetilde{GL_2}}(\delta_B^{-s/4}(\widetilde{\eta}\boxtimes{}^{w_2}\widetilde{1}_{GL_1}))$

$$M(s,\eta)f_s(\widetilde{g}) = \int_F f_s\left(\mathfrak{s}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\mathfrak{s}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\widetilde{g}\right)\, dx$$

  for $f_s \in V_{hol}(s,\eta)$.
- $M(s,\eta)f_s(g)$ converges absolutely for $\mathrm{Re}(s) \gg 0$.
- Involution (Kable): $g \mapsto {}^{\iota}g := w_2\, {}^{t}g^{-1}w_2$, $g \in GL_2$
  $\rightsquigarrow$ Construct a lift on $\widetilde{GL_2}$: $\widetilde{g} \mapsto {}^{\iota}\widetilde{g}$
- Normalized $\mathbb{C}$-linear maps
  $N(s,\eta,\psi) : V(s,\eta) \to V(s,\eta^{-1})$

$$N(s,\eta,\psi)f_s(\widetilde{g}) = \gamma(s,\eta^{-2},\psi)M(s,\eta)f_s({}^{\iota}\widetilde{g})$$

Good Sections

Proposition (Functional Equations, Gao-Shahidi-Szpruch)

$$N(-s,\eta^{-1},\psi^{-1})\circ N(s,\eta,\psi)=\mathrm{Id}.$$

Definition

$f_s \in V(s,\eta)$ is called a good section if

- $f_s \in V_{hol}(s,\eta)$
- $f_s \in N(-s,\eta^{-1},\psi^{-1})(V_{hol}(-s,\eta^{-1}))$

Remark

- *Piatetski-Shapiro and Rallis - Rankin triple product L-functions*
- *Kaplan - Rankin-Selberg L-functions for $SO_{2l}\times GL_n$*
- *The good section is closed under normalized intertwining operator.*

Integral representations

- θ : the exceptional representation of $\widetilde{GL}_2$ by Kazhdan and Patterson
- $\mathcal{W}(\theta, \psi^{-1})$: the Whittaker model for θ

$W \in \mathcal{W}(\pi, \psi)$, $W_\theta \in \mathcal{W}(\theta, \psi^{-1})$ and $f_{2s-1} \in V_{good}(2s-1, \omega_\pi^{-1})$

$$I(W, W_\theta, f_{2s-1}) = \int_{Z^2 N \backslash GL_2} W(g) W_\theta(\mathfrak{s}(g)) f_{2s-1}(\mathfrak{s}(g))\, dg$$

$I(W, W_\theta, f_{2s-1})$ converges absolutely for $\mathrm{Re}(s) \gg 0$.

Theorem (Bump-Ginzburg, Yamana)

1. *For* $W \in \mathcal{W}(\pi, \psi)$, $W_\theta \in \mathcal{W}(\theta, \psi^{-1})$ *and* $f_{2s-1} \in V_{good}(2s-1, \omega_\pi^{-1})$, $I(W, W_\theta, f_{2s-1}) \in \mathbb{C}(q^{-s/2})$.
2. $\langle I(W, W_\theta, f_{2s-1}) \rangle$ *is a* $\mathbb{C}[q^{\pm s/2}]$*-fractional ideal in* $\mathbb{C}(q^{-s/2})$.
3. $\langle I(W, W_\theta, f_{2s-1}) \rangle = \left\langle \dfrac{1}{P(q^{-s/2})} \right\rangle$ *such that* $P(X) \in \mathbb{C}[X]$ *and* $P(0) = 1$.

L-functions

Definition

$$L(s,\pi,\mathrm{Sym}^2)=\frac{1}{P(q^{-s/2})}.$$

Proposition (Functional Equation, Bump-Ginzburg, Yamana)

For $W\in\mathcal{W}(\pi,\psi)$, $W_\theta\in\mathcal{W}(\theta,\psi^{-1})$, *and* $f_{2s-1}\in V_{good}(2s-1,\omega_\pi^{-1})$,

- $$I(\widetilde{W},\widetilde{W}_\theta,N(2s-1,\omega_\pi^{-1},\psi)f_{2s-1}) = \gamma(s,\pi,\mathrm{Sym}^2,\psi)I(W,W_\theta,f_{2s-1})$$

- $$\varepsilon(s,\pi,\mathrm{Sym}^2,\psi):=\gamma(s,\pi,\mathrm{Sym}^2,\psi)\frac{L(s,\pi,\mathrm{Sym}^2)}{L(1-s,\widetilde{\pi},\mathrm{Sym}^2)}$$

$$\frac{I(\widetilde{W},\widetilde{W}_\theta,N(2s-1,\omega_\pi^{-1},\psi)f_{2s-1})}{L(1-s,\widetilde{\pi},\mathrm{Sym}^2)}=\varepsilon(s,\pi,\mathrm{Sym}^2,\psi)\frac{I(W,W_\theta,f_{2s-1})}{L(s,\pi,\mathrm{Sym}^2)}$$

Regular L-functions

- Spherical representations (Bump-Ginzburg):

$$I(W^\circ, W_\theta^\circ, f_{2s-1}^\circ) = \frac{L(s,\pi,\mathrm{Sym}^2)}{L(2s,\omega_\pi^2)}$$

- The lack of the multiplicativity of $\gamma(s,\pi,\mathrm{Sym}^2)$
- Cogdell and Piatetski-Shapiro's interpretation of the theory of derivatives and exceptional poles

Definition

- $\mathcal{I}_{reg}(\pi) = \langle I(W, W_\theta, f_{2s-1}) \mid W \in \mathcal{W}(\pi,\psi), W_\theta \in \mathcal{W}(\theta,\psi^{-1}), f_{2s-1} \in V_{hol}(2s-1,\omega_\pi^{-1}) \rangle$
- $\mathcal{I}_{reg}(\pi) = \left\langle \dfrac{1}{Q(q^{-s/2})} \right\rangle$
- $L_{reg}(s,\pi,\mathrm{Sym}^2) = \dfrac{1}{Q(q^{-s/2})}$

Exceptional L-functions

- π is θ-distinguished if $\mathrm{Hom}_{GL_n}(\pi \otimes \theta \otimes \theta, \mathbb{C}) \neq 0$.

Proposition

π : a discrete series representation of GL_n
$L(s, \pi, \mathrm{Sym}^2)$ has a pole at $s = 0$ if and only π is θ-distinguished.

- Langlands-Shahidi method : Kaplan (Local-global)
- Rankin-Selberg method : Yamana (Local)

Proposition (Kaplan)

π : irreducible admissible generic representations of GL_n
If π is θ-distinguished, $\pi \simeq \widetilde{\pi}$.

Exceptional and regular L-functions

Definition

- $s = s_0$ is said to be exceptional if $\frac{I(W, W_\theta, f_{2s-1})}{L_{reg}(s, \pi, \mathrm{Sym}^2)}$ has a pole for some $W \in \mathcal{W}(\pi, \psi)$, $W_\theta \in \mathcal{W}(\theta, \psi^{-1})$ and $f_{2s-1} \in V_{good}(2s-1, \omega_\pi^{-1})$.
- $L_{ex}(s, \pi, \mathrm{Sym}^2) := \frac{L(s, \pi, \mathrm{Sym}^2)}{L_{reg}(s, \pi, \mathrm{Sym}^2)}$

- Iwasawa decompositions $GL_2 = ZPK$

$$\rightsquigarrow \mathcal{I}_{reg}(\pi) = \left\langle \int_{F^\times} \underbrace{W \begin{pmatrix} a & \\ & 1 \end{pmatrix}}_{\pi|_P} \underbrace{W_\theta \left(\mathfrak{s} \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right)}_{\theta|_{\tilde{P}}} |a|^{\frac{s}{2} - \frac{3}{4}} d^\times a \right\rangle$$

Bernstein and Zelevinsky derivatives

$\leftsquigarrow$ Induction / Jacquet functors

- Exceptional representations θ of $\widetilde{GL}_n$: Kable
- $\{0\} \subset \tau_2 \subset \tau_1 := \theta|_{\widetilde{P}}$
- The Kirillov model (Cogdell, Gelbart, and Piatetski-Shapiro)

$$-\mathcal{W}(\tau_1, \psi^{-1}) = \left\{ W_\theta \left(\mathfrak{s} \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right) \middle| W_\theta \in \mathcal{W}(\theta, \psi^{-1}), a \in F^\times \right\} = K(\theta, \psi^{-1})$$

$$-\mathcal{W}(\tau_2, \psi^{-1}) = \left\{ W_\theta \left(\mathfrak{s} \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right) \middle| \; W_\theta \in \mathcal{W}(\theta, \psi^{-1}), a \in F^\times \right.$$

$$\left. \text{there exists } N > 0 \text{ such that } \quad W_\theta \left(\mathfrak{s} \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right) = 0 \quad \text{if} \quad |a| < q^{-N} \right\}$$

- $\tau_1/\tau_2 \rightsquigarrow \theta^{(i)}$: derivatives, representations of $\widetilde{GL}_r$

Factorizations

Proposition (J.)

Let π be an irreducible admissible generic representation of GL_2 such that all of its derivatives are completely reducible. Then

$$L(s,\pi,\mathrm{Sym}^2)^{-1} = \underset{i,j}{\textit{l.c.m.}}\{L_{ex}(s,\pi_j^{(i)},\mathrm{Sym}^2)^{-1}\}$$

where the least common multiple is with respect to the divisibility in $\mathbb{C}[q^{\pm s/2}]$ and is taken over all j with $0 \leqslant j \leqslant 1$ and for all constituents $\pi_i^{(1)}$ of $\pi^{(1)}$