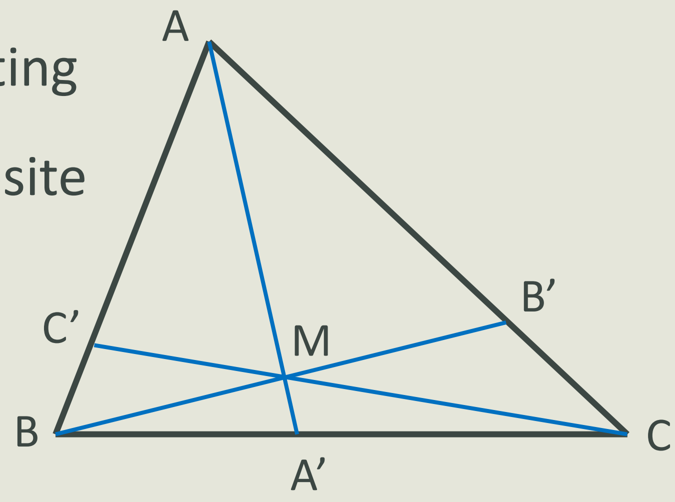
Ceva and Menelaus

Lecture 5   Feb 7, 2021

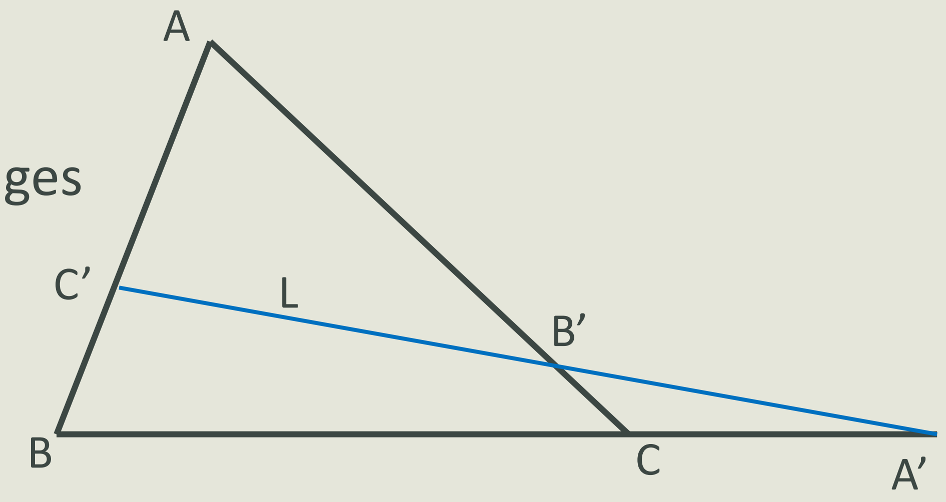
Ceva's Theorem

Theorem. Suppose AA', BB', CC', are line segments connecting vertices A,B,C of a triangle Δ to points A', B', C' on the opposite edges. Then $\frac{AB'}{B'C} \times \frac{CA'}{A'B} \times \frac{BC'}{C'A} = 1$ if and only if AA', BB', and CC' pass through a common M point in Δ (we say AA', BB', and CC' **concurrent**)



Menelaus' Theorem (Version 1)

Theorem. Suppose A' is a point on the line BC, outside the segment BC, and B' and C' are points on the edges AC and AB of a triangle Δ=ABC.



Then $\frac{AB'}{B'C} \times \frac{CA'}{A'B} \times \frac{BC'}{C'A} = 1$ if and only if

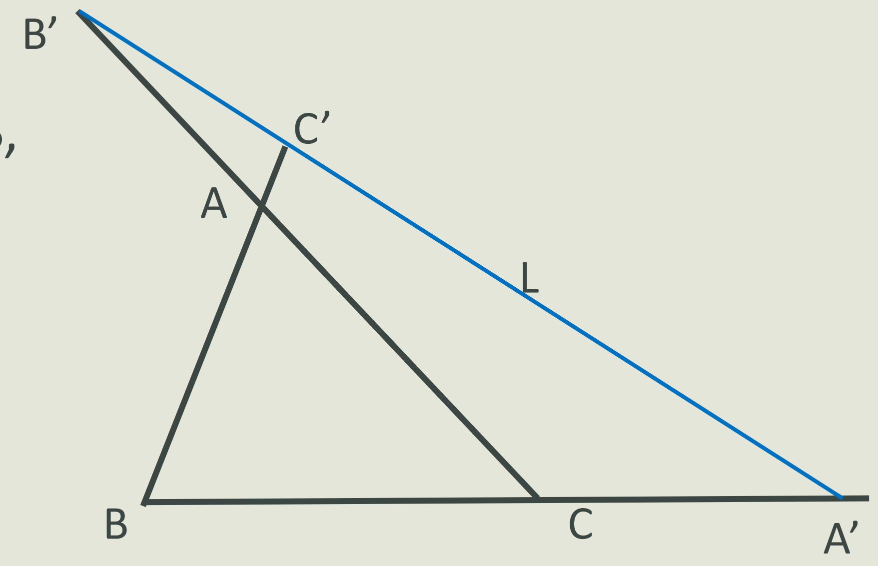
A', B', C' are on one line L (we say A', B', C' are **collinear**)

Menelaus' Theorem (Version 2)

Theorem. Suppose A', B', C' are points on the lines BC, CA, AB, outside the segment BC, CA, AB of a triangle Δ=ABC.

Then $\frac{AB'}{B'C} \times \frac{CA'}{A'B} \times \frac{BC'}{C'A} = 1$ if and only if

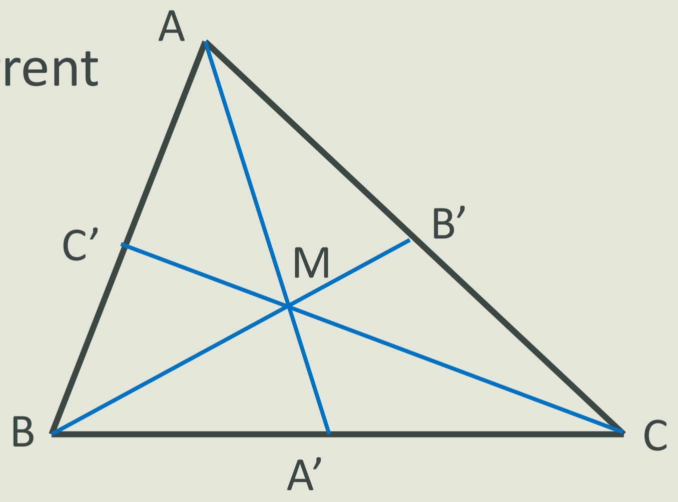
A', B', C' are on one line L (they are **collinear**)



Special cases

Before going over the proof(s), we discuss some special cases

- **Corollary 1 of Ceva:** The medians of a triangle are concurrent



- **Proof:**

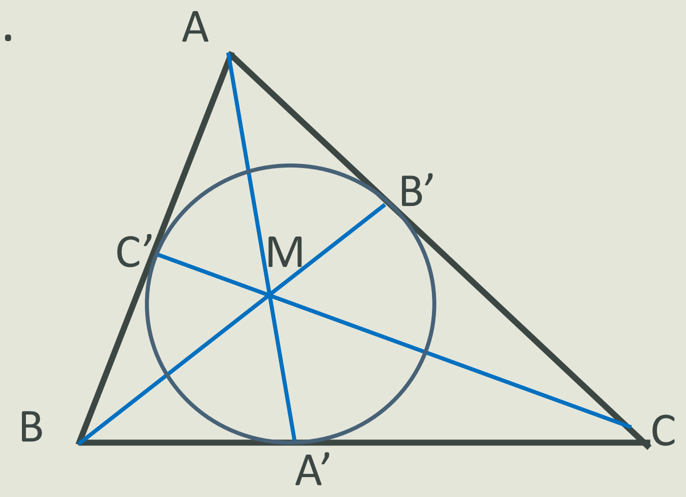
median: the line connecting a vertex to the middle of the opposite edge →

AC’=C’B, BA’=A’C, CB’=B’A → $\frac{AB'}{B'C} \times \frac{CA'}{A'B} \times \frac{BC'}{C'A} = 1 \times 1 \times 1 = 1$ → AA’, BB’, and CC’ are concurrent

- **HW:** Show that AM= 2 A’M, BM= B’M, CM= 2 C’M (M is known as the **centroid** of ABC)

Special cases

- **Corollary 4:** Let ΔABC be a triangle, and let A’, B’, C’ be the points of tangency of the circle inscribed in ΔABC. Then AA’, BB’ , and CC’ are concurrent.

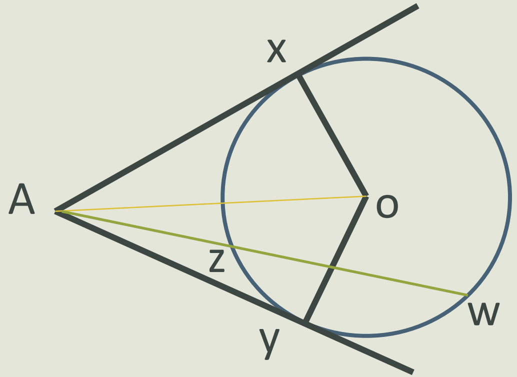


- **Proof:**

AC’=AB’, BA’=BC’, CA’=CB’ →

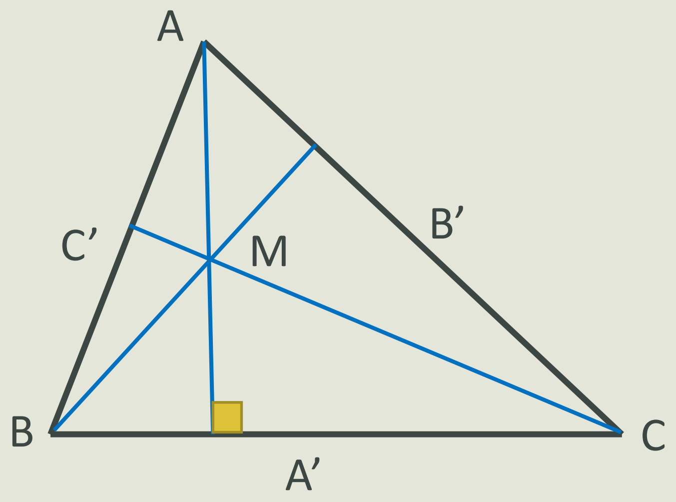
$$\frac{AB'}{B'C} \times \frac{CA'}{A'B} \times \frac{BC'}{C'A} = \frac{AB'}{AC'} \times \frac{CA'}{CB'} \times \frac{BC'}{BA'} = 1 \times 1 \times 1 = 1 \rightarrow$$

AA’, BB’, and CC’ are concurrent



Special cases

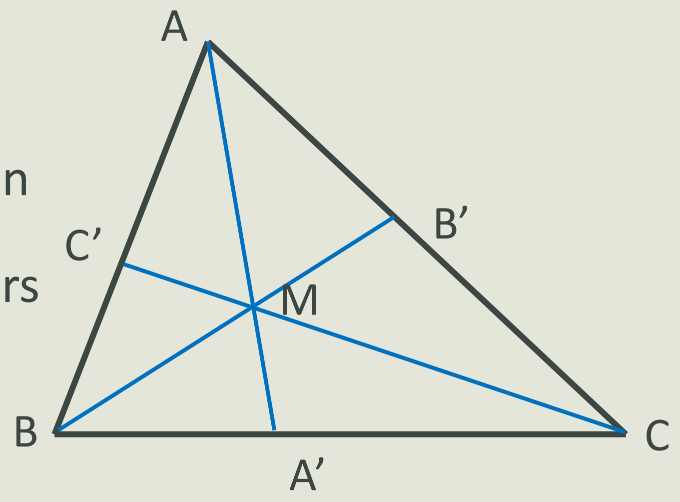
- **Corollary 2 of Ceva:** The altitudes of a triangle are concurrent
- **Proof:**



Special cases

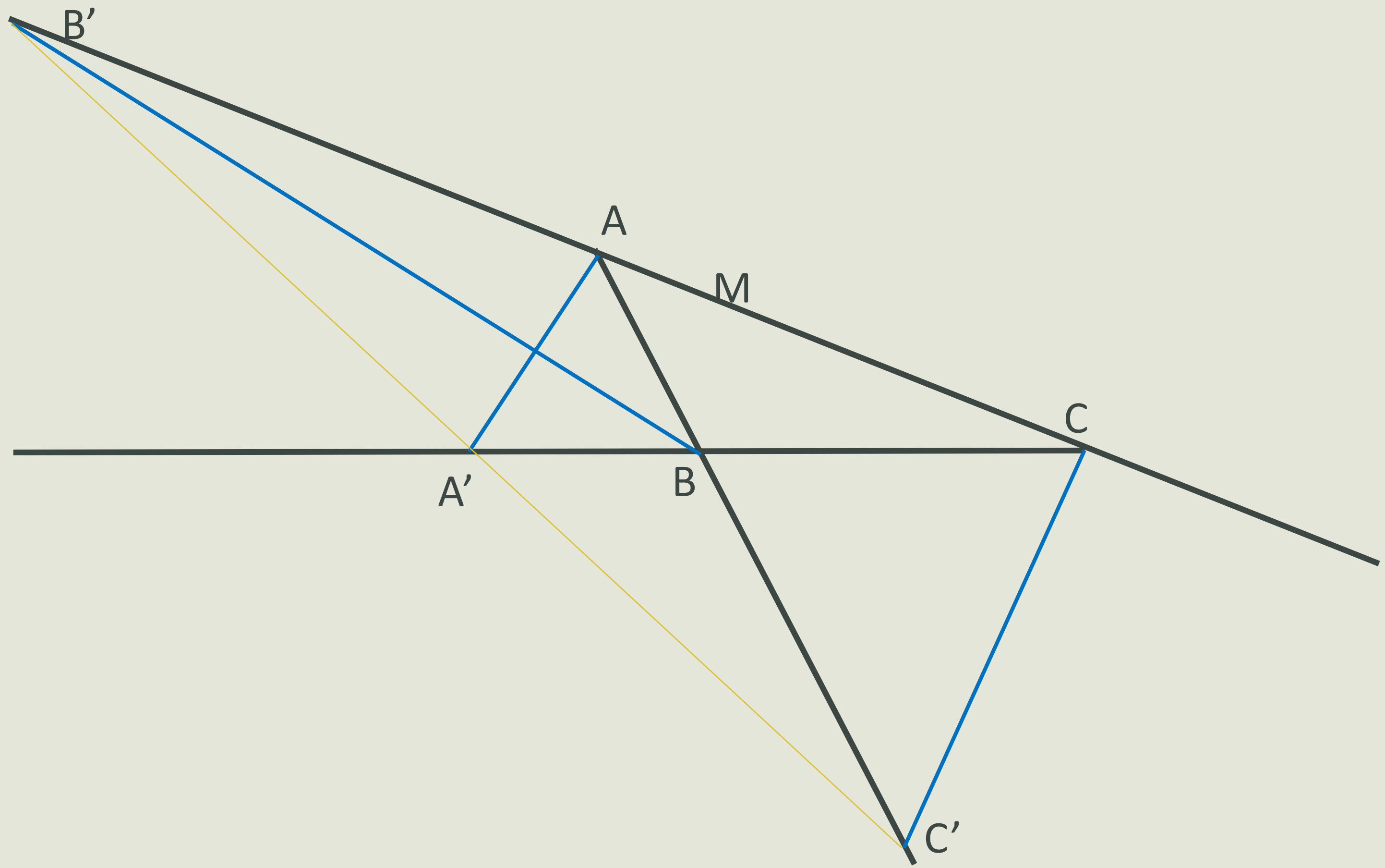
- **Corollary 3 of Ceva:** The (interior) angle bisectors of a triangle are concurrent.
- **Proof:**

Hint. Calculate areas of AA’B and AA’C in 2 ways to find a relation between BA’/A’C and BA/AC .Repeat this for other angle bisectors



Special cases

- **Corollary 1 of Menelaus:** The external angle bisectors of a triangle intersect their opposite sides at three collinear points

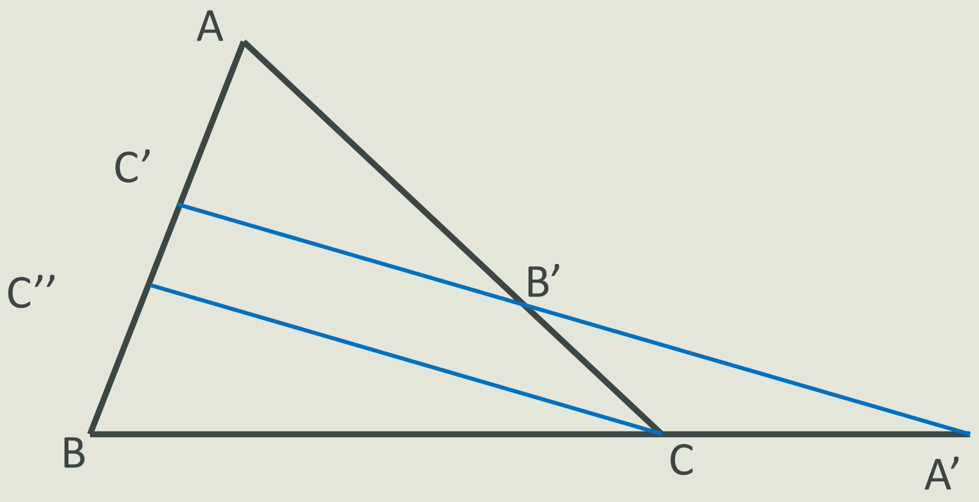


Proofs

Proof of Menelaus.

- AB’/B’C = AC’/C’C’’
- CA’/A’B=C’C’’/C’B

$$\rightarrow \frac{AB'}{B'C} \times \frac{CA'}{A'B} \times \frac{BC'}{C'A} = \frac{AC'}{C'C''} \times \frac{C'C''}{C'B} \times \frac{BC'}{C'A} = 1$$

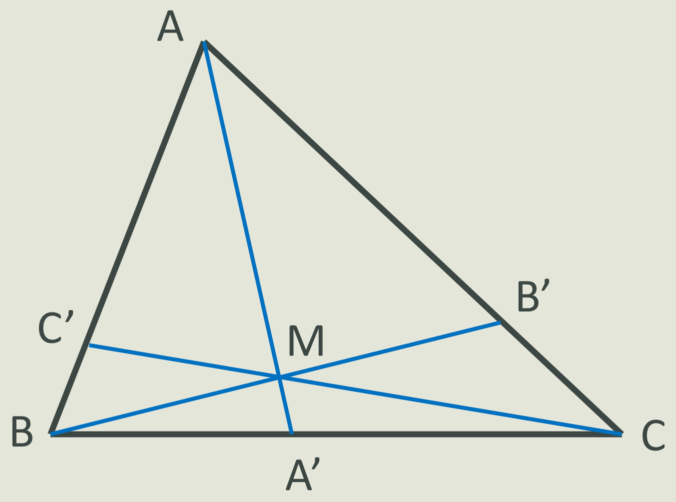


Proofs

Proof of Ceva. We use Menelaus's theorem 2 times

1- Triangle AA'C and line BB'

2- Triangle AA"B and line CC'



1->

2->

1+2->

More questions to practice at home

1. http://math.fau.edu/yiu/MPS2016/PSRM2016I.pdf
2. https://math.osu.edu/sites/math.osu.edu/files/ceva-menelaus.pdf
3. https://math.stackexchange.com/questions/2608959/find-the-ratio-of-segments-using-cevas-theorem

More advanced discussion:

- https://www.mathpages.com/home/kmath442/kmath442.htm

Practice question from Source 1

4. Given three circles with centers A, B, C and distinct radii, show that the exsimilicenters of the three pairs of circles are collinear.