Complex Geometry

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1 Preliminaries in complex analysis

In this section, we briefly review some definitions and basic facts from the theory of holomorphic functions in one and several variables.

There are a few different (but equivalent) ways to define a holomorphic function. We start from a definition that would naturally extend to almost complex geometry. Let

$$F\colon \Omega \subset \mathbb{C}^n \to \mathbb{C}^m$$

be a smooth function defined on an open domain Ω. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way, we can think of F as a smooth function $F\colon \Omega \subset \mathbb{R}^{2n} \to \mathbb{R}^{2m}$. For every $z \in \Omega$, let

$$D_zF\colon T_z\mathbb{R}^{2n} \cong \mathbb{R}^{2n} \longrightarrow T_{F(z)}\mathbb{R}^{2m} \cong \mathbb{R}^{2m}$$

denote the $2m \times 2n$ derivative (matrix) of F.

Definition 1.1. We say F is holomorphic if, at every point $z \in \Omega$, D_zF is complex-linear with respect to the canonical complex structures on $T_z\mathbb{R}^{2n} \cong \mathbb{C}^n$ and $T_{F(z)}\mathbb{R}^{2m} \cong \mathbb{C}^m$.

In other words, writing

$$z=(z_1,\dots,z_n)=(x_1+\mathfrak{i}y_1,\dots,x_n+\mathfrak{i}y_n),$$

we can think of the multiplication by $\mathfrak{i}=\sqrt{-1}$ on $T_z\mathbb{C}^n$ as the real linear map

$$\mathfrak{i}_{\mathbb{R}^{2n}}=\begin{bmatrix}0 & -I_n\\ I_n & 0\end{bmatrix}: T_{(x,y)}\mathbb{R}^{2n}\longrightarrow T_{(x,y)}\mathbb{R}^{2n}, \qquad \text{with} \quad (x,y)=(x_1,\dots,x_n,y_1,\dots,y_n),$$

that sends $\frac{\partial}{\partial x_a}$ to $\frac{\partial}{\partial y_a}$ and $\frac{\partial}{\partial y_a}$ to $-\frac{\partial}{\partial x_a}$, for all $a=1,\dots,n$. Similarly, we can identify the multiplication by $\mathfrak{i}$ on $T_{F(z)}\mathbb{C}^m$ with the real linear map

$$\mathfrak{i}_{\mathbb{R}^{2m}}=\begin{bmatrix}0 & -I_m\\ I_m & 0\end{bmatrix}: T\mathbb{R}^{2m}\longrightarrow T\mathbb{R}^{2m}.$$

Then, F is holomorphic if and only if the matrix identity

$$D_{(x,y)}F\circ\mathfrak{i}_{\mathbb{R}^{2n}}=\mathfrak{i}_{\mathbb{R}^{2m}}\circ D_{(x,y)}F$$

holds at every point $z=(x,y)\in\Omega$. If we write $F=(f_1,\dots,f_m)$ with

$$f_b\colon\Omega\longrightarrow\mathbb{C} \qquad \forall\ b=1,\dots,m,$$

then F is holomorphic if and only if each component f_b of F is holomorphic. This reduces the discussion to $m=1$.

For $z\in\mathbb{C}$, define

$$\frac{\partial}{\partial z}=\frac{1}{2}\Big(\frac{\partial}{\partial x}-\mathfrak{i}\frac{\partial}{\partial y}\Big) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}}=\frac{1}{2}\Big(\frac{\partial}{\partial x}+\mathfrak{i}\frac{\partial}{\partial y}\Big).$$

These are $\mathbb{C}$-valued tangent vectors that are dual to $\mathbb{C}$-valued 1-forms

$$\mathrm{d}z=\mathrm{d}x+\mathfrak{i}\mathrm{d}y \quad \text{and} \quad \mathrm{d}\overline{z}=\mathrm{d}x-\mathfrak{i}\mathrm{d}y.$$

Note that we have

$$\frac{\partial}{\partial z}z=\mathrm{d}z\Big(\frac{\partial}{\partial z}\Big)=\frac{\partial}{\partial \overline{z}}\overline{z}=\mathrm{d}\overline{z}\Big(\frac{\partial}{\partial \overline{z}}\Big)=1 \quad \text{and} \quad \frac{\partial}{\partial z}\overline{z}=\frac{\partial}{\partial \overline{z}}z=0.$$

Let $f\colon\Omega\subset\mathbb{C}^n\to\mathbb{C}$ be a smooth function. Then, $\mathrm{d}f$ can be decomposed into $\mathbb{C}$-linear and anti $\mathbb{C}$-linear parts,

$$\mathrm{d}f=\partial f+\overline{\partial}f, \tag{1.1}$$

where

$$\partial f:=\sum_{a=1}^{n}\frac{\partial f}{\partial z_a}\,\mathrm{d}z_a \quad \text{and} \quad \bar{\partial}f:=\sum_{a=1}^{n}\frac{\partial f}{\partial \overline{z}_a}\,\mathrm{d}\overline{z}_a.$$

Thus, f is holomorphic if and only if $\bar{\partial}f\equiv 0$. The equation $\bar{\partial}f\equiv 0$ is a set of identities

$$\frac{\partial}{\partial \overline{z}_a}f=\frac{1}{2}\Big(\frac{\partial}{\partial x_a}f-\mathfrak{i}\frac{\partial}{\partial y_a}f\Big)\equiv 0 \qquad \forall\ a=1,\dots,n.$$

that are known as the Cauchy-Riemann equation(s). It is straight-forward to check that

$$4\partial\bar{\partial}f=\Delta f\cdot\sum_{a=1}^{n}\mathrm{d}x_a\wedge\mathrm{d}y_a\,,$$

where Δ is the Laplace equation. Therefore, real and imaginary parts of every holomorphic function, as well as its norm-square $|f|^2$, are real harmonic functions. As a result, if $f\colon \Omega\subset \mathbb{C}^n \longrightarrow \mathbb{C}$ is holomorphic, then $|f|$ has no local minimum or maximum in Ω. If Ω is bounded and f can be continuously extended to $\overline{\Omega}$, then $|f|$ takes its maximal values on $\partial\overline{\Omega}$.

It is also straight-forward to check that a function $f\colon \Omega\subset\mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if it is holomorphic with respect to every variable at every point of its domain.

Example 1.2. Examples of holomorphic functions include polynomials, exponential functions, and their inverses, compositions, and linear combinations (on a suitable domain). We will mainly be concerned with polynomials.

HW 1.3. An invertible smooth map $f\colon U\subset\mathbb{R}^2 \longrightarrow V\subset\mathbb{R}^2$ between open subsets U and V is called conformal if it preserves angles at every point. Via the standard identification $\mathbb{C}=\mathbb{R}^2$, show that conformal=biholomorphic.

Holomorphic functions are rigid in the sense that knowing their values on some subdomain of a connected domain will uniquely specify their value elsewhere. Also, holomorphic functions do not admit partition of unity. Here are some basic and important facts about holomorphic functions of one and several variables:

• **Cauchy integral formula**. Suppose $f\colon \Omega\subset\mathbb{C}\to\mathbb{C}$ is a holomorphic function and $U\subset\Omega$ is the interior region[1] of a smooth embedded circle $C\subset\Omega$. Orient C in the counter-clock wise direction (with respect to any point in U). Then, for every $w\in U$ we have

$$f(w)=\frac{1}{2\pi\mathfrak{i}}\oint_C \frac{f(z)}{z-w}\mathrm{d}z. \tag{1.2}$$

In particular, for any $w\in U$, we can take U to be a sufficiently small disk around w to find a formula for $f(w)$ in terms of a line integral over a small circle around w.

HW 1.4. Use Green's theorem to verify that the right-hand side of (1.2) is independent of the choice of the curve C going around w counter-clock wise once.

For an arbitrary smooth function $f\colon \Omega\longrightarrow\mathbb{C}$, the Cauchy integral formula reads

$$f(w)=\frac{1}{2\pi\mathfrak{i}}\oint_C \frac{f(z)}{z-w}\mathrm{d}z+\frac{1}{2\pi\mathfrak{i}}\iint_U \frac{\partial f}{\partial\bar z}\frac{\mathrm{d}z\wedge\mathrm{d}\bar z}{z-w}. \tag{1.3}$$

Note that

$$\mathrm{d}z\wedge\mathrm{d}\bar z=-2\mathfrak{i}\mathrm{d}x\wedge\mathrm{d}y.$$

Thus, the second component on the righthand side of (1.3) is the double integral

$$\frac{-1}{\pi}\iint_U \frac{\partial f}{\partial\bar z}\frac{1}{z-w}\ \mathrm{d}x\mathrm{d}y.$$

The proof of (1.3) is based on Stoke's theorem for differential forms with singularities; see [**?**]. The identity (1.3) can be interpreted in the following way. The righthand side of (1.3) gives a decomposition

$$f=h+\widetilde{f}$$

[1] Recall that by the Jordan Curve Theorem, every such curve C divides the plane into an "interior" and an "exterior" region so that every continuous path connecting a point of one region to a point of the other intersects with that loop somewhere. Moreover, the interior region is simply-connected

such that h is holomorphic and $\bar{\partial}\widetilde{f} = \bar{\partial} f$. Furthermore, the $\bar{\partial}$-Poincare lemma (in one variable) states that given any smooth function $g\colon \overline{U} \longrightarrow \mathbb{C}$, the equation

$$f(w) = \frac{1}{2\pi \mathfrak{i}} \iint_U g(z) \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{z - w},$$

which comes from the second part of the righthand side in (1.3), defines a smooth function in U such that $\frac{\partial}{\partial \overline{z}} f = g$; i.e. locally, the equation $\frac{\partial}{\partial \overline{z}} f = g$ is solvable.

Note the right-hand side of (1.2) only depends on the values of f on the curve C. Not every continuous function on C has a continous extension to a holomorphic function in interior U of C, but if it exists then by (1.2) it is unique. The necessary and sufficient condition for the existence of such an extension is the following. Given any continuous functions $f\colon C \longrightarrow \mathbb{C}$ such that

$$\int_C (z-w)^n f(z) \mathrm{d}z = 0 \qquad \forall\, n \geq 0 \tag{1.4}$$

for some fixed $w \in U$, there exists a unique (and similarly denoted) continuous extension $f\colon \overline{U} \to \mathbb{C}$ of f that is holomorphic on U. Otherwise, if one of the conditions in (1.4) is not satisfied, the Cauchy integral formula defines a holomorphic function on U that does not continuously extend to the given f on C; see HW 1.7 below.

HW 1.5. With notation as above, if $f\colon \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic, use Green's theorem to show that

$$\int_C (z-w)^n f(z) \mathrm{d}z = 0 \qquad \forall\, n \geq 0$$

HW 1.6. Let C be the circle of radius 1 in $\mathbb{C}$ and thus U be the open disk of radius 1. For

$$f\colon C \longrightarrow \mathbb{C}, \qquad f(z) = \frac{1}{z} \quad \forall\, z \in C,$$

what is the holomorphic function on U arising from the Cauchy integral formula (1.2).

HW 1.7. Let C be the circle of radius 1 in $\mathbb{C}$ and thus U be the open disk of radius 1. Every smooth function

$$f\colon C \longrightarrow \mathbb{C}$$

has a Fourier expansion

$$f(\mathrm{e}^{\mathfrak{i}\theta}) = \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathfrak{i}n\theta},$$

where θ is the angle coordinate on C. What is the relation between the holomorphic function on U arising from the Cauchy integral formula (1.2) and the Fourier expansion of f on C?

For holomorphic functions of more than one variable, the Cauchy integral formula is obtained by repeating (1.2). For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^n_{>0}$, let

$$B_\varepsilon(w) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon |z_i - w_i| < \varepsilon_i \quad \forall\, i = 1, \ldots, n\}$$

be the product of disks of radius ε_i around w_i, and $f\colon \overline{B_\varepsilon(w)} \to \mathbb{C}$ be a continuous function that is holomorphic on $B_\varepsilon(w)$. Then,

$$f(w) = \frac{1}{(2\pi\mathfrak{i})^n} \oint_{|z_n - w_n| = \varepsilon_n} \cdots \oint_{|z_1 - w_1| = \varepsilon_1} \frac{f(z_1, \ldots, z_n)}{(z_1 - w_1) \cdots (z_n - w_n)} \mathrm{d}z_1 \cdots \mathrm{d}z_n. \tag{1.5}$$

• **Holomorphic functions are analytic.** It can be proved using the Cauchy integral formula that a function $f\colon \Omega \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if for every $w \in \Omega$ and any $\varepsilon \in \mathbb{R}^n_{>0}$ such that $B_\varepsilon(w) \subset \Omega$, the restriction of f to $B_\varepsilon(w)$ can be written as a convergent power series

$$\sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1\cdots i_n}(z_1-w_1)^{i_1}\cdots(z_n-w_n)^{i_n} \tag{1.6}$$

with

$$a_{i_1\cdots i_n} = \frac{1}{i_1!\cdots i_n!}\cdot\frac{\partial^{i_1+\cdots+i_n} f}{\partial z_1^{i_1}\cdots\partial z_n^{i_n}}.$$

In the one variable case, this is simply

$$f(z) = \sum_{i=0}^{\infty} a_i (z-w)^i, \tag{1.7}$$

where

$$a_i = \frac{1}{i!}\frac{\partial^i f}{\partial z^i} = \frac{1}{2\pi \mathrm{i}}\int_{|z-w|=\varepsilon}\frac{f(z)}{(z-w)^{i+1}} \qquad \forall\ i \geq 0.$$

• **Identity Principle.** As we mentioned above, it follows from the harmonicity property that if $f, g\colon \Omega \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ are holomorphic, Ω is a connected open set, and $f = g$ on a non-empty open subset $V \subset U$, then $f = g$ everywhere. In other words, the (discrete data of) coefficients of the Taylor expansion at any point $w \in \Omega$, uniquely determine f at every other point of a given connected domain Ω.

• **Riemann mapping theorem.** Let $U \subset \mathbb{C}$ be a simply-connected proper open subset. Then U is biholomorphic to the open unit disk $B_1(0)$; i.e. there is a holomorphic function $f\colon U \longrightarrow B_1(0)$ that is invertible (the inverse will be automatically holomorphic).

Remark 1.8. In the case of a simply connected-bounded domain with smooth boundary, the Riemann mapping function and all its derivatives extend continuously to the closure of the domain.

Remark 1.9. (Prove that) The group of biholomorphisms (or automorphisms) of the upper-half plane

$$\mathcal{H} = \{z \in \mathbb{C}\colon \mathrm{Im}(z) > 0\} \tag{1.8}$$

is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$, where

$$\mathrm{PSL}(2,\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R})\colon \det(A) = 1\}/\pm I_{2\times 2},$$

and every

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

acts on $\mathcal{H}$ by the so called mobius transformation

$$z \longrightarrow \frac{az+b}{cz+d}.$$

By Riemann mapping theorem, the automorphism group of every other simply-connected domain is conjugate to $\mathrm{PSL}(2,\mathbb{R})$.

HW 1.10. Describe a biholomorphic function that maps a square to a disk. Hint: search Google for Schwarz-Christoffel-Transformations.

• **Liouville theorem.** It can be proved using the Cauchy integral formula that every bounded holomorphic function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is constant. This explains why U must be a proper domain in the Riemann extension theorem.

• **Residue theorem.** Suppose w is a point in the open domain $\Omega \subset \mathbb{C}$ and $f\colon \Omega - \{w\} \longrightarrow \mathbb{C}$ is a holomorphic function. Then, for every $\varepsilon > 0$ such that $B_\varepsilon(w) \subset \Omega$, restricted to $B_\varepsilon(w) - \{w\}$, the function f admits a Laurent series expansion

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} a_n (z-w)^{n+1} \\ a_n &= \frac{1}{2\pi \mathrm{i}} \int_{|z-w|=\varepsilon} \frac{f(z)}{(z-w)^{n+1}} \mathrm{d}z \qquad \forall\, n \in \mathbb{Z}. \end{aligned} \tag{1.9}$$

In the integral above we can replace the circle $|z| = \varepsilon$ with any other loop in Ω around w that has winding number 1. The coefficient a_{-1} is called the residue of the function $f(z)$ (or the holomorphic differential 1-form $f(z)\mathrm{d}z$) at w. The function is f is said to have a pole of order $n > 0$ at w if $a_{-n} \neq 0$ and $a_m = 0$ for all $m > n$. The function is f is said to have an essential singularity at w if the set

$$\{n > 0\colon a_{-n} \neq 0\}$$

is unbounded.

The Laurent series formula (1.9) remains valid if f is only defined on an annulus

$$A_{\varepsilon,\delta}(w) := B_\varepsilon(w) - B_\delta(w)$$

for some $\delta < \varepsilon$ and the integral in (1.9) can be evaluated on any loop in $A_{\varepsilon,\delta}(w)$ that has winding number 1 around w.

Definition 1.11. A meromorphic function on an open domain $\Omega \subset \mathbb{C}$ is a holomorphic function $f\colon \Omega - S \longrightarrow \mathbb{C}$ for some discrete subset $S \subset \Omega$ such that f has poles of finite order at the points of S.

Remark 1.12. We will learn late that a meromorphic function $f\colon \Omega - S \longrightarrow \mathbb{C}$ can be seen as a holomorphic function $f\colon \Omega \longrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$, where $\mathbb{CP}^1 \cong S^2$ is a closed holomorphic manifold.

HW 1.13. Suppose $f\colon \Omega \subset \mathbb{C} \longrightarrow \mathbb{C} \cup \infty$ is a meromorphic function. Show that f'/f is meromorphic function with simple poles at zeros and poles of f. Furthermore, show that the residue of f'/f at any zero/pole of f is equal to the signed order of f at the point; i.e. the residue of f'/f at any zero of order d of f is d and the residue of f'/f at any pole of order d of f is $-d$. Use the formula for residue formula to prove that for any Jordan curve $C \subset \Omega$, the integral

$$\oint \frac{f'(z)}{f(z)} \mathrm{d}z$$

is equal to sum of the (signed) orders of the zeros and poles of f inside C.

• **Hartog's theorem.** In the context of the Residue theorem, if we increase the dimension, we will see a very much different behavior.

Theorem 1.14. *Suppose* $n>1$*,* w *is a point in the open domain* $\Omega\subset\mathbb{C}^n$*,* $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_n)\in\mathbb{R}^n_{>0}$ *such that* $\overline{B_\varepsilon(w)}\subset\Omega$ *and* $f\colon\Omega-\overline{B_\varepsilon(w)}\longrightarrow\mathbb{C}$ *is a holomorphic function. Then* f *has a unique holomorphic extension to* Ω*.*

Proof. Since $\overline{B_\varepsilon(w)}\subset\Omega$, there exists a sufficiently small $\delta>0$ such that

$$B_{\varepsilon+\delta}(w):=B_{\varepsilon_1+\delta,\ldots,\varepsilon_n+\delta}(w)\subset\Omega.$$

We restrict to the open subset

$$V_1=\{(z_1,\ldots,z_n)\in B_{\varepsilon+\delta}(w)\colon \varepsilon_1<|z_1|\}\subset B_{\varepsilon+\delta}(w)-\overline{B_\varepsilon(w)}$$
$$V_2=\{(z_1,\ldots,z_n)\in B_{\varepsilon+\delta}(w)\colon \varepsilon_2<|z_2|\}\subset B_{\varepsilon+\delta}(w)-\overline{B_\varepsilon(w)}$$

and $V=V_1\cup V_2$. Note that the image of the natural projection map

$$\pi\colon V\longrightarrow\mathbb{C}^{n-1},\qquad z=(z_1,\ldots,z_n)\longrightarrow z'=(z_2,\ldots,z_n) \tag{1.10}$$

is the full polydisk

$$B_{\varepsilon_2+\delta,\ldots,\varepsilon_n+\delta}(w')\subset\mathbb{C}^{n-1},\quad w'=\pi(w),$$

and the fiber of π over any z' contains the annulus $A_{\varepsilon_1+\delta,\varepsilon}(w_1)\subset\mathbb{C}$. For each $z'=(z_2,\ldots,z_n)\in B_{\varepsilon_2+\delta,\ldots,\varepsilon_n+\delta}(w')$, define

$$f_{z'}\colon\pi^{-1}(z')\longrightarrow\mathbb{C},\quad f_{z'}(z_1)=f(z_1,z_2,\ldots,z_n)$$

to be the restriction of f to the fiber $\pi^{-1}(z')$. Since $\pi^{-1}(z')$ contains the annulus $A_{\varepsilon_1+\delta,\varepsilon}(w_1)$, by the Residue theorem (over an annulus), each $f_{z'}$ has a Laurent series expansion

$$f_{z'}(z_1)=\sum_{n=-\infty}^{\infty}a_n(z')z_1^n.$$

By the (multivariable version of the) Cauchy integral formula, for each $n\in\mathbb{Z}$, the coefficient $a_n(z')$ itself is a holomorphic function of z'. On the other hand, if z' belongs to the subdomain $\pi(V_2)\subset B_{\varepsilon_2+\delta,\ldots,\varepsilon_n+\delta}(w')$, then $\pi^{-1}(z')$ is the entire disk $B_{\varepsilon_1+\delta}(w_1)$. Therefore, for such z',

$$a_n(z')=0\qquad\forall\, n<0.$$

For $n<0$, since $a_n(z')=0$ on a subdomain of the connected domain $B_{\varepsilon_2+\delta,\ldots,\varepsilon_n+\delta}(w')$, by the Identity Principle, it vanishes for all $z'\in B_{\varepsilon_2+\delta,\ldots,\varepsilon_n+\delta}(w')$. However, that implies that we can define the holomorphic extension of f to $B_{\varepsilon+\delta}(w)$ by the power series

$$\sum_{n=0}^{\infty}a_n(z')z_1^n.$$

The Taylor series converges uniformly since the coefficients a_n are holomorphic and achieve their maximum absolute value at the boundary. Clearly, the holomorphic extension to $B_{\varepsilon+\delta}(w)$ given by this power series glues with f on Ω along the overlapping region $B_{\varepsilon+\delta}(w)-\overline{B_\varepsilon(w)}$ to give the desired holomorphic function. □

The main reason for the very different results in the Residue and Hartog's theorems is topological. If $n = 1$, then $\mathbb{C} - \overline{B_\varepsilon(w)}$ is not simply connected, while if $n > 1$, then $\mathbb{C}^n - \overline{B_\varepsilon(w)}$ is simply connected. In other words, Cauchy Integral Formula sees loops in non-simply-connected domains of definition of a holomorphic 1-form. In order to extend the Residue theorem to higher dimensions and state the Hartog's theorem more generally, we first need to review Weierstrass Theorem and define a few concepts.

Suppose $\Omega \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ is an open domain and $g\colon \Omega \longrightarrow \mathbb{C}$ is a holomorphic function. The zero set

$$Z(g) := \{z \in \Omega\colon g(z) = 0\}$$

is called the zero-divisor of g. If $\mathrm{d}g \neq 0$ along $Z(g)$, then $Z(g)$ is a complex hypersurface (complex codimension 1) sub-space of Ω (possibly empty). Otherwise, as Weierstrass Theorem 1.18 illustrates, $Z(g)$ will still be a complex codimension 1 analytic subspace of Ω possibly with singularities. This is quite different from the real analysis where, for example, the zero set of the function

$$\mathbb{R}^n \to \mathbb{R}, \qquad (x_1, \ldots, x_n) \longrightarrow x_1^2 + \ldots + x_n^2$$

is just a single point (i.e. has codimension n). Over complex numbers, the zero set of $z_1^2 + \ldots + z_n^2$ is a complex codimension 1 "variety" that is singular only at the origin.

HW 1.15. Show that $Z(z_1^2 + z_2^2)$ is a union of two complex lines.

In one variable, by factoring out the largest common factor of $(z - w)^i$, we conclude that every holomorphic function g has a unique local decomposition

$$g(z) = (z - w)^d u(z), \qquad u(w) \neq 0.$$

From this we conclude that the zero locus of g is discrete (countably many points) and every zero of g has a well-defined multiplicity $d > 0$. Similarly, in higher dimensions, the Weierstrass Preparation Theorem below gives a local decomposition of any holomorphic function, using which we can study its zero divisor $Z(g)$.

Definition 1.16. A Weierstrass polynomial of degree d in a neighborhood of $w \in \mathbb{C}^n$, with respect to the coordinate z_1 in a coordinate system $z = (z_1, z_2, \ldots, z_n)$, is a holomorphic function of the form

$$p(z) = p(z_1, z') = (z_1 - w_1)^d + \alpha_1(z')(z_1 - w_1)^{d-1} + \ldots + \alpha_d(z'), \qquad z' = (z_2, \ldots, z_n),$$

such that the coefficients $\alpha_i(z')$ are holomorphic function on some sufficiently small neighborhood of w' and they vanish at w'. In particular, we have $p(w) = 0$ and, restricted to the z_1-axis at w, the function p is simply $(z_1 - w_1)^d$.

Example 1.17. The function

$$p(z_1, z_2) = z_1^2 + z_2^2 + z_1 z_2$$

is a Weierstrass polynomial of degree 2 in a neighborhood of the origin with respect to both z_1 and z_2 in the standard coordinates. With respect to z_1, we have

$$\alpha_1(z_2) = z_2 \quad \text{and} \quad \alpha_2(z_2) = z_2^2.$$

Suppose $f\colon \Omega \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ is holomorphic, $f(w) = 0$, and f does not vanish identically on the z_1-axis. The latter implies that the power series expansion (1.6) of f near w contains a term of the form $a(z_1 - w_1)^d$ with $a \neq 0$ and $d \geq 1$.

Theorem 1.18. *Suppose $f\colon \Omega \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ is holomorphic, $f(w) = 0$, f does not vanish identically on the z_1-axis through w. Then, there exists a unique Weierstrass polynomial p of degree d in a neighborhood of $w \in \mathbb{C}^n$ such that $f = p \cdot u$ on a sufficiently small neighborhood of w for some holomorphic function u with $u(w) \neq 0$.*

Proof. For each $z' = (z_2, \ldots, z_n) \in \mathbb{C}^n$ sufficiently close to $w' = (w_2, \ldots, w_n)$ let

$$f_{z'}\colon \Omega \cap \big(\mathbb{C} \times \{z'\}\big) \longrightarrow \mathbb{C}, \qquad f_{z'}(z_1) := f(z_1, z'),$$

denote the restriction of f to fiber over z' via the projection map (1.10). By assumption $f_{w'} \not\equiv 0$, $f_{w'}(w_1) = 0$, and the leading order term of the Taylor expansion of $f_{w'}$ around w_1 is a monomial $a(z_1 - w_1)^d$ for some integer $d > 0$. Thus, there exist sufficiently small $\varepsilon_1, \delta > 0$ and $\varepsilon' = (\varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{R}^{n-1}_{>0}$ such that

(1) the only zero of $f_{w'}$ in $B_{\varepsilon_1}(w_1)$ is w_1;

(2) $f_{w'}(z_1) \geq \delta$ whenever $|z_1 - w_1| = \varepsilon_1$;

(3) and consequently, $f_{z'}(z_1) \geq \delta/2$ whenever $|z_i - w_i| \leq \varepsilon_i$ for $i = 2, \ldots, n$ and $|z_1 - w_1| = \varepsilon_1$.

First, it follows from the Item 3 above and the last statement of HW 1.13 that, for each $z' \in B_{\varepsilon'}(w')$, $f_{z'}$ has exactly d zeros, say $\zeta_1(z'), \ldots, \zeta_d(z')$, in the disk $|z_1 - w_1| = \varepsilon_1$, counted with multiplicities. For each $z' \in B_{\varepsilon'}(w')$, consider the degree d polynomial

$$p_{z'}(z_1) = \prod_{i=1}^{d} (z_1 - \zeta_i(z')).$$

We show that these polynomials deform homomorphically with z'; i.e. the overall function

$$p\colon B_\varepsilon(w) \longrightarrow \mathbb{C}, \qquad p(z) = p(z_1, z') = p_{z'}(z_1) \quad \forall\ z = (z_1, z') \in B_\varepsilon(w) \tag{1.11}$$

where $\varepsilon = (\varepsilon_1, \varepsilon')$, is holomorphic. In order to prove this statement, note that, for each $z' \in B_{\varepsilon'}(w')$, we have

$$p_{z'}(z_1) = (z_1 - w_1)^d + \alpha_1(z')(z_1 - w_1)^{d-1} + \cdots + \alpha_d(z'),$$

where each α_m is a symmetric polynomial in $\zeta_1(z'), \ldots, \zeta_d(z')$. It is known that every such symmetric polynomial can be written (via a universal polynomial formula) in terms of the elementary symmetric polynomials

$$\zeta_1(z')^b + \ldots + \zeta_d(z')^b, \qquad b \in \mathbb{N}.$$

On the other hand, it can be shown as in HW 1.13 that

$$\zeta_1(z')^b + \ldots + \zeta_d(z')^b = \frac{1}{2\pi \mathrm{i}} \oint_{|z_1 - w_1| = \varepsilon_1} \frac{z_1^b (\partial f_{z'}/\partial z_1)(z_1, z')}{f(z_1, z')} \mathrm{d}z_1;$$

see the HW below. The righthand side is the integral of a holomorphic function with respect to z_1. Thus, it defines a holomorphic function of z'. In conclusion, the functions $\alpha_1(z'), \ldots, \alpha_d(z')$ are holomorphic. This hows that the function p defined in (1.11) is a holomorphic function on $B_\varepsilon(w)$. Finally, the quotient

$$u(z_1, z') = \frac{p(z_1, z')}{f(z_1, z')}$$

has removable singularities at $\zeta_1, \ldots, \zeta_d$. Thus, it is defined on the entire $B_\varepsilon(w)$ and is non-zero on it. □

The so-called Weierstrass Preparation Theorem above shows that, unlike for smooth real-valued functions, locally in a suitable coordinate system, the zero set of a non-trivial holomorphic function on an open set of $\mathbb{C}^n$ is a finite-sheeted branched cover of $\mathbb{C}^{n-1}$. The branch loci is the discriminant locus of the corresponding Weierstrass polynomial!

HW 1.19. Confirm the following statement used in the proof above. Suppose $f\colon \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$ is a holomorphic function. Let C be a Jordan curve in Ω. Suppose $f|_C$ is non-zero and $\zeta_1, \ldots, \zeta_d$ are the zeros of f inside C, counted with multiplicity. Then, for each $b \in \mathbb{N}$, we have

$$\zeta_1^b + \ldots + \zeta_d^b = \frac{1}{2\pi\mathfrak{i}} \oint_C \frac{z^b f'(z)}{f(z)} \mathrm{d}z.$$

In the one variable case, we observed that a holomorphic function defined on the complement of a point $w \in \Omega \subset \mathbb{C}$ either has a pole at w or has a holomorphic extension to w. The latter happens if and only if f is bounded near w. In light of Hartog's theorem, we realized that points in $\mathbb{C}$ generalize to complex hypersurfaces in higher dimensions. Therefore, it is natural to have the following result known as the Riemann Extension Theorem.

Theorem 1.20. *Suppose $g\colon \Omega \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ is a holomorphic function. If $f\colon \Omega \setminus Z(g) \longrightarrow \mathbb{C}$ is holomorphic and locally bounded near $Z(g)$, then f uniquely extends to a holomorphic function on the entire Ω.*

Proof. Around every point $w \in Z(g)$, we can choose a suitable coordinate system $(z_1, \ldots, z_n)$ such that the z_1-axis through w is not contained in $Z(g)$; see Remark 1.22 below. As in the proof above, we can find $\varepsilon_1, \delta > 0$ and $\varepsilon' = (\varepsilon_2, \ldots, \varepsilon_n)$ such that $f(z_1, w') \geq \delta$ on $|z_1 - w_1| = \varepsilon_1$ and $f_{z'}(z_1) \geq \delta/2$ for $z' \in B_{\varepsilon'}(w')$ and $|z_1 - w_1| = \varepsilon_1$. By the one-variable Riemann Extension Theorem and the boundedness assumption, on each slide $B_{\varepsilon_1}(w_1) \times \{z'\}$, we can extend $f_{z'}$ to a similarly denoted holomorphic function. As before, by Cauchy Integral Formula, we have

$$f(z_1, z') = \frac{1}{2\pi\mathfrak{i}} \oint_{|z_1 - w_1| = \varepsilon_1} \frac{f(z_1, z')}{z_1 - w_1} \mathrm{d}z_1$$

which shows that the extended function depends homomorphically on z' as well. □

Remark 1.21. As in the one-variable case, the result remains true if we replace $Z(g)$ with a sufficiently small neighborhood of that in the statement of Theorem 1.20.

Remark 1.22. Just as in Calculus of real functions, we have some flexibilities in choosing a good coordinate system around points. Suppose $w \in U \subset \mathbb{C}^n$ and $F\colon U \longrightarrow \mathbb{C}^n$ is a holomorphic function. Then the complex Jacobian of F is the complex linear matrix

$$\mathrm{d}_z F\colon T_z\mathbb{C}^n \cong \mathbb{C}^n \longrightarrow T_{F(z)}\mathbb{C}^n \cong \mathbb{C}^n.$$

If $\mathrm{d}_z F$ is full-rank, then restricted to a sufficiently small neighborhood V of w, $F|_U$ is a biholomorphic map onto its image $V' = F(V)$ (Inverse Function Theorem). Therefore, $(z_1', \ldots, z_n') = F(z)$ can be used as a new coordinate system around z. Similarly, by Implicit Function Theorem, if

$$F\colon U \subset \mathbb{C}^n \longrightarrow \mathbb{C}^m$$

is holomorphic, $n \geq m$, and $\mathrm{d}_z F$ is full-rank, then there exists

- sufficiently small neighborhoods $V_1 \supset z$ and $V_2 \supset F(z)$,

- local coordinates $(z_1, \ldots, z_n)$ on V_1,
- and local coordinates $(x_1, \ldots, x_m)$ on V_2,

such that $f(z_1, \ldots, z_n) = (z_1, \ldots, z_m)$.

Remark 1.23. In the world of smooth real functions, bijectivity of a function does not imply its regularity. For example $x \longrightarrow x^3$ is a bijective map from $\mathbb{R}$ to $\mathbb{R}$ but its derivative is zero at $x = 0$ and thus does not admit a smooth inverse. Contrary to the real world, a holomorphic function is bi-holomorphic if and only if it is bijective.

Before stating the last results of this section, we need to recall some elementary definitions from commutative algebra. Let $\mathcal{R}$ be ring, then we say

- $\mathcal{R}$ is an integral domain if $u, v \in \mathcal{R}$ and $uv = 0$ implies $u = 0$ or $v = 0$;
- An element $u \in \mathcal{R}$ is a unit if there exists $v \in \mathcal{R}$ such that $uv = 1$;
- $u \in \mathcal{R}$ is irreducible (prime) if $u = vw$ implies one of v or w is a unit;
- $\mathcal{R}$ is Unique Factorization Domain (UFD) every u can be uniquely written as a product of irreducible (up to taking product with unit elements).

Some of the main results are

(1) if $\mathcal{R}$ is UFD then so is the polynomial ring $\mathcal{R}[t]$;

(2) if $\mathcal{R}$ is UFD and $p, q \in \mathcal{R}[t]$ are relatively prime, then there exists relatively prime elements $\alpha, \beta \in R[t]$ and $0 \neq \gamma \in \mathcal{R}$ (called resultant of p and q) such that

$$\alpha p + \beta q = \gamma.$$

Let $\mathcal{O}_{\mathbb{C}^n}$ denote the sheaf of holomorphic functions on $\mathbb{C}^n$; c.f. my Lecture notes from Math 6410 for the definition of sheaf. Briefly, a sheaf $\mathcal{F}$ on a topological space X is a way of assigning a set $\mathcal{F}(U)$ to each open set $U \subset X$ such that

- if $V \subset U$, then there is a restriction map $r_{U,V} \colon \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$;
- if $U \supset V \supset W$, then $r_{V,W} \circ r_{U,V} = r_{U,W}$;
- if $\{s_\alpha \in \mathcal{F}(U_\alpha)\}_{\alpha \in \mathcal{I}}$ is a collection of local "sections" of $\mathcal{F}$ such that

  $$s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta} \qquad \forall\ \alpha, \beta \in \mathcal{I},$$

  then we can uniquely glue them to obtain a section

  $$s \in \mathcal{F}(U), \qquad U = \bigcup_{\alpha \in \mathcal{I}} U_\alpha,$$

  such that $s|_{U_\alpha} = s_\alpha$ for all $\alpha \in \mathcal{I}$.

Here, $\mathcal{O}_{\mathbb{C}^n}$ is the sheaf that assigns to each open set $U \subset \mathbb{C}^n$ the ring $\mathcal{O}_{\mathbb{C}^n}(U)$ of holomorphic functions on U.

Formally, the stalk of a sheaf $\mathcal{F}$ at a point $x \in X$, denoted by $\mathcal{F}_x$, is the direct limit of $\mathcal{F}(U)$ over all open sets U such that $x \in U$; i.e.

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U).$$

In other words, an element in $\mathcal{F}_x$ is simply the germ of a section around x. If $x \in U$, every section $s \in \mathcal{F}(U)$ defines an element $[s]$ of $\mathcal{F}_x$. Furthermore, if $x \in U_1 \cap U_2$, then $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ define the same element $[s_1] = [s_2] \in \mathcal{F}_x$ iff they agree on possibly smaller neighborhood $x \in V \subset U_1 \subset U_2$. For the sheaf holomorphic functions, by the Identity Principle, the values of $f \in \mathcal{O}_{\mathbb{C}^n}(U)$ in an arbitrary small neighborhood of any point $x \in U$ will uniquely determine its values on U. Therefore, it is safe to denote the element in $\mathcal{O}_{\mathbb{C}^n,x}$ corresponding to f by f (and not $[f]$).

An important consequence of the Weierstrass Preparation Theorem is the following result. Each $\mathcal{O}_{\mathbb{C}^n,z}$ is an integral domain by the Identity Principle. Moreover, it is a local ring (i.e. it has a unique maximal ideal) whose maximal ideal is the set of locally-defined holomorphic functions vanishing at z. The unit elements of $\mathcal{O}_{\mathbb{C}^n,z}$ correspond to locally-defined holomorphic functions f such that $f(z) \neq 0$.

Theorem 1.24. *The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ (and thus any other $\mathcal{O}_{\mathbb{C}^n,z}$) is a UFD.*

Proof. We prove this statement by induction on $n \in \mathbb{N}$. For $n = 0$, $\mathcal{O}_{\mathbb{C}^0,0} = \mathbb{C}$ which is UFD. Assume $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is UFD. Let f be a holomorphic function on a sufficiently small neighborhood of 0 defining an element in $\mathcal{O}_{\mathbb{C}^n,0}$. Just as in the arguments above, by choosing a suitable coordinate system around 0, we may assume that the z_1-axis is not contained in $Z(f)$. By Weierstrass Preparation Theorem, we can write $f = p \cdot u$ where u is a unit in $\mathcal{O}_{\mathbb{C}^n,0}$ and $p \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. By Property (1) in Page 11 (known as Gauss Lemma) and the induction assumption, $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a UFD. Therefore, after possibly shrinking the domain of f, we have

$$f = p_1 \cdots p_m \cdot u$$

such that $p_i \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ are irreducible and are uniquely determined from p up to multiplication by units and permutation. It is only left to show that each irreducible factor p_i is also irreducible as an element of $\mathcal{O}_{\mathbb{C}^n,0}$. That would follow from the uniqueness statement of Theorem 1.18 and that every Weierstrass has a finite positive degree. □

Proposition 1.25. *Suppose $f \in \mathcal{O}_{\mathbb{C}^n,0}$ and $p \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial of degree d. Then, there exist $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ of degree less than d and $h \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $f = ph + r$. The functions h and r are uniquely determined.*

The result can be used to show that $\mathcal{O}_{\mathbb{C}^n,0}$ is a Noetherian ring meaning that every ideal in $\mathcal{O}_{\mathbb{C}^n,0}$ is finitely generated.

Proposition 1.26. *The local UFD $\mathcal{O}_{\mathbb{C}^n,0}$ is Noetherian.*

Corollary 1.27. *(Weak Nullstellensatz) Suppose $g \in \mathcal{O}_{\mathbb{C}^n,0}$ is irreducible and $f \in \mathcal{O}_{\mathbb{C}^n,0}$. If f vanishes on $Z(g)$, then g divides f.*

Instead of going through the proofs of these statements, we explain the geometric meaning of them. The main purpose of these results is to characterize the local properties of *analytic subvarieties* in $\mathbb{C}^n$.

Definition 1.28. Suppose $\Omega \subset \mathbb{C}^n$ is an open domain. We say $X \subset \Omega$ is an analytic variety if, for every $w \in X$, there exists a sufficiently small neighborhood $U \ni w$ such that $X \cap U$ is the common zero locus of a finite collection of holomorphic functions $f_1, \ldots, f_k$ on U. In particular, we say X is an analytic divisor/hypersurface if it is locally the zero set of just one holomorphic function.

Remark 1.29. Since the definition is local, it will readily extends to arbitrary complex manifolds defined in Section 2.

Remark 1.30. An Analytic variety $X \subset \mathbb{C}^n$ can be singular. We denote the set of singular points by X^{sing}. Away from X^{sing}, (each component of) X is a complex submanifold of $\mathbb{C}^n$ and the number of local defining equations will be exactly the codimension (this can vary from component to component).

Example 1.31. Let $X_\lambda \subset \mathbb{C}^2$ be the analytic hypersurface defined by

$$f_\lambda(z_1, z_2) = z_1^2 + z_2^3 - \lambda = 0.$$

Since

$$\mathrm{d}f_\lambda = 2z_1 \mathrm{d}z_1 + 3z_2^2 \mathrm{d}z_3,$$

the only possible singular point is the origin $(0, 0)$. Therefore, for $\lambda \neq 0$, X_λ is non-singular (it is an open Riemann surface diffeomorphic to $T^2 -$ point), and for $\lambda = 0$ it has a so-called cusp singularity at the origin.

Definition 1.32. An analytic subvariety $X \subset \Omega \subset \mathbb{C}^n$ is called irreducible if it is not a union of two or more analytic subvarieties. It is called to be locally irreducible at $w \in X$, if for every sufficiently small open set $w \in U \subset \mathbb{C}^n$, the intersection $X \cap U$ is irreducible.

Lemma 1.33. *Let $X \subset \Omega \subset \mathbb{C}^n$ be an analytic hypersurface given by the single equation $f \equiv 0$. Then, for every $w \in X$, X is locally irreducible at w if and only if $f \in \mathcal{O}_{\mathbb{C}^n,w}$ is irreducible.*

Proof. ($\Leftarrow$) Suppose $f \in \mathcal{O}_{\mathbb{C}^n,w}$ is irreducible but in a sufficiently small neighborhood U of w, $X \cap U = V_1 \cup V_2$, with $V_1, V_2 \neq X \cap U$ being analytic subvarieties of U. By definition and the last assumption, there are $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^n,w}$ such that f_i vanishes identically on V_i but not on the other component. Since $f_1 f_2$ vanishes on X around w, by Corollary 1.27, f divides $f_1 f_2$ in $\mathcal{O}_{\mathbb{C}^n,w}$. Since f is irreducible, it must divide one of f_1 or f_2.

($\Rightarrow$) This direction is obvious.

□

More generally, by Theorem 1.24, we have the following lemma.

Lemma 1.34. *Let $X \subset \Omega \subset \mathbb{C}^n$ be an analytic hypersurface given by the single equation $f \equiv 0$. Then, for every $w \in X$, restricted to a sufficiently small neighborhood U of w we have a unique decomposition*

$$X \cap U = V_1 \cup \ldots \cup V_k$$

such that V_i is irreducible.

As Definition 1.28 shows, for every open set $U \subset \mathbb{C}^n$, there is a correspondence between the analytic subvarieties of U and ideals of the ring $\mathcal{O}_{\mathbb{C}^n}(U)$. Similarly, by taking limit, there is a relation between the the germs of analytic subvarieties of $\mathbb{C}^n$ around any point $w \in \mathbb{C}^n$ and ideals of the local ring $\mathcal{O}_{\mathbb{C}^n,w}$. More precisely, to an analytic subvariety X of U we assign the ideal

$$I(X) = \{f \in \mathcal{O}_{\mathbb{C}^n}(U) \colon X \subset f^{-1}(0)\},$$

and associated with every ideal of $I \subset \mathcal{O}_{\mathbb{C}^n}(U)$ we define its support or zero set to be

$$Z(I) = \{w \in U \colon f(w) = 0 \quad \forall\, f \in I\}.$$

By Proposition 1.26, $Z(I)$ is locally defined by a finite set of equations. This correspondence allows us to obtain an algebraic interpretation of certain geometric properties of X. For example, complex dimension or smoothness of X can be read purely algebraically from $I(X)$. That is what algebraic geometry is about: to use methods of algebra to define and study geometric objects. For example, taking union $X \cup Y$ between two varieties X and Y corresponds to taking product between the corresponding ideals $I(X) \cdot I(Y)$. If $X \subset Y$ then $I(Y) \subset I(X)$.

HW 1.35. What does taking intersection correspond to?

We have $Z(I(X)) = X$ but the assignment $I \to Z(I)$ is not one-to-one. For instance, both $f(z_1, z_2) = z_1$ and $g = (z_1, z_2) = z_1^2$ have the same zero set while the ideal generated by the later is a sub-ideal of the ideal generated by the former in $\mathbb{C}[z_1, z_2]$. Therefore, we usually consider the largest ideal that defines the same variety X. For every ideal I in a commutative ring $\mathcal{R}$, the radical of I, denoted by $\sqrt{I}$, is defined as

$$\sqrt{I} = \{r \in \mathcal{R} \colon r^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

A semi-prime or radical ideal is an ideal I for which $\sqrt{I} = I$. So, in the correspondence between varieties and ideals, we can effectively restrict to radical ideals of $\mathcal{O}_{\mathbb{C}^n}(U)$ and $\mathcal{O}_{\mathbb{C}^n,w}$. In other words, $I(Z(I)) = \sqrt{I}$.

An proper ideal P of a commutative ring $\mathcal{R}$ is called prime if for every $a, b \in \mathcal{R}$ such that their product ab is an element of P, then either $a \in P$ or $b \in P$. This definition generalizes the following property of prime numbers in $\mathbb{N}$: p is a prime number if whenever p divides a product ab of two integers, then p divides a or p divides b. Every prime ideal is obviously radical. Furthermore, every radical ideal $I \subset \mathcal{R}$ is the intersection of all the prime ideals P of $\mathcal{R}$ that contain I. In the dictionary above, prime ideals of $\mathcal{O}_{\mathbb{C}^n}(U)$ and $\mathcal{O}_{\mathbb{C}^n,w}$ correspond to irreducible subvarieties and the prime factorization of an ideal corresponds to decomposing a variety into a union of irreducible components.

For hypersurfaces, the Weierstrass Preparation Theorem shows that they are locally a finite-sheeted branched cover of a polydisk in $\mathbb{C}^{n-1}$. More or less, the following statement is a generalization of this statement to higher codimension.

Theorem 1.36. *Let $X \subset \mathbb{C}^n$ be an irreducible variety. Then, locally around every point of X there exists an open set U with a coordinate system $(z_1, \ldots, z_n)$ such that the natural projection map*

$$\pi \colon \mathbb{C}^n \longrightarrow \mathbb{C}^d, \qquad (z_1, \ldots, z_n) \longrightarrow (z_1, \ldots, z_m)$$

maps $U \cap X$ surjectively to a polydisk $\Delta \subset \mathbb{C}^m$ and realizes $X \cap U$ as a finite-sheeted cover of Δ branched over an analytic hypersurface of Δ.

The integer m is the dimension of X. Algebraically, if $X \cap U$ corresponds to the prime ideal $P \subset \mathcal{O}_{\mathbb{C}^n,0}$, then the statement above means that the induced ring homomorphism

$$\pi^* \colon \mathcal{O}_{\mathbb{C}^d,0} \longrightarrow \mathcal{O}_{\mathbb{C}^n,0}/P$$

is finite integral ring extension.

The proof of Theorem 1.36 is rather technical. Thus, we only discuss a special case where X is locally the zero locus of just two functions.
Suppose $0 \in X$ and, in a neighborhood of 0, X is the zero of set of two holomorphic functions $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$. We may assume X contains no analytic hypersurface through 0; otherwise, f and g will have a common factor. We may also assume that the zero set of f and g does not contain the z_1-axis, and hence that f and g are Weierstrass polynomials in z_1. Since f and g are relatively prime, by Theorem 1.24 and Item (1) in Page 11, there are Weierstrass polynomials $\alpha, \beta \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ and $0 \neq \gamma \in \mathcal{O}^{\mathbb{C}^{n-1},0}$ such that

$$\alpha f + \beta g = \gamma.$$

HW 1.37. Use Proposition 1.25 to show that the image of X under the projection map

$$\pi \colon \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}, \qquad (z_1, z') \longrightarrow z',$$

is $Z(\gamma)$.

We conclude that $\pi(X)$ is a hypersurface in $\mathbb{C}^{n-1}$ around the origin. But then we know that $Z(\gamma)$ is a finite-branched cover of a polydisk Δ in $\mathbb{C}^{n-2}$ around the origin. The covering $X \longrightarrow Z(\gamma)$ is also a finite-sheeted covering map. Therefore, in a neighborhood of the origin, the composition $X \to \mathbb{C}^{n-2}$ is a finite-sheeted covering of a polydisk.

So far, we have been studying local (thus non-compact) complex varieties. In the next section, we learn about arbitrary complex varieties and we will be mainly work with closed (=compact) ones. It follows from the Liouville Theorem that if X is a compact analytic variety, the only holomorphic functions defined on the entire X are the constant functions. Therefore, it is necessary to consider meromorphic functions. Recall that in the one-variable case, we can view a meromorphic function as a function from X to $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. As we explain below, the same is not quite true in higher dimensions. One needs to pass to a so-called blowup space to define a function to $\mathbb{P}^1$.

Definition 1.38. Let $\Omega \subset \mathbb{C}^n$ be an open subset. A meromorphic function f on U is a holomorphic function defined on the complement of a nowhere dense subset $S \subset \Omega$ with the following property: locally near every point of S, there exists an open neighborhood U and holomorphic functions $h, g \colon U \longrightarrow \mathbb{C}$ such that $f = \frac{h}{g}$ on U.

More explicitly, the set S in the definition above is a divisor. Moreover, if S is locally the zero set of a holomorphic function $g \colon U \longrightarrow \mathbb{C}$, and $g = g_1 \cdots g_k$ is a decomposition of g into irreducible factors (with distinct zero sets), then

$$f = \frac{h}{g_1^{m_1} \cdots g_k^{m_k}} \tag{1.12}$$

such the h and g_i are relatively prime. The integer $m_i > 0$ is the polar-order of f along the irreducible divisor $S_i = Z(g_i)$. The formal sum

$$\sum_{i=1}^{k} m_i Z(g_i)$$

is called the polar-divisor of f. Similarly, just like holomorphic functions, the zero set $Z(f)$ of a meromorphic function f is locally defined to be the zero set of the numerator, i.e. $Z(h)$. If $h = h_1 \cdots h_\ell$ is the decomposition of h into irreducible factors, then the formal sum

$$\sum_{i=1}^{\ell} Z(h_i)$$

is called the zero-divisor of f. Note that, for a reason that will be explained later, we are replacing the union of $Z(h_i)$ with a weighted formal sum.

Example 1.39. The function

$$f = \frac{e^{z_1+z_2}}{z_1 z_2^2} \colon \mathbb{C}^2 \longrightarrow \mathbb{C}$$

is a meromorphic function with a pole of order 2 along $\mathbb{C} \times \{0\} = (z_2 \equiv 0)$ and a pole of order 1 along $\{0\} \times \mathbb{C} = (z_1 \equiv 0)$.

The assignment U to

$$\mathcal{K}(U) = \text{the set of meromorphic functions on } U$$

is a sheaf of fields (if U is connected). It is the field of fractions of the ring $\mathcal{O}(U)$.

Remark 1.40. If $f \colon \Omega \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ is a meromorphic and $n > 1$, the zero and polar divisors of f have complex codimension 1 and can intersect along a complex codimension 2 variety $Y \subset \Omega$. Outside $Y \subset \Omega$, the meromorphic function f can be seen as a holomorphic function to $\mathbb{CP}^1$. However, along Y we have $\frac{0}{0}$ situation in (1.12). For every $p \in Y$, the limit $\lim_{z \longrightarrow p} f(z)$ depends on the direction of approaching p. For this reason, in order to think of f as a function to $\mathbb{CP}^1$, we will replace Ω with a larger space that instead of Y has a larger subvariety parametrizing all the directions normal to Y.

2 Complex manifolds and vector bundles

Roughly speaking, a manifold is a (topological) space that locally resembles Euclidean space, and globally, it is obtained by attaching countably many such local pieces (known as charts). Globally most manifolds are not homeomorphic to Euclidean space or an open subset of that. For example, the sphere is not homeomorphic to the plane. In the following section, we will learn about complex manifolds/varieties, complex submanifolds/subvarieties, and holomorphic vector bundles. In particular, we will be interested in complex subvarieties of dimension one (=complex curves) and codimension one (=divisors), and holomorphic line bundles.

Remark 2.1. Recall from the previous section that the terminology "variety" refers to complex analytic spaces that can be singular. A variety that is not singular is a complex manifold.

Definition 2.2. A complex manifold X is a topological manifold admitting an atlas

$$\mathcal{A} = \{\varphi_\alpha \colon U_\alpha \longrightarrow \mathbb{C}^n\}_{\alpha\in\mathcal{I}}$$

such that the change of coordinate maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are holomorphic. A function $f \colon U \subset X \longrightarrow \mathbb{C}$ is holomorphic if $f \circ \varphi_\alpha^{-1}$ is holomorphic for all $\alpha \in \mathcal{I}$. A coordinate system $(z_1, \ldots, z_n)$ on $U \subset X$ refers to the coordinate values of a chart $\varphi \colon U \longrightarrow X$ (in the maximal atlas containing $\mathcal{A}$). A map $F \colon X \longrightarrow Y$ between two complex manifolds is holomorphic if for every two charts $\varphi \colon U \subset X \longrightarrow \mathbb{C}^n$ and $\psi \colon V \subset Y \longrightarrow \mathbb{C}^m$, the composition $\psi \circ F \circ \varphi^{-1}$ is holomorphic. We say $X \subset Y$ is a holomorphic submanifold if X is the image of a holomorphic embedding in Y.

Note that, in our definition, complex manifolds do not have boundary. So compact=closed. Complex manifolds with boundary can be defined by including charts with boundary. We will explicitly mention that if we ever need to work with manifolds with boundary. By Implicit Function Theorem (c.f. Remark 1.22), if $X \subset Y$ is a holomorphic submanifold of $\mathbb{C}$-dimension n, locally around every point of N there are coordinates $(z_1, \ldots, z_m)$ on X such that Y is the transverse intersection of zero sets $(z_{n+1} \equiv 0) \cap \cdots \cap (z_m \equiv 0)$.

Remark 2.3. Many other important concepts such as transversality are defined similarly in the complex category.

Similarly to the previous section, associated to every complex manifold we have the sheaves $\mathcal{O}_X$ and $\mathcal{K}_X$. The first associates to each open set U the ring $\mathcal{O}_X(U)$ consisting of the holomorphic functions on U. The second one associates to each connected open set U the field $\mathcal{K}_X(U)$ consisting of the meromorphic functions on U. The latter is the field of fractions of the former. The stalks of these sheaves are defined locally as before. As we stated before, if X is closed (compact), the vector space $\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$ consisting of the global sections of $\mathcal{O}_X$ (i.e. holomorphic functions defined on the entire X) is isomorphic to $\mathbb{C}$. However, (assuming X is connected),

$$\mathcal{K}(X) := \Gamma(X, \mathcal{K})$$

is called the function field of X and is quite large. Note that $\mathcal{K}(X)$ is a field-extension of $\mathbb{C}$.

Proposition 2.4. *Let X be a compact complex manifold of dimension n. Then*

$$\mathrm{trdeg}_{\mathbb{C}}\mathcal{K}(X) \leq n.$$

The function field $\mathcal{K}(X)$ and its transcendence degree $\mathrm{trdeg}_{\mathbb{C}}\mathcal{K}(X)$ play a major role in classification of X.

In addition to pasting local charts, there are two major global methods for constructing manifolds: either as the zero set of a set of equations or as the quotient of some easier manifolds. To define quotient complex manifolds, we consider the holomorphic action of a discrete or complex Lie group G on a complex manifold X in the following sense. Suppose X is a complex manifold and G is a discrete group (probably finite). By a holomoprhic (right-) action of G on X we mean a function

$$\varphi \colon X \times G \longrightarrow X, \qquad (x, g) \longrightarrow x \cdot g := \varphi(x, g) \in X$$

such that $\varphi(-, g) \colon X \longrightarrow X$ is holomorphic for all $g \in G$ and $\varphi(-, g_1 g_2) = (\varphi(-, g_1), g_2)$ for all $g_1, g_2 \in G$. In particular, by the second property, each $\varphi(-, g)$ is a biholomorphic map. Let

$$X/\varphi \equiv X/G := X/\big(x \sim x \cdot g \colon \forall\ x \in X,\ g \in G\big)$$

denote the quotient space with the quotient topology.

Theorem 2.5. *With notation as above, suppose G is a discrete group that acts freely and properly on X in the following sense:*

- *(freely means) for every point $x \in X$ the stabilizer subgroup $G_x = \{g \in G \colon x \cdot g = x\}$ is the trivial subgroup;*
- *(properly means) for every compact subset $K \subset X$, the subset $G_K = \{g \in G : (K \cdot g) \cap K \neq \emptyset\}$ is finite.*

Then the holomorphic manifold structure on X induces a unique holomorphic manifold structure on X/G.

Example 2.6. For every $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$, the action of $\mathbb{Z}^2$ on $\mathbb{C}$ by

$$\mathbb{C} \times \mathbb{Z}^2 \longrightarrow C, \qquad z \times (m, n) \longrightarrow z + m + n\tau,$$

is holomorphic, free, and proper. The quotient manifold $\mathbb{T}_\tau = \mathbb{C}/\mathbb{Z}^2$ is the 2-dimensional torus $\mathbb{T}^2$ with a complex structure that depends on τ. The complex tori $\mathbb{T}^2_\tau$ and $\mathbb{T}^2_{\tau'}$ are biholomorphic if and only if

$$\tau' = A \cdot \tau = \frac{a\tau + b}{c\tau + d} \tag{2.1}$$

for some

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{Z}) = \{A \in M_{2\times 2}(\mathbb{Z}) : \det(A) = 1\}/ \pm I_{2\times 2}.$$

Therefore, the space parametrizing complex tori up to biholomorphism is itself the quotient space

$$\mathcal{H}/\mathrm{PSL}(2, \mathbb{Z}),$$

where $\mathcal{H}$ is the upper-half plane (1.8). The action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathcal{H}$ is proper but not free. The quotient space $\mathcal{H}/\mathrm{PSL}(2, \mathbb{Z})$ is thus a singular variety (an orbifold) and not a manifold.

A complex Lie group G is a complex manifold with a group structure such that the product map

$$G \times G \longrightarrow G$$

is holomorphic. Examples of Lie groups include $\mathbb{C}^*$ and $\mathrm{GL}(N, \mathbb{C})$.

Theorem 2.7. *Suppose G is a complex Lie group that acts freely and properly on a holomorphic manifold X in the following sense:*

- *(freely) for every point $x \in X$ the stabilizer subgroup $G_x = \{g \in G \colon x \cdot g = x\}$ is the trivial subgroup;*
- *(properly) for every compact subset $K \subset X$, the subset $G_K = \{g \in G : (K \cdot g) \cap K \neq \emptyset\}$ is compact.*

Then the manifold structure on X induces a unique holomorphic manifold structure on X/G such that the quotient map $\pi \colon X \longrightarrow X/G$ is a holomorphic submersion.

HW 2.8. Prove that every compact, connected, complex Lie group is a torus.

The most important example of a complex manifold that can be obtained as a Lie group quotient space is the example of the complex projective space. This example generalizes in different ways to Grassmannians and Toric varieties. They all fit into the framework of GIT quotients.

Example 2.9. The n-dimensional complex projective space $\mathbb{CP}^n$ is the set of lines through origin in $\mathbb{C}^{n+1}$. If V is an abstract complex vector space, we use the notation $\mathbb{P}(V)$ to denote the set of lines through origin in V. Therefore, $\mathbb{CP}^n = \mathbb{P}(\mathbb{C}^{n+1})$. We have

$$\mathbb{P}(V) = V - \{0\}/\mathbb{C}^*$$

where $\lambda \in \mathbb{C}^*$ acts by

$$v \to \lambda v.$$

For $\mathbb{CP}^n$, the x_is in the equivalence class $[x_0, \ldots, x_n] \in \mathbb{CP}^n$ of $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} - \{0\}$ are called the homogenous or projective coordinates of $\mathbb{CP}^n$. They are not coordinates in the usual sense, but they have a geometric meaning that we will learn about below. The open sets

$$U_i = \{[x_0, \ldots, x_n] \in \mathbb{CP}^n \colon x_i \neq 0\}, \quad i = 0, \ldots, n,$$

cover $\mathbb{CP}^n$. For each $i = 0, \ldots, n$, the map

$$\varphi\colon U_i \longrightarrow \mathbb{C}^n, \qquad [x_0, \ldots, x_n] \longrightarrow \Big(z_0, \ldots \widehat{z_i} \ldots, z_n\Big) = \Big(\frac{x_0}{x_i}, \ldots \frac{\widehat{x_i}}{x_i} \ldots, \frac{x_n}{x_i}\Big)$$

is a holomorphic chart onto $\mathbb{C}^n$. The collection $\{\varphi_i \colon U_i \longrightarrow \mathbb{C}^n\}_{i=0}^n$ is a holomorphic atlas on $\mathbb{CP}^n$. Every embedding $W \subset V$ of complex linear spaces gives rise to an embedding of complex manifolds $\mathbb{P}(W) \subset P(V)$. For instance, the complement H_i of each open set U_i is a complex hypersurface $H_i \cong \mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ that corresponds to the $\mathbb{C}$-codimension 1 subspace

$$(x_i \equiv 0) \subset \mathbb{C}^{n+1}.$$

□

A 1-dimensional complex manifold has real dimension 2 and is called a Riemann surface or a complex curve. As we will briefly explaine in Remark 2.19 below, every orientable 2-manifold can be given the structure of a Riemann surface. Usually, this can be done in more than one way and we get a family of non-equivalent complex structures on the underlying topological manifold. This family is often a complex variety itself and is called a "moduli space".

For $n = 1$ in Example 2.9, $\mathbb{CP}^1 = S^2$ is obtained by gluing two copies of $\mathbb{C}$, say with coordinates $z = \frac{x_1}{x_0}$ and $w = \frac{x_0}{x_1}$, along $\mathbb{C}^* \subset \mathbb{C}$ via the gluing map

$$w = \frac{1}{z}.$$

Topologically, $\mathbb{CP}^1$ is the one-point compactification of $\mathbb{C}$ that adds the point $z = \infty \sim w = 0$. There is a unique way to equip S^2 with a holomorphic structure.

HW 2.10. Show the the manifolds M_λ obtained by gluing the two charts above via

$$w = \lambda/z$$

are all biholomorphic for different values of $\lambda \in \mathbb{C}^*$.

Closed Riemann surfaces are topologically/smoothly classified by their genus $g \in \mathbb{N}$ or equivalently their Euler characteristic $\chi = 2 - 2g$. Starting with $g = 0$, we have S^2 which is simply connected and admits a unique complex structure $\mathbb{CP}^1$. The automorphism (biholomorphisms) group of $\mathbb{CP}^1$ is $\mathrm{PSL}(2, \mathbb{C})$, acting on $\mathbb{C} \cup \infty$ as in (2.1).
Next in the list, we have the 2-torus with $g = 1$ and $H_1(\mathbb{T}^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. As we explained in Example 2.6, the universal cover of $\mathbb{T}^2$ is $\mathbb{C}$ and there is complex one-dimensional space of complex structures on $\mathbb{T}^2$. The automorphism group of generic $\mathbb{T}^2_\tau$ is $\mathbb{T}^2_\tau \times \mathbb{Z}_2$, where the first component corresponds to translations and the second corresponds to $z \longrightarrow -z$ on $\mathbb{C}$.
The universal cover of every closed Riemann surface Σ with the genus $g > 1$ is the upper-half plane $\mathcal{H}$, and

$$S = \mathcal{H}/\Gamma$$

where Γ is a finite-index subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Here, $\mathrm{PSL}(2, \mathbb{R})$ acts on $\mathcal{H}$ as in (2.1). The moduli space of holomorphic structures on Σ is $(3g - 3)$-dimensional and is denoted by $\mathcal{M}_g$ (called the Deligne-Mumford space).

For every Riemann surface X, a meromorphic function $f\colon X - S \longrightarrow \mathbb{C}$ is equivalent to a holomorphic function $f\colon X \longrightarrow \mathbb{CP}^1$ such that $f^{-1}(\infty)$ is the polar divisor of f. More precisely, for every $p \in S$, in any local coordinate x around p (p corresponds to $x = 0$), f has the form

$$\mathbb{CP}^1 \ni z(f(x)) = a_{-n}x^{-n} + a_{-n+1}x^{-n+1} + \cdots = x^{-n}h(x)$$

such that $h(0) \neq 0$. In the coordinate $w = z^{-1}$ around $\infty \in \mathbb{CP}^1$, we have

$$w(f(x)) = x^n \frac{1}{h(x)}.$$

Therefore, $f(p) = \infty$ with multiplicity n. In general, if $f\colon X \longrightarrow Y$ is a holomorphic between two Riemann surfaces, $q \in X$, x is a local coordinate near q, and y is a local coordinate near $f(q)$, then f locally has the form

$$f(z) = z^d h(z)$$

such that $h(0) \neq 0$. The number $d = \mathrm{mult}_q(f) \in \mathbb{Z}_{>0}$ called the multiplicity of f at q and is independent of the choice of local coordinates.

HW 2.11. Show that the function field of $\mathbb{CP}^1$ is $\mathbb{C}(z) = \mathbb{C}(w)$, the field of polynomial fractions in one variable; i.e. show that every holomorphic functions $\mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ is the quotient of two polynomials. Similarly, show that the function field of $\mathbb{CP}^n$ is $\mathbb{C}(z_1, \ldots, z_n)$, the field of polynomial fractions in n variables.

HW 2.12. Let $f(z) = p(z)/q(z)$ be a holomorphic function given by the quotient of two degree m and n polynomials $p(z)$ and $q(z)$, respectively. Let $d = \max(m, n)$. Show that f is generically a $d : 1$-covering map. Describe the set of branched points, i.e. the points p in the target where the size of $f^{-1}(p)$ is smaller than d. Describe the zero and polar divisors of f. The former is the formal linear sum

$$\sum_{q \in f^{-1}(0)} \mathrm{mult}_q(f) \cdot q$$

and the later is the formal linear sum

$$\sum_{q \in f^{-1}(\infty)} \mathrm{mult}_q(f) \cdot q.$$

HW 2.13. With notation as in Example 2.6, note the action of $\mathbb{Z}_2$ on $\mathbb{C}$ by $z \longrightarrow -z$ descends to a $\mathbb{Z}_2$ action on each T^2_τ with exactly 4 fixed points: $[0], [1/2], [\tau/2], [(1+\tau)/2]$. Even though the action is not free, show that the quotient space $Y = T^2_\tau/\mathbb{Z}_2$ has the structure of a smooth Riemann surface. Prove that $Y = \mathbb{CP}^1$. Conclude that there is a holomorphic map

$$\pi\colon T^2_\tau \longrightarrow \mathbb{CP}^1$$

that is generically $2:1$ and is ramified at 4 points. Can you use π to find the find the function field of a torus?

The (local) definition of analytic varieties in $\mathbb{C}^n$ (Definition 1.28) extends to a similar definition for arbitrary ambient space in the following way.

Definition 2.14. Suppose Y is a complex manifold. We say $X \subset Y$ is an analytic subvariety of Y if, for every $w \in X$, there exists a sufficiently small neighborhood $U \ni w$ such that $X \cap U$ is the common zero locus of a finite collection of holomorphic functions $f_1, \ldots, f_k$ on U. In particular, we say X is a complex projective manifold/variety if it can be embedded in $\mathbb{CP}^n$.

One may ask *which analytic manifolds/varieties are projective*? At the end of this section, we will learn about the Kodaira Embedding Theorem that answers this question.

Example 2.15. The complex manifold

$$X = \frac{\mathbb{C}^2 - \{0\}}{z \sim 2z}$$

is topologically an S^1-bundle over S^3 and is known as Hopf (complex) surface. We will learn later that X is not projective.

While the definition of an analytic subvariety $X \subset M$ locally describes X as the zero set of finite set of functions, it is desirable to describe a subvariety by a set of globally defined equations. However, if M is closed, because $\mathcal{O}_M(M) = \mathbb{C}$, the only globally defined holomorphic functions are the constant functions. We do not want to consider meromorphic functions either because they have poles. For example, each hyperplane $H_i \subset \mathbb{CP}^n$ in Example 2.9 is described by a single equation $x_i \equiv 0$, but x_i is not a function on $\mathbb{CP}^n$. The idea to resolve this issue and makes sense of x_i as some sort of function is to consider sections of holomorphic vector bundles.

Definition 2.16. A holomorphic vector bundle $E \longrightarrow M$ on a complex manifold M is a $\mathbb{C}$-rank r complex vector bundle admiting local trivializations

$$\left\{\Phi_\alpha\colon E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r\right\}_{\alpha\in\mathcal{I}}$$

such that the change of trivialization maps

$$\Phi_\beta \circ \Phi_\alpha^{-1}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(r)$$

are holomorphic.

Remark 2.17. Some complex vector bundles may not admit any holomorphic structure, and whenever they do, they may admit several non-isomorphic holomorphic structures. There are cohomological groups/relations that determine whether a complex vector bundle admits any holomorphic structure and in how many ways. We will learn about these later.

Example 2.18. The tangent and cotangent bundles of every complex manifold admit natural holomorphic structures. The relation with the smooth tangent and cotangent bundle structures is as follows. Suppose M is a holomorphic manifold of complex dimension n. We can think of M as a smooth manifold of real dimension $2n$. The linearization of the complex structure on M can be seen as a real endomorphism

$$J\colon TM \longrightarrow TM \tag{2.2}$$

satisfying $J^2 = -\mathrm{id}$. Let TM denote the smooth tangent bundle of M. The complexifiation

$$TM \otimes_{\mathbb{R}} \mathbb{C}$$

of TM is $\mathbb{C}$-vector bundle of rank $2n$. It admits a decomposition

$$TM \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}M \oplus T^{0,1}M \equiv \mathcal{T}M \oplus \overline{\mathcal{T}}M$$

such that J acts on $T^{1,0}M = \mathcal{T}M$ complex linearly and on $T^{0,1}M = \overline{\mathcal{T}}M$ anti-compex linearly. In terms of local coordinates $(z_1 = x_1 + \mathrm{i}y_1, \ldots, z_n = x_n + \mathrm{i}y_n)$ on M, TM is generated by

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n},$$

J acts by $J\frac{\partial}{\partial x_a} = \frac{\partial}{\partial y_a}$ and $J\frac{\partial}{\partial y_a} = -\frac{\partial}{\partial x_a}$, $\mathcal{T}M$ is generated by

$$\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n},$$

and $\overline{\mathcal{T}}M$ is generated by their complex conjugates. The case of cotangent bundle is similar.

Remark 2.19. An almost complex structure on a real $2n$-dimensional smooth manifold M is an endomorphism $J\colon TM \longrightarrow TM$ covering the identity map on M such that $J^2 = -\mathrm{id}$. A natural question is *whether given an almost complex structure J on M, there is a holomorphic structure on M whose first order action is J*? A theorem of Newlander-Nirenberg provides an explicit if-and-only-if answer to this question. For example, given a Riemannian metric g on a Riemann surface Σ, the rotation by $\pi/2$ defines an almost complex structure J on $T\Sigma$. Furthermore, it follows from the theorem of Newlander and Nirenberg that every almost complex structure in real dimension 2 is integrable to a holomorphic structure. Therefore, every orientable Riemann surface admits a holomorphic structure.

Example 2.20. Consider the complex projective spaces $\mathbb{P}(V)$ in Example 2.9. The so called tautological line bundle $\gamma \longrightarrow \mathbb{P}(V)$ is the complex line bundle whose fiber over the point $[v] \in \mathbb{P}(V)$ is the complex line $\mathbb{C} \cdot v \subset V$. By definition, γ is a complex subline bundle of the trivial bundle $\mathbb{P}(V) \times V$. For $\mathbb{CP}^n$, with respect to the atlas in Example 2.9, γ is given by transition maps

$$\Phi_j \circ \Phi_i^{-1}\colon U_i \cap U_j \longrightarrow \mathrm{GL}(1) = \mathbb{C}^*, \quad \Phi_j \circ \Phi_i^{-1}(z_0, \ldots \widehat{z_i} \ldots z_n) = z_j. \tag{2.3}$$

Therefore, γ is a holomorphic line bundle. The facts is that every complex line bundle on $\mathbb{P}(V)$ admits a unique holomorphic structure and is isomorphic to a tensor power $\gamma^{\otimes m}$ of γ for some $m \in \mathbb{Z}$. The line bundle γ is often denoted by $\mathcal{O}(-1)$ or $\mathcal{O}_{\mathbb{P}(V)}(-1)$. Its dual

$$\mathcal{O}_{\mathbb{P}(V)}(1) = \gamma^{-1} = \gamma^*$$

and its higher powers $\mathcal{O}_{\mathbb{P}(V)}(m) = \gamma^{\otimes -m}$ play a major role in the study of complex projective varieties.

Suppose $\mathcal{L} \longrightarrow M$ is a holomorphic line bundle. The assignment

$$U \longrightarrow \Gamma(U, \mathcal{L}|_U)$$

for every open set $U \subset M$ defines a sheaf of complex vector spaces on M. We will mainly be interested in the group $\Gamma(M, \mathcal{L})$ of global holomorphic sections of $\mathcal{L}$.

HW 2.21. Following Example 2.20, show that, for $m < 0$, $\Gamma(\mathbb{CP}^n, \mathcal{O}(m)) = 0$. For $m \geq 0$, show that $\Gamma(\mathbb{CP}^n, \mathcal{O}(m))$ can be identified with the $\mathbb{C}$-vector space of homogenous polynomials of degree m in the projective coordinates $x_0, \ldots, x_n$. In particular, $x_0, \ldots, x_n$ can be seen as the global holomorphic sections of $\mathcal{O}(1)$. We conclude that, for $m > 0$, the line bundle $\mathcal{O}(m)$ has plenty of sections. For instance, for every two points $p, q \in \mathbb{CP}^n$, there is a section $s \in \Gamma(\mathbb{CP}^n, \mathcal{O}(m))$ such that $s(p) = 0$ and $s(q) \neq 0$.

HW 2.22. Suppose $E \longrightarrow M$ is a holomorphic vector bundle and $s_1, \ldots, s_k \in \Gamma(M, \mathcal{L})$ are k holomorphic sections. Show that

$$X = \bigcap_{i=1}^{k} s_i^{-1}(0) = \{p \in M \colon s_i(p) = 0 \quad \forall i = 1, \ldots, k\}$$

is an analytic subvariety.

The construction above gives a recipe for constructing plenty of analytic subvarieties from sections of holomorphic vector bundle. We will be interested in bundles that have plenty of sections. By HW 2.21, in Example 2.9, H_i is the zero set of the section $x_i \in \Gamma(\mathbb{CP}^n, \mathcal{O}(1))$.

Remark 2.23. Suppose $\pi \colon E \longrightarrow M$ is a holomorphic vector bundle and $s \colon M \longrightarrow E$ is a holomorphic section of that. A priori, the $\mathbb{C}$-derivative of s is a linear map

$$\mathrm{d}s \colon \mathcal{T}M \longrightarrow \mathcal{T}E$$

where $\mathcal{T}E$ denotes the complex tangent space of the total space of E. The projection map π gives rise to a short-exact sequence of complex vector bundles

$$0 \longrightarrow \pi^*E \longrightarrow \mathcal{T}E \xrightarrow{\mathrm{d}\pi} \pi^*TM.$$

A complex linear connection ∇ on E gives a splitting

$$\mathcal{T}E \cong \pi^*E \oplus \pi^*TM$$

of this exact sequence and thus allows us to extract the non-trivial vertical component

$$\mathrm{d}_x^{\nabla} s \colon \mathcal{T}_x M \longrightarrow E_x, \qquad \forall\, x \in M,$$

of $\mathrm{d}s$ at every point $x \in M$. However, recall from differential topology that, along the zero set of s (i.e. if $s(x) = 0$), $\mathrm{d}_x^{\nabla} s$ is independent of the choice of ∇ and we can simply denote it by $\mathrm{d}_x^{\perp} s$. Then, a section s is called transverse if $\mathrm{d}_x s$ is full rank at every $x \in s^{-1}(0)$. If s is transverse, then $s^{-1}(0)$ is a complex submanifold of $\mathbb{C}$-codimension equal to the $\mathbb{C}$-rank of E.

For example, if $M = \mathbb{CP}^2$ and $E = \mathcal{O}(d)$ for some $d > 0$, then a section s of $\mathcal{O}(d)$ is a homogenous polynomial of degree d

$$s = \sum_{i_0 + i_1 + i_2 = d} x_0^{i_0} x_1^{i_1} x_2^{i_2}.$$

We have

$$\mathrm{d}s = \frac{\partial}{\partial x_0}\mathrm{d}x_0 + \frac{\partial}{\partial x_1}\mathrm{d}x_1 + \frac{\partial}{\partial x_2}\mathrm{d}x_2,$$

and s is transverse if the only common zero of the degree $d-1$ homogenous polynomials $\frac{\partial}{\partial x_0}$, $\frac{\partial}{\partial x_1}$, and $\frac{\partial}{\partial x_2}$, is $(0,0,0)$ (which does not define a point in $\mathbb{CP}^2$!). Generic s is transverse (why?!) and (we will learn that) the zero set of that is a closed Riemann surface of genus

$$\frac{(d-1)(d-2)}{2}.$$

Furthermore, every complex submanifold of dimension 1 in $\mathbb{CP}^2$ is the zero set of such a section (we will learn this later).

The set of isomorphism classes of holomorphic (and smooth complex) line bundles $\mathcal{L}$ on a complex manifold M is a group where product structure comes from taking the tensor product and $\mathcal{L}^{-1} = \mathcal{L}^*$. This is called the Picard group and is denoted by $\mathrm{Pic}(X)$. Those holomorphic lines bundles that are smoothly trivial form a subgroup that is denoted by $\mathrm{Pic}^0(X)$. If $f\colon X \longrightarrow Y$ is a holomorphic map, then the pull-back under f defines a group of homomorphism

$$f^*\colon \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X).$$

In the next section, we will use sheaf theory and some cohomological arguments to understand holomorphic line bundles on X and thus $\mathrm{Pic}(X)$.

HW 2.24. Suppose X is a compact complex manifold. Show the complex vector space $\Gamma(X,\mathcal{L})$ of the holomorphic sections of any holomorphic vector bundle $\mathcal{L} \longrightarrow X$ is finite dimensional.

Suppose $\mathcal{L} \longrightarrow X$ is a holomorphic line bundle such that:
($\star$) for every $p \in X$ there exists $s \in \Gamma(X,\mathcal{L})$ such that $s(p) \neq 0$; i.e., the $\mathbb{C}$-linear homomorphism

$$\Gamma(X,\mathcal{L}) \longrightarrow \mathcal{L}|_p \cong \mathbb{C}, \qquad s \longrightarrow s(p),$$

is surjective.

Let $s_0,\ldots,s_N$ be a basis for the complex vector space $\Gamma(X,\mathcal{L})$. For each $p \in X$ and $i,j = 0,\ldots,N$, the ratio $s_i(p)/s_j(p) \in \mathbb{C}\cup\infty$ is defined. Therefore, by ($\star$), the point

$$[s_0(p),\ldots,s_N(p)] \in \mathbb{CP}^N$$

is defined. If $\mathcal{L}|_U \cong U \times \mathbb{C}$ is a local trivialization of $\mathcal{L}$, then the restriction of each section to U is given by a similarly denoted holomorphic function

$$s_i\colon U \longrightarrow \mathbb{C}.$$

This shows that the map

$$\iota_{\mathcal{L}}\colon X \longrightarrow \mathbb{CP}^N, \qquad p \longrightarrow [s_0(p),\ldots,s_N(p)], \tag{2.4}$$

is holomorphic. In conclusion, if we can find a holomorphic line bundle with lots of sections, we can use these sections to map X into $\mathbb{CP}^N$. Suppose further that

($\star\star$,i) for every $p \neq q \in X$ there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(p) = 0$ and $s(q) \neq 0$; i.e., the $\mathbb{C}$-linear homomorphism

$$\Gamma(X, \mathcal{L}) \longrightarrow \mathcal{L}|_p \oplus \mathcal{L}|_p \cong \mathbb{C}^2, \qquad s \longrightarrow s(p) \oplus s(q),$$

is surjective; and

($\star\star$, ii) for every $p \in X$ and $v \in T_pX$ there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(p) = 0$ and $\mathrm{d}^{\perp}s(p) = v$.

If ($\star\star$,i) holds, then ϱ will be one-to-one. If ($\star\star$,ii) holds, then ϱ will be an immersion. Note that ($\star\star$,ii) can be thought of as the limit of ($\star\star$,i) when q converges to p in the direction of v. There is a notion of positivity for holomorphic line bundles that ensures the properties ($\star$), ($\star\star$,i) ($\star\star$,ii) hold. In conclusion, we have the following result.

Theorem 2.25. *(Kodaira's Embedding Theorem) Let X be a compact complex manifold and $\mathcal{L}$ be a* ***positive*** *line bundle. Then there exists $k_0 > 0$ such that for every $k \geq k_0$, the map $\iota_{\mathcal{L}^{\otimes k}}$ corresponding to the line bundle $\mathcal{L}^{\otimes k}$ is an embedding. Therefore, X is a complex projective variety if and only if it admits a positive line bundle.*

We will learn about the notion of positivity in the coming sections. A positive line bundle is also called ample. Then a line bundle $\mathcal{L}$ for which $\iota_{\mathcal{L}}$ is an embedding is called very ample. So a sufficiently high tensor power of any ample line bundle is very ample. As an example, the line bundle $\mathcal{O}(1)$ over $\mathbb{CP}^n$ is very ample. It gives rise to the trivial embedding of $\mathbb{CP}^n$ into itself. However, for $m > 1$, we get an emebdding

$$\iota_{\mathcal{O}(m)} \colon \mathbb{CP}^n \longrightarrow \mathbb{CP}^{\binom{n+m}{n}-1}$$

For example, the line bundle $\mathcal{O}(2)$ over $\mathbb{P}^1$ gives rise to the embedding

$$\iota_{\mathcal{O}(2)} \colon \mathbb{CP}^1 \longrightarrow \mathbb{CP}^2, \quad [x_0, x_1] \longrightarrow [y_0, y_1, y_2] = [x_0^2, x_0x_1, x_1^2].$$

Clearly, the image of $\iota_{\mathcal{O}(2)}$ is the degree 2 curve/hypersurface

$$(y_1^2 - y_0y_2 = 0) \subset \mathbb{CP}^2.$$

The map

$$\iota_{\mathcal{O}(m)} \colon \mathbb{CP}^1 \longrightarrow \mathbb{CP}^m$$

is called the Veronese map.

3 Some cohomological results

For any manifold X, let $\Lambda^r(X, \mathbb{R})$ and $\Lambda^r(X, \mathbb{C})$ denote the spaces of smooth $\mathbb{R}$-valued and $\mathbb{C}$-valued r-forms on X. Recall that the de Rham cohomology $H^*_{\mathrm{dR}}(X, \mathbb{R})$ is the cohomology of the cochain complex

$$0 \longrightarrow \Lambda^0(X, \mathbb{R}) \xrightarrow{\mathrm{d}} \Lambda^1(X, \mathbb{R}) \xrightarrow{\mathrm{d}} \cdots;$$

the $\mathbb{C}$-valued de Rham cohomology groups are defined similarly and

$$H^*_{\mathrm{dR}}(X, \mathbb{C}) = H^*_{\mathrm{dR}}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Over complex manifolds, $\Lambda^r(X,\mathbb{C})$ further decomposes as

$$\Lambda^r(X,\mathbb{C}) = \bigoplus_{r=p+q} \Lambda^{p,q}(X,\mathbb{C}),$$

where $\Lambda^{p,q}(X,\mathbb{C})$ is the space smooth differential forms of type (p,q) in the sense that it every $\eta \in \Lambda^{p,q}(X,\mathbb{C})$ locally has the form

$$\eta = \sum_{|I|=p,|J|=q} a_{I;J} \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J = \sum a_{i_1,\ldots,i_p;j_1,\ldots,j_q} \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z}_{i_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{i_q}.$$

As in (1.1), the exterior derivative operator d decomposes as $\mathrm{d} = \partial + \overline{\partial}$ such that

$$\partial\colon \Lambda^{p,q}(X,\mathbb{C}) \longrightarrow \Lambda^{p+1,q}(X,\mathbb{C}) \quad \text{and} \quad \overline{\partial}\colon \Lambda^{p,q}(X,\mathbb{C}) \longrightarrow \Lambda^{p,q+1}(X,\mathbb{C}).$$

The decomposition above into (p,q)-types is preserved under holomorphic maps; in particular, $\bar{\partial}$-operator commutes with pull back by holomorphic maps. Also, we have $\partial^2 = 0$ and $\overline{\partial}^2 = 0$. Therefore, for each $p \geq 0$,

$$0 \longrightarrow \Lambda^{p,0}(X,\mathbb{C}) \xrightarrow{\bar{\partial}} \Lambda^{p,1}(X,\mathbb{C}) \xrightarrow{\bar{\partial}} \cdots;$$

is a co-chain complex. The cohomology groups of this sequence are denoted by $H^{p,q}_{\bar{\partial}}(X,\mathbb{C})$ and are called the Dolbeaut cohomology groups of X. The ordinary Poincare lemma states that every closed form of positive degree on (a contractible domain in) $\mathbb{R}^n$ is exact. It shows us that the positive degree de Rham cohomology groups are locally trivial and $H^*_{\mathrm{dR}}(X,\mathbb{C})$ indeed captures the global topology of X. Similarly, the $\bar{\partial}$-Poincare lemma states that, for every polydisk $\Delta \subset \mathbb{C}^n$,

$$H^{p,q}_{\bar{\partial}}(\Delta,\mathbb{C}) = 0 \qquad \forall\ q > 0;$$

i.e. every (p,q)-form η that satisfies $\bar{\partial}\eta = 0$ and $q > 0$ is locally $\bar{\partial}$ of some $(p,q-1)$-form. The proof relies on Cauchy Integral formula. The $\bar{\partial}$-Poincare lemma would also enable us to show that the Dolbeaut cohomology groups of X coincide with sheaf cohomology groups of Ω^p_X, where Ω^p_X is the sheaf of holomorphic p-forms on X; see below.

Example 3.1. We will learn that, on every genus g Riemann surface X, the non-trivial Dolbeaut cohomology groups are

$$H^{0,0}_{\bar{\partial}}(X,\mathbb{C}) = \mathbb{C},$$
$$H^{1,0}_{\bar{\partial}}(X,\mathbb{C}) = \mathbb{C}^g, \quad H^{0,1}_{\bar{\partial}}(X,\mathbb{C}) \cong \overline{H^{1,0}_{\bar{\partial}}(X,\mathbb{C})} = \mathbb{C}^g,$$
$$H^{1,1}_{\bar{\partial}}(X,\mathbb{C}) = \mathbb{C}.$$

The $\mathbb{C}$-vector space $H^{1,0}_{\bar{\partial}}(X,\mathbb{C})$ is the space of holomorphic 1-forms on X; i.e. 1-forms that are locally of the form $f(z)\mathrm{d}z$ for some holomorphic function z. □

Suppose $E \longrightarrow X$ is a holomorphic vector bundle. For each $p,q \geq 0$, let $\Lambda^{p,q}(X,E)$ denote the space of (p,q)-forms with values in E; i.e. $\Lambda^{p,q}(X,E)$ is the tensor product of $\Lambda^{p,q}(X,\mathbb{C})$ and the space of smooth sections of E.

HW 3.2. Prove that the $\bar{\partial}$-operator on differential forms natural extends to a well-defined $\bar{\partial}$-operator

$$\Lambda^{p,q}(X,E) \xrightarrow{\bar{\partial}} \Lambda^{p,q+1}(X,E)$$

satisfying $\bar{\partial}^2 = 0$. Explain why the same does not hold for ∂.

By the homework above, for each $p \geq 0$,

$$0 \longrightarrow \Lambda^{p,0}(X,E) \xrightarrow{\bar{\partial}} \Lambda^{p,1}(X,E) \xrightarrow{\bar{\partial}} \cdots;$$

is a co-chain complex. The cohomology groups of this sequence are denoted by $H^{p,q}_{\bar{\partial}}(X,E)$ and are called the Dolbeaut cohomology groups of E. For example, $H^{0,0}_{\bar{\partial}}(X,E)$ is precisely the vector space $\Gamma(X,E)$ of the holomorphic sections of E.

Example 3.3. Suppose X is a compact Riemann surface of genus g and $E \longrightarrow X$ is a holomorphic vector bundle. For dimensional reasons, the only non-trivial $(0,q)$ Dolbeaut cohomology groups of E are

$$H^{0,0}_{\bar{\partial}}(X,E) \quad \text{and} \quad H^{0,1}_{\bar{\partial}}(X,E).$$

The famous Riemann-Roch theorem, that we will learn later, states that

$$\dim H^{0,0}_{\bar{\partial}}(X,E) - \dim H^{0,1}_{\bar{\partial}}(X,E) = \deg(E) + \text{rank}(E)(1-g).$$

Here,

$$\deg(E) = c_1(E) \in H^2(X,\mathbb{Z}) \cong \mathbb{Z}$$

is given by the first Chern class $c_1(E)$. For instance, if $X = \mathbb{P}^1$ and $E = \mathcal{O}(m)$ with $m \geq 0$, then $g = 0$, $\text{rank}(E) = 1$, and

$$\dim H^{0,0}_{\bar{\partial}}(\mathbb{P}^1,\mathcal{O}(m)) - \dim H^{0,1}_{\bar{\partial}}(\mathbb{P}^1,\mathcal{O}(m)) = m+1.$$

From HW 2.21, we conclude

$$H^{0,0}_{\bar{\partial}}(\mathbb{P}^1,\mathcal{O}(m)) = \mathbb{C}^{m+1} \quad \text{and} \quad H^{0,1}_{\bar{\partial}}(\mathbb{P}^1,\mathcal{O}(m)) = 0.$$

Similarly, if $m = -n < 0$, we get

$$H^{0,0}_{\bar{\partial}}(\mathbb{P}^1,\mathcal{O}(m)) = 0 \quad \text{and} \quad H^{0,1}_{\bar{\partial}}(\mathbb{P}^1,\mathcal{O}(m)) = \mathbb{C}^{-1-m} = \mathbb{C}^{n-1}.$$

The case $m = -1$, i.e. $E = \gamma = \mathcal{O}(-1)$ is special in the sense that

$$H^{0,0}_{\bar{\partial}}(\mathbb{P}^1,\gamma) = 0 \quad \text{and} \quad H^{0,1}_{\bar{\partial}}(\mathbb{P}^1,\gamma) = 0.$$

The calculations above show that there is a symmetry around $m = -1$ in the sense that $H^{0,0}_{\bar{\partial}}(\mathbb{P}^1,\mathcal{O}(m))$ coincides with $H^{0,1}_{\bar{\partial}}(\mathbb{P}^1,\mathcal{O}(2-m))$ and vice versa. This is a special case of Serre duality that we will learn later. □

Next, we will review sheaf theory and cech cohomology. Suppose $\mathcal{F}$ is a sheaf on X and $\mathcal{A} = \{U_\alpha\}_{\alpha\in\mathcal{I}}$ is an open covering of X. Define

$$U_I = \bigcap_{\alpha\in I} U_\alpha \qquad \forall\ I \subset \mathcal{I},$$

$$C^q(\mathcal{A},\mathcal{F}) = \bigoplus_{I\subset\mathcal{I},\ |I|=q+1} \mathcal{F}(U_I).$$

It will be convenient to write an element $\eta \in C^q(\mathcal{A},\mathcal{F})$ as

$$\eta = \bigoplus_{i_0,\dots,i_q\in\mathcal{I}} \eta_{i_0 i_1 \dots i_q}$$

with the convention that

$$\eta_{i_0 i_1 \ldots i_q} = (-1)^{\varepsilon(\sigma)} \eta_{i_{\sigma(0)} i_{\sigma(1)} \ldots i_{\sigma(q)}}$$

for every permutation σ of $(i_0, \ldots, i_q)$. In particular, just like differential forms,

$$\eta_{i_0 i_1 \ldots i_q} = 0$$

whenever $i_a = i_b$ for some $a \neq b$; otherwise,

$$\eta_{i_0 i_1 \ldots i_q} \in \mathcal{F}(U_{i_0 i_1 \ldots i_q}).$$

The co-boundary map $\partial\colon C^{q-1}(\mathcal{A}, \mathcal{F}) \longrightarrow C^q(\mathcal{A}, \mathcal{F})$ is defined by restriction in the following way

$$\big(\partial \eta\big)_{i_0 i_1 \ldots i_q} = \sum_{a=0}^{q} (-1)^a \, \eta_{i_0 \ldots i_{a-1} i_{a+1} \ldots i_q}|_{U_{i_0 \ldots i_q}}.$$

Note that when the group structure on $\mathcal{F}$ is indeed multiplicative, the summation above should be replaced by product. It is easy to check that $\partial^2 = 0$ (again, remember that 0 means the trivial homomorphism of the category in the question. If the group structure is a product, this will be the trivial homomorphism the maps everything to 1). Therefore $\big(C^\bullet(\mathcal{A}, \mathcal{F}), \partial\big)$ is a cochain complex. The cech cohomology groups of $\mathcal{F}$ with respect to $\mathcal{A}$ are the cohomology groups of this complex:

$$\check{H}^k(\mathcal{A}, \mathcal{F}) = \frac{\operatorname{Ker}\Big(C^k(\mathcal{A}, \mathcal{F}) \xrightarrow{\partial} C^{k+1}(\mathcal{A}, \mathcal{F})\Big)}{\operatorname{Image}\Big(C^{k-1}(\mathcal{A}, \mathcal{F}) \xrightarrow{\partial} C^k(\mathcal{A}, \mathcal{F})\Big)}. \tag{3.1}$$

The covering-independent cech cohomology groups $\check{H}^k(X, \mathcal{F})$ are defined to be the direct limit of $\check{H}^k(\mathcal{A}, \mathcal{F})$ under refinements of $\mathcal{A}$. In most examples, the limit is achieved on a sufficiently refined $\mathcal{A}$ in the sense that all U_I are sufficiently small polydisks.

We say that a sequence of sheaf maps

$$0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{E} \longrightarrow 0$$

is exact if $\mathcal{F}$ is $\operatorname{Ker}(g)$ and $\mathcal{E}$ is the sheafification of $\operatorname{Coker}(f)$. In this situation, we also say $\mathcal{F}$ is a sub-sheaf of $\mathcal{G}$ and $\mathcal{E}$ is a quotient of $\mathcal{G}$. The exactness essentially means that, (only) for sufficiently small U, the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U) \xrightarrow{g_U} \mathcal{E}(U) \longrightarrow 0$$

is exact. This may not be true for larger and more complicated open sets, especially $U = X$. More generally, we say

$$\cdots \longrightarrow \mathcal{F}_{k-1} \xrightarrow{f_{k-1}} \mathcal{F}_k \xrightarrow{f_k} \mathcal{F}_{k+1} \longrightarrow \cdots$$

is exact if $\operatorname{Ker}(f_k)$ is equal to the sheafification of $\operatorname{Coker}(f_{k-1})$. The main result is the following.

Theorem 3.4. *Corresponding to every short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{E} \longrightarrow 0$$

on X, there is a long exact sequence of cech cohomology groups

$$\begin{aligned} 0 &\longrightarrow \check{H}^0(X, \mathcal{F}) \xrightarrow{f_*} \check{H}^0(X, \mathcal{G}) \xrightarrow{g_*} \check{H}^0(X, \mathcal{E}) \\ &\xrightarrow{\delta_*} \check{H}^1(X, \mathcal{F}) \xrightarrow{f_*} \check{H}^1(X, \mathcal{G}) \xrightarrow{g_*} \check{H}^1(X, \mathcal{E}) \\ &\longrightarrow \cdots. \end{aligned}$$

The following examples and facts will be the most relevant ones to our discussion of complex manifolds.

- The zero-th cech cohomology group $\check{H}^0(X, \mathcal{F})$ is the space of global sections $\mathcal{F}(X)$.
- To every holomorphic vector bundle $E \longrightarrow X$, we associate its sheaf of holomorphic sections

$$U \longrightarrow \Gamma(U, E).$$

  The sheaf of sections of the trivial holomorphic line bundle $X \times \mathbb{C}$ is the sheaf of holomorphic functions on X. It is denoted by $\mathcal{O}_X$ and is called the structure sheaf of X. For every other holomorphic vector bundle E, $\Gamma(U, E)$ is a free $\mathcal{O}_X(U)$-module. There are other sheaves of $\mathcal{O}_X$-modules that are important but every sheaf of $\mathcal{O}_X$-modules that is locally free is the sheaf of sections of a holomorphic vector bundle.
- If X is a complex manifold, let $\mathcal{O}^*_X$ denote the sheaf of $\mathbb{C}^*$-valued (nowhere-zero) holomorphic functions. Here, the group structure is the multiplication and the unit is 1. Suppose $\mathcal{A} = \{U_\alpha\}$ is a sufficiently refined holomorphic atlas on X. By definition, a 1-cocycle

$$\varphi \in C^1(\mathcal{A}, \mathcal{O}^*_X)$$

  is a collection of holomorphic functions

$$\varphi_{\alpha\beta} \colon U_{\alpha\beta} = U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$$

  such that

$$(\partial\varphi)_{\alpha\beta\gamma} = \varphi_{\beta\gamma}\varphi^{-1}_{\alpha\gamma}\varphi_{\alpha\beta}|_{U_{\alpha\beta\gamma}} = 1.$$

  Note that this is the multiplicative version of the definition of the cochain map of the cech cohomology. So $\partial\phi$ is trivial if and only if

$$\varphi_{\alpha\gamma}(z) = \varphi_{\beta\gamma}(z)\varphi_{\alpha\beta}(z) \qquad \forall\ z \in U_{\alpha\beta\gamma}.$$

  This is exactly the cocycle condition of the transition maps

$$(U_\alpha \times \mathbb{C})|_{U_{\alpha\beta}} \longrightarrow (U_\beta \times \mathbb{C})|_{U_{\alpha\beta}}, \qquad (z, v) \longrightarrow (z, \varphi_{\alpha\beta}(z)v)$$

  of a holomorphic line bundle $\mathcal{L} \longrightarrow X$. Conversely, if $\mathcal{L} \longrightarrow X$ is a holomorphic line bundle such that $\mathcal{L}_{U_\alpha} \cong U_\alpha \times \mathbb{C}$, the transition maps $\{\varphi_{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow \mathrm{End}(\mathbb{C}) = \mathbb{C}^*\}$ define a cech 1-coccyle φ. Two cech 1-cocyles $\varphi = \{\varphi_{\alpha\beta}\}$ and $\varphi' = \{\varphi'_{\alpha\beta}\}$ define the same cohomology group if and only if they differ by a coboundary; i.e. if

$$\varphi'_{\alpha\beta}\varphi^{-1}_{\alpha\beta} = (\partial\theta)_{\alpha\beta} = \theta_\beta\theta^{-1}_\alpha \tag{3.2}$$

  for some

$$\theta = (\theta_\alpha)_{\alpha\in\mathcal{I}} \in C^0(\mathcal{A}, \mathcal{O}^*_X)$$

  Let $\mathcal{L}$ and $\mathcal{L}'$ denote the holomorphic line bundles corresponding to φ and φ', respectively. By (3.2), the following diagram commutes

$$\begin{array}{ccc} (U_\alpha \times \mathbb{C})|_{U_{\alpha\beta}} & \xrightarrow{\theta_\alpha} & (U_\alpha \times \mathbb{C})|_{U_{\alpha\beta}} \\ \Big\downarrow \varphi_{\alpha\beta} & & \Big\downarrow \varphi'_{\alpha\beta} \\ (U_\beta \times \mathbb{C})|_{U_{\alpha\beta}} & \xrightarrow{\theta_\beta} & (U_\beta \times \mathbb{C})|_{U_{\alpha\beta}} \end{array}.$$

Which means the local isomorphisms

$$\mathcal{L}|_{U_\alpha} \cong U_\alpha \times \mathbb{C} \longrightarrow \mathcal{L}'|_{U_\alpha} \cong U_\alpha \times \mathbb{C}, \qquad (z, v) \longrightarrow (z, \theta_\alpha(z)v)$$

are compatible on the overlaps and define a global holomorphic isomorphism $\mathcal{L} \xrightarrow{\theta} \mathcal{L}'$. We conclude that there is a one-to-one correspondence between the elements of the cech cohomology group $\check{H}^1(\mathcal{A}, \mathcal{O}_X^*)$ and the isomorphism classes of holomorphic complex line bundles on X; i.e.,

$$\operatorname{Pic}(X) \cong \check{H}^1(\mathcal{A}, \mathcal{O}_X^*). \tag{3.3}$$

- For any topological space X, let $\underline{\mathbb{Z}}_X$, $\underline{\mathbb{R}}_X$, and $\underline{\mathbb{C}}_X$ denote the sheaves of locally constant functions taking values in $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. If X is a complex manifold, let Ω_X^p denote the sheaf of holomorphic p-forms on X. Then, we have

$$\begin{aligned} \check{H}^*(X, \underline{\mathbb{Z}}) &\cong H^*_{\text{sing}}(X, \underline{\mathbb{Z}}), \\ \check{H}^*(X, \underline{\mathbb{R}}) &\cong H^*_{\text{sing}}(X, \underline{\mathbb{R}}) \cong H^*_{\text{dR}}(X, \mathbb{R}), \\ \check{H}^*(X, \underline{\mathbb{C}}) &\cong H^*_{\text{sing}}(X, \underline{\mathbb{C}}) \cong H^*_{\text{dR}}(X, \mathbb{C}), \\ \check{H}^*(X, \Omega_X^p) &\cong H^{p,*}_{\bar{\partial}}(X, \mathbb{C}). \end{aligned} \tag{3.4}$$

The proof of $\check{H}^* \cong H^*_{\text{dR}}$ above uses Poincare lemma; see Proposition 3.42 in my lecture notes for MATH6410. Similalrly, if $\Lambda_X^{p,q}$ denotes the sheaf of smooth (p,q)-forms on a complex manifold X (note that $\Lambda^{p,q}(X, \mathbb{C}) = \Lambda_X^{p,q}(X)$ is the space of global sections of $\Lambda_X^{p,q}$), then by $\bar{\partial}$-Poincare Lemma, the sequence

$$0 \longrightarrow \Omega_X^p \xrightarrow{\iota} \Lambda_X^{p,0} \xrightarrow{\bar{\partial}} \Lambda_X^{p,1} \xrightarrow{\bar{\partial}} \Lambda_X^{p,2} \longrightarrow \dots$$

is exact and a similar proof yields the last isomorphism. The isomorphism $\check{H}^* \cong H^*_{\text{sing}}$ is Theorem 3.65 in my lecture notes for MATH6410. It uses an open covering derived from a triangulation of X.

Example 3.5. For $X = \mathbb{CP}^n$, we have

$$\check{H}^q(X, \Omega_X^p) \cong H^{p,q}_{\bar{\partial}}(X, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if} \quad 0 \le p = q \le n; \\ 0 & \text{otherwise.} \end{cases}$$

- If X is a complex manifold, the sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0,$$

where the first map is the inclusion map and the second map is the exponential map

$$f(z) \longrightarrow \mathrm{e}^{2\pi \mathrm{i} f(z)},$$

is a short exact sequence of sheaf. The associated long exact sequence of cech cohomology classes in Theorem 3.4 reads

$$\begin{aligned} 0 &\longrightarrow \check{H}^0(X, \underline{\mathbb{Z}}_X) \longrightarrow \check{H}^0(X, \mathcal{O}_X) \longrightarrow \check{H}^0(X, \mathcal{O}_X^*) \\ &\xrightarrow{\delta_1} \check{H}^1(X, \underline{\mathbb{Z}}_X) \longrightarrow \check{H}^1(X, \mathcal{O}_X) \longrightarrow \check{H}^1(X, \mathcal{O}_X^*) \\ &\xrightarrow{\delta_2} \check{H}^2(X, \underline{\mathbb{Z}}_X) \longrightarrow \check{H}^2(X, \mathcal{O}_X) \longrightarrow \check{H}^2(X, \mathcal{O}_X^*) \\ &\longrightarrow \cdots . \end{aligned}$$

By the isomorphisms above, this sequence can be re-written as

$$
\begin{aligned}
&0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow 0\\
&0 \longrightarrow H^1_{\mathrm{sing}}(X,\mathbb{Z}) \longrightarrow H^{0,1}_{\bar\partial}(X,\mathbb{C}) \longrightarrow \mathrm{Pic}(X)\\
&\quad \xrightarrow{\delta_2} H^2_{\mathrm{sing}}(X,\mathbb{Z}) \longrightarrow H^{0,2}_{\bar\partial}(X,\mathbb{C}) \longrightarrow \check{H}^2(X,\mathcal{O}^*_X)\\
&\quad \longrightarrow \cdots .
\end{aligned}
$$

For any holomorphic line bundle $\mathcal{L} \in \mathrm{Pic}(X)$, $\delta_2(\mathcal{L}) \in H^2_{\mathrm{sing}}(X,\mathbb{Z})$ is precisely the first Chern class $c_1(\mathcal{L})$. The quotient

$$
\frac{\check{H}^1(X,\mathcal{O}_X)}{\check{H}^1(M,\underline{\mathbb{Z}}_X)} \cong \frac{H^{0,1}_{\bar\partial}(X,\mathbb{C})}{H^1_{\mathrm{sing}}(X,\mathbb{Z})}
$$

is a torus known as $\mathrm{Pic}^0(X)$ or Jacobian of X; it is the subgroup of holomorphic line bundles on X that are smoothly trivial but homomorphically non-trivial. For example, if X is a compact Riemann surface of genus g, then

$$
\mathrm{Pic}^0(X) = \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}}
$$

is a g-dimensional torus. For $g=0$, $\mathrm{Pic}^0(\mathbb{CP}^1)=0$ because every holomorphic line bundle is tensor power of the tautological line bundle.

If we forget about the holomorphicity, a similar long exact sequence shows that every smooth complex line bundle is uniquely determined by its first Chern class and every cohomology class in $H^2_{\mathrm{sing}}(X,\mathbb{Z})$ is the first Chern class of some complex line bundle. In the holomorphic category, however, from the exact sequence

$$
0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow H^2_{\mathrm{sing}}(X,\mathbb{Z}) \longrightarrow H^{0,2}_{\bar\partial}(X,\mathbb{C}),
$$

we conclude that only if $H^{0,2}_{\bar\partial}(X,\mathbb{C})=0$, then every element of $H^2_{\mathrm{sing}}(X,\mathbb{Z})$ is the first chern class of some holomorphic line bundle. Otherwise, there are (smooth) complex line bundles on X that do not admit any holomorphic structure. If X is a Riemann-surface, for dimensional reasons, $H^{0,2}_{\bar\partial}(X,\mathbb{C})=0$. On the other hand, in complex dimension 2, there are interesting complex manifolds with non-trivial $H^{0,2}_{\bar\partial}(X,\mathbb{C})$. For instance a K3 surface is a simply connected (Kähler) holomorphic surface X with $c_1(TX)=0$ and $H^{0,2}_{\bar\partial}(X,\mathbb{C}) \cong \mathbb{C}$. All K3 surfaces are smoothly identical (diffeomorphic). The complex line

$$
\mathbb{C} \cong H^{0,2}_{\bar\partial}(X,\mathbb{C}) \subset H^2_{\mathrm{dR}}(X,\mathbb{C}) \cong \mathbb{C}^{22}
$$

defines a point in the projective space $\mathbb{P}(H^2_{\mathrm{dR}}(X,\mathbb{C})) \cong \mathbb{CP}^{21}$. This gives us a map

$$
\{\text{The space of all K3 surfaces}\} \longrightarrow \mathbb{CP}^{21}
$$

that helps us parametrize the "moduli space" of complex structures on K3 surfaces. The image of the map above belongs to a 20-dimensional hypersurface.

4 Hermitian metrics and Kähler structures

Definition 4.1. A Hermitian metric on a complex vector bundle $E \longrightarrow X$ is a (smoothly varying) fiber-wise bilinear map

$$h = \langle -, - \rangle : E \otimes_{\mathbb{R}} \overline{E} \longrightarrow \mathbb{C},$$

such that

(1) h is complex linear in the first input and anti-complex linear in the second input,

(2) (symmetry) $h(u, \overline{v}) = \overline{h(v, \overline{u})}$,

(3) (positive definiteness) $h(u, \overline{u}) > 0$ for all $0 \neq u \in E$.

For each fiber E_p and with respect to any trivialization $E_p \cong \mathbb{C}^r$, we have

$$h(u, \overline{v}) = u^T H \overline{v},$$

and H is an $r \times r$ matrix satisfying $H^T = \overline{H}$, with positive eigenvalues.

If we think of E as a real vector bundle and think of the complex multiplication by $\mathfrak{i}$ as an almost complex structure, we get

$$\frac{1}{2} h = g - \mathfrak{i}\omega,$$

such that g is a Riemannian metric on the real vector space underlying E, and $\omega \in \Lambda^2_{\mathbb{R}} E^*$. Each of h, g, and ω, will determine the rest. We will elaborate on this later in this section.

If X is a complex manifold, a Hermitian metric on X is a Hermitian metric on its complex tangent bundle $\mathcal{T}X$. Locally, with respect to a coordinate chart $(z_1, \ldots, z_n)$, we can write

$$h = \sum_{i,j} h_{i\overline{j}} \mathrm{d}z_i \otimes \mathrm{d}\overline{z_j}$$

for which

$$H = (h_{i\overline{j}})_{1 \leq i,j \leq n}. \tag{4.1}$$

The so called fundamental real $(1,1)$-form ω associated to h is given by

$$\omega = \frac{\mathfrak{i}}{2} \sum_{i,j} h_{i\overline{j}} \mathrm{d}z_i \wedge \mathrm{d}\overline{z_j} \in \Lambda^{1,1}(X, \mathbb{C}) \cap \Lambda^2(X, \mathbb{R}).$$

The $2n$-form

$$\frac{1}{n!} \omega^n$$

is nonzero everywhere and thus defines a volume form on X. Note that every complex manifold is canonically oriented. We will use this volume form to define integration on X.

HW 4.2. Show that ω is indeed a real 2-form; i.e., when written in terms of $\mathrm{d}x_i$ and $\mathrm{d}y_i$, all the coefficients are real.

In terms of ω, the Condition (3) in Definition 4.1 is equivalent to the following.

Definition 4.3. A 2-form $\omega \in \Lambda^2(X,\mathbb{R}) \cap \Lambda^{1,1}(X,\mathbb{C})$ on a complex manifold X is called positive if $-\mathfrak{i}\omega(u,\overline{u}) > 0$ for all $0 \neq u \in \mathcal{T}X$. Equivalently, thinking of ω as a real 2-form in $\Lambda^2(X,\mathbb{R})$, the positivity condition is equivalent to

$$\omega(v, iv) > 0 \qquad \forall\ 0 \neq v \in TX,$$

where $i\colon TX \longrightarrow TX$ is considered as an almost complex structure as in (2.2).

Definition 4.4. A Hermitian metric h on a complex manifold is called Kähler if $\mathrm{d}\omega = 0$.

Since working with differential forms is easier than working with metrics, and ω and h carry the same information, we usually work with ω and, if $\mathrm{d}\omega = 0$, we say that ω is a Kähler structure/form on X.

Example 4.5. The standard Kähler form on $\mathbb{C}^n$ is

$$\omega = \frac{\mathfrak{i}}{2}\sum_i \mathrm{d}z_i \wedge \mathrm{d}\overline{z_i} = \sum_i \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

with the standard volume form

$$\frac{1}{n!}\omega^n = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

Example 4.6. Suppose X is a Riemann surface. Since every 2-form on X is closed, every Hermitian metric on X is Kähler. Locally, in a coordinate chart z on X, we have

$$\omega = \frac{\mathfrak{i}}{2}\varrho(z)\mathrm{d}z \wedge \mathrm{d}\overline{z} = \varrho(x,y)\mathrm{d}x \wedge \mathrm{d}y$$

for some positive real-valued function ϱ. If we write $\varrho = \mathrm{e}^{\varphi}$, then the scalar curvature of ω is equal to

$$-\varrho^{-1}\Delta \ln \varrho = -\mathrm{e}^{-\varphi}\Delta\varphi. \tag{4.2}$$

For instance, the Poincare metric

$$\omega = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2} = \frac{\mathfrak{i}}{2}\frac{1}{\mathrm{Im}(z)^2}\mathrm{d}z \wedge \mathrm{d}\overline{z} \tag{4.3}$$

on the upper half plane $\mathcal{H}$ has the curvature

$$-\varrho^{-1}\Delta \ln \varrho = 2y^2\Delta \ln y = -2.$$

Note that the Gaussian curvature is half of the Scalar curvature.

Example 4.7. Let $z = (z_0, \ldots, \widehat{z_i}, \ldots z_n)$ be the affine coordinates on the open set $U_i \subset \mathbb{CP}^n$ in Example 2.9. The 2-forms

$$\omega_{\mathrm{FS},i} = \frac{\mathfrak{i}}{2}\partial\bar{\partial}\ln(1+|z|^2) = \frac{\mathfrak{i}}{2}\frac{(1+|z|^2)\sum \mathrm{d}z_a \wedge \mathrm{d}\overline{z}_a - (\sum \overline{z}_a \mathrm{d}z_a) \wedge (\sum z_a \mathrm{d}\overline{z}_a)}{(1+|z|^2)^2} \tag{4.4}$$

are compatible on the overlaps $U_i \cap U_j$ and define a Kähler structure ω_{FS} on $\mathbb{CP}^n$ called the Fubini-Study metric/Kähler form. For $n = 1$, we have

$$\omega_{\mathrm{FS}} = \frac{\mathfrak{i}}{2}\frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1+|z|^2)^2} = \frac{\mathrm{d}x \wedge \mathrm{d}y}{(1+x^2+y^2)^2} = \frac{r\mathrm{d}r \wedge \mathrm{d}\theta}{(1+r^2)^2}.$$

Therefore,

$$\int_{S^2} \omega_{FS} = 2\pi \int_0^\infty \frac{r\mathrm{d}r}{(1+r^2)^2} = \pi \int_0^\infty \frac{\mathrm{d}s}{(1+s)^2} = \pi \frac{-1}{1+s}\,|_0^\infty = \pi(0-(-1)) = \pi. \tag{4.5}$$

Also, using (4.2), we compute that the scalar curvature of this metric to be

$$2(1+r^2)^2 \Delta \ln(1+r^2) = 2(1+r^2)^2 \Big(\frac{\partial^2 \ln(1+r^2)}{\partial r^2} + \frac{1}{r} \frac{\partial \ln(1+r^2)}{\partial r} \Big) = 8.$$

The metric above on S^2 corresponds to the induced metric on the sphere of radius $\frac{1}{2}$ in $\mathbb{R}^3$. That explains the missing 4 in (4.5) and the extra factor of 4 in the scalar curvature.

If (X,ω) is a Kähler manifold and $Y \subset X$ is a complex submanifold, then $\omega|_Y$ defines an induced Kähler structure on X. In conclusion, every smooth complex projective variety has a Kähler structure. It also follows from the discussion above that the volume of Y is given by the integral

$$\frac{1}{\dim_{\mathbb{C}} Y!} \int_Y \omega^{\dim_{\mathbb{C}} Y}.$$

The fact that the volume of a complex submanifold Y of a complex manifold X is expressed as the integral over Y of a globally defined differential form is quite different from what we have in Riemannian geometry. Furthermore, by Stokes' Theorem, if (X,ω) is Kähler, i.e. $\mathrm{d}\omega = 0$, then this integral only depends on the homology class $[Y] \in H_{\dim_{\mathbb{R}} Y}(X,\mathbb{Z})$. If $Y \subset X$ is a singular variety, the singular locus $Y^{\mathrm{sing}} \subset Y$ has at most real codimension 2. Therefore, similarly, the integral

$$\int_Y \omega^{\dim_{\mathbb{C}} Y} = \int_{Y - Y^{\mathrm{sing}}} \omega^{\dim_{\mathbb{C}} Y}$$

is defined and depends on the homology class $[Y] \in H_{\dim_{\mathbb{R}} Y}(X,\mathbb{Z})$. The same applies to other differential forms on X in the following sense.

Lemma 4.8. *(Stokes' Therem for analytic varieties) Suppose X is a complex manifold and $Y \subset X$ is an analytic subvariety of complex dimension k. For every differential $(2k-1)$-form η on X, the integral*

$$\int_Y \mathrm{d}\eta = \int_{Y - Y^{\mathrm{sing}}} \mathrm{d}\eta$$

is defined and is equal to 0. Consequently, for every closed $(2k)$-form η on X, the integral

$$\int_Y \eta = \int_{Y - Y^{\mathrm{sing}}} \eta$$

is defined and only depends on the homology class of Y.

In complex dimension 1, all compact Riemann surfaces are complex projective varieties. However, in complex dimension 2 and higher, there are Kähler manifolds that are not projective varieties. Therefore, we have the hierarchy

Complex Manifolds $\subset$ Kähler Manifolds $\subset$ Smooth Complex Projective Varieties.

For example, as we mentioned at the end of previous section, there are simply connected complex surfaces X known as $K3$ surfaces that are topologically characterized by $c_1(TX) \equiv 0$. We described a map

$$\{\text{The space of all K3 surfaces}\} \longrightarrow \mathbb{CP}^{21}$$

that parametrizes the space of complex structures on K3 surfaces. The image of this map belongs to a 20-dimensional hypersurface $Q \subset \mathbb{CP}^{21}$. While all $K3$ surfaces are Kähler, those that are projective correspond to a union of 19-dimensional components in Q. Every quartic surface in $\mathbb{CP}^3$, e.g. the Fermat quartic

$$X \equiv (x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0) \subset \mathbb{CP}^3,$$

is a K3 surface.

Every Kähler form defines a cohomology class

$$[\omega] \in H^2_{\rm dR}(X,\mathbb{R}) \cap H^{1,1}_{\bar\partial}(X,\mathbb{C}).$$

If X is compact, since $\int_X \omega^{\dim_{\mathbb{C}} X} = \mathrm{vol}(X) \neq 0$, the cohomology class of ω will be non-trivial. A non-zero multiple of every Kähler form is a Kähler form. The set of all Kähler classes $[\omega]$ on a compact complex manifold X (together with the zero-class) is an open convex cone $\mathcal{K}ah(X)$ in

$$H^2_{\rm dR}(X,\mathbb{R}) \cap H^{1,1}_{\bar\partial}(X,\mathbb{C}).$$

The intersection above in the smoothly invariant group $H^2_{\rm dR}(X,\mathbb{R})$ can change as we change the complex structure on X. For example, in the example of K3 surfaces above, the inclusion $H^2_{\rm dR}(X,\mathbb{R}) \cap H^{1,1}_{\bar\partial}(X,\mathbb{C}) \subset H^2_{\rm dR}(X,\mathbb{R})$ specifies the complex structure on X.

Given a complex vector bundle $E \longrightarrow X$, a $\mathbb{C}$-linear connection ∇ is an $\mathbb{C}$-linear map

$$\nabla\colon \Lambda^0(X,E) = \Gamma(X,E) \longrightarrow \Lambda^1(X,E)$$

that satisfies the Leibniz rule

$$\nabla(f\zeta) = f\nabla\zeta + \mathrm{d}f \otimes \zeta \qquad \forall\ f \in C^\infty(X,\mathbb{C}),\ \zeta \in \Gamma(X,E). \tag{4.6}$$

With respect to any local trivialization $E|_U \cong U \times \mathbb{C}^r$, ∇ can be written as

$$\nabla := \mathrm{d} + \Theta \qquad \text{s.t.} \qquad \Theta \in \Lambda^1\big(U, \mathrm{End}(\mathbb{C}^r)\big); \tag{4.7}$$

i.e. Θ is an $(r \times r)$ matrix of 1-forms. Changing one trivialization to another, with a change of trivialization map

$$U \times \mathbb{C}^r \longrightarrow U \times \mathbb{C}^r, \qquad (x,v) \longrightarrow (x, \Phi(x)v),$$

changes Θ to

$$\Theta' = \mathrm{d}\Phi \circ \Phi^{-1} + \Phi \circ \Theta \circ \Phi^{-1}. \tag{4.8}$$

Using a partition of unity, it is easy to show that every (smooth) vector bundle admits a plethora of (smooth) connections.

HW 4.9. Show that if ∇ and ∇' are two connections on E, then

$$\nabla - \nabla' \in \Lambda^1(X, \mathrm{End}(E));$$

i.e. the difference of every two connections is a globally defined $\mathrm{End}(E)$-valued 1-form.

By the homework above, the space of connections on E is an affine space with tangent space $\Lambda^1(X,\mathrm{End}(E))$.

Example 4.10. In (4.8), if $E \longrightarrow X$ is a holomorphic line bundle, since Θ is a single 1-form and Θ is a $\mathbb{C}^*$-valued holomorphic function, we get

$$\Theta' - \Theta = \frac{\mathrm{d}\Phi}{\Phi}.$$

For any connection ∇, the expression

$$F^{\nabla}(\zeta_1, \zeta_2)\xi = \nabla_{\zeta_1}\nabla_{\zeta_2}\xi - \nabla_{\zeta_2}\nabla_{\zeta_1}\xi - \nabla_{[\zeta_1,\zeta_2]}\xi, \qquad \forall\, \zeta_1, \zeta_2 \in \Gamma(X, TX \otimes_{\mathbb{R}} \mathbb{C}),\ \xi \in \Gamma(X, E), \tag{4.9}$$

is $C^\infty(M, \mathbb{C})$-linear in all three inputs. Hence it defines an element

$$F^{\nabla} \in \Lambda^2(X, \mathrm{End}(E)),$$

called the curvature of ∇. In terms of the local connection matrices Θ in (4.7), the $\mathrm{End}(E)$-valued 2-form F^{∇} has the form

$$F = \mathrm{d}\Theta - \Theta \wedge \Theta. \tag{4.10}$$

A connection ∇ can be seen as a way of extending the exterior derivative d to E-valued differential forms by

$$\mathrm{d}_{\nabla} \colon \Lambda^k(X, E) \longrightarrow \Lambda^{k+1}(X, E), \quad \alpha = \eta \otimes \zeta \longrightarrow \mathrm{d}_{\nabla}\alpha = \mathrm{d}\eta \otimes \zeta + (-1)^k \eta \otimes \nabla\zeta.$$

for all $k \geq 0$. Unlike the exterior derivative d, $\mathrm{d}_{\nabla} \circ \mathrm{d}_{\nabla} \neq 0$; thinking of F^{∇} as a map

$$\Lambda^k(X, E) \longrightarrow \Lambda^{k+2}(X, E), \quad \alpha \longrightarrow F^{\nabla} \wedge \alpha;$$

we have $\mathrm{d}_{\nabla} \circ \mathrm{d}_{\nabla}(\alpha) = F^{\nabla} \wedge \alpha$; so F^{∇} measures how much $\mathrm{d}_{\nabla} \circ \mathrm{d}_{\nabla}$ deviates from being a cochain map.

If $\mathrm{rank}_{\mathbb{C}}\, E = 1$, then Θ is an honest 2-form and $\Theta \wedge \Theta = 0$; thus, F defines a global closed 2-form on X such that the cohomology class

$$[\frac{\mathrm{i}}{2\pi} F] \in H^2(X, \mathbb{C}) \tag{4.11}$$

is independent of the choice of ∇. This cohomology class is the deRham representative of $c_1(E)$. More generally, If $\mathrm{rank}_{\mathbb{C}}(E) = r$, the Chern classes of E, as de Rham cohomology classes, are defined by

$$1 + tc_1(E) + t^2 c_2(E) + \ldots + t^r c_r(E) = \det(I + \frac{\mathrm{i}t}{2\pi} F).$$

These cohomology classes are weighted by $\frac{\mathrm{i}}{2\pi}$ to become real-valued for suitable choice of ∇; see below. In particular,

$$c_1(E) = [\frac{\mathrm{i}}{2\pi}\mathrm{trace}(F)] \in H^2_{\mathrm{dR}}(M, \mathbb{R}) \quad \text{and} \quad c_r(E) = [\Big(\frac{\mathrm{i}}{2\pi}\Big)^r \det(F)] \in H^{2r}_{\mathrm{dR}}(M, \mathbb{R}). \tag{4.12}$$

Given a Hermitian metric $h \equiv \langle -, - \rangle$ on E, we say that a $\mathbb{C}$-linear connection ∇ is compatible with h if

$$\mathrm{d}\left\langle \xi, \overline{\zeta} \right\rangle = \left\langle \nabla\xi, \overline{\zeta} \right\rangle + \left\langle \xi, \overline{\nabla\zeta} \right\rangle. \tag{4.13}$$

There are plenty of connections that are compatible with a given h.

Now suppose X is a complex manifold and E is a holomorphic vector bundle. Then the exterior derivative d and the matrix Θ in (4.7) decompose as

$$\mathrm{d} + \Theta = (\partial + \Theta^{1,0}) + (\bar{\partial} + \Theta^{0,1}).$$

We denote the first and second summands on the righthand side by $\nabla^{1,0}$ and $\nabla^{0,1}$. We have

$$\nabla^{1,0} \colon \Gamma(X,E) \longrightarrow \Lambda^{1,0}(X,E), \quad \text{and}$$
$$\nabla^{0,1} \colon \Gamma(X,E) \longrightarrow \Lambda^{1,0}(X,E).$$

We say ∇ is a connection compatible with the complex (holomorphic) structure, or a Chern connection, if $\nabla^{0,1} = \bar{\partial}$, i.e. $\Theta^{0,1} \equiv 0$ on each local chart.

Lemma 4.11. *Suppose $E \longrightarrow X$ is a holomorphic vector bundle with a Hermitian metric h. Then, there exists a unique connection ∇ on E that is compatible with both the metric and the complex structure.*

Proof. Fix a local holomorphic trivialization $E|_U \cong U \times \mathbb{C}^r$. Let $H = (h_{i\bar{j}})$ be the matrix of h with respect to this trivialization as in (4.1). The equation (4.13) reads

$$\mathrm{d}H = \Theta^T H + H\overline{\Theta}.$$

Since, by assumption, $\Theta = \Theta^{1,0}$, we have

$$\partial H = \Theta^T H \qquad \text{and} \qquad \bar{\partial} H = H\overline{\Theta}.$$

We find that

$$\Theta = (\partial H H^{-1})^T = \overline{H}^{-1} \partial \overline{H}. \tag{4.14}$$

□

In the situation above, decomposing the curvature form(s) into different (p,q)-types,

$$F = F^{(2,0)} + F^{(1,1)} + F^{(0,2)},$$

since $\nabla^{0,1} \circ \nabla^{0,1} = \bar{\partial}^2 = 0$, we conclude that

$$F^{0,2} \equiv 0 \quad \text{and} \quad F^{2,0} = -(F^{0,2})^* = 0.$$

Therefore, $F = F^{1,1}$ is a matrix of $(1,1)$-forms.

Example 4.12. Suppose $\mathcal{L} \longrightarrow X$ is a holomorphic vector bundle with a Hermitian metric h. Therefore, with respect to any holomorphic trivialization $\mathcal{L}|_U \cong U \times \mathbb{C}$, given by a nowhere zero section $\zeta \in \Gamma(U, \mathcal{L}|_U)$, h is simply given by one real function

$$H = (h_{1\bar{1}}(z)), \quad H = h_{1\bar{1}} = \langle \zeta, \overline{\zeta} \rangle : U \longrightarrow \mathbb{R}.$$

By the equation above, the connection 1-form Θ is given by

$$\Theta = \frac{\partial H}{H} = \partial \ln(H). \tag{4.15}$$

Furthermore, by (4.10) and (4.10),

$$F = \mathrm{d}\Theta = \bar{\partial}\partial \ln(H) \tag{4.16}$$

is a global closed 2-form and

$$c_1(\mathcal{L}) = [\frac{\mathfrak{i}}{2\pi}\bar{\partial}\partial \ln(H)] \in H^2_{\mathrm{dR}}(X,\mathbb{R}) \cap H^{(1,1)}_{\bar{\partial}}(X,\mathbb{C}).$$

□

Notice that there is a similarity between the equation above and (4.4) in the sense that both 2-forms are defined as $\partial\bar{\partial}$ or $\bar{\partial}\partial$ of some function.

Lemma 4.13. *There is a Hermitian metric on the holomorphic line bundle $\mathcal{O}(1) \longrightarrow \mathbb{CP}^n$ such that*

$$\frac{\mathfrak{i}}{2}F = \omega_{\mathrm{FS}}.$$

Proof. We have $\mathcal{O}(1) = \gamma^*$ and $\gamma^* \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$. The restriction of the standard Hermitian metric on $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ to γ is the Hermitian metric

$$h_\gamma\big(([x],x),([x],x)\big) = |x|^2.$$

In terms of the local affine coordinates $z = (z_0,\ldots,\widehat{z_i},\ldots,z_n)$ on the open set $U_i \subset \mathbb{CP}^n$, and the local trivialization $\gamma|_{U_i} \cong U_i \times \mathbb{C}$ in Example 2.20, we have

$$h_\gamma\big((z,1),(z,1)\big) = 1 + |z|^2.$$

Therefore, the induced metric on the dual bundle $\mathcal{O}(1) = \gamma^*$ satisfies

$$h_{\mathcal{O}(1)}\big((z,1),(z,1)\big) = (1+|z|^2)^{-1}.$$

By (4.16),

$$\frac{\mathfrak{i}}{2}F = \frac{\mathfrak{i}}{2}\bar{\partial}\partial \ln\big((1+|z|^2)^{-1}\big) = -\frac{\mathfrak{i}}{2}\bar{\partial}\partial \ln\big(1+|z|^2\big) = \frac{\mathfrak{i}}{2}\partial\bar{\partial} \ln\big(1+|z|^2\big) = \omega_{\mathrm{FS}}.$$

□

Lemma 4.13 shows that the curvature of the Chen connection of some Hermitian metric on $\mathcal{O}(1)$ is positive in the sense that

$$F(u,\overline{u}) = -2\mathfrak{i}\omega_{\mathrm{FS}}(u,\overline{u}) > 0 \qquad \forall\ 0 \neq u \in \mathcal{T}X$$

The following is the positivity notion used in the Kodaira's Embedding Theorem.

Definition 4.14. A holomorphic line bundle $\mathcal{L} \longrightarrow X$ is called positive, if it admits a Hermitian metric h such that the curvature $(1,1)$-form of the Chern connection of h is positive (i.e. $\mathfrak{i}F$ is a Kähler form on X). In other words, $\mathcal{L}$ is called positive if its first Chern class $c_1(\mathcal{L}) = [\frac{\mathfrak{i}}{2\pi}F]$ can be represented by a Kähler form. The last statement can also be rephrased as $c_1(\mathcal{L})$ belongs to the Kähler cone of $X(c_1(\mathcal{L}) \in \mathcal{K}ah(X))$.

Thus, Theorem 2.25 reads: *Let X be a compact complex manifold and $\mathcal{L}$ be a holomorphic line bundle such that $c_1(\mathcal{L}) \in \mathcal{K}al(X)$. Then there exists $k_0 > 0$ such that for every $k \geq k_0$, the map $\iota_{\mathcal{L}^{\otimes k}}$ corresponding to the line bundle $\mathcal{L}^{\otimes k}$ is an embedding.*

There is an alternative description of complex vector bundles where we consider a real rank $2r$ vector bundle $E \longrightarrow X$ together with a real linear endomorphism $J\colon E \longrightarrow E$, covering the identity map on X, satisfying $J^2 = -\mathrm{id} \in \mathrm{End}_{\mathbb{R}}(E)$; see Example 2.18. Then, the complexfication $E \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as a direct sum of complex vector bundles,

$$E \otimes_{\mathbb{R}} \mathbb{C} \cong E^{1,0} \otimes E^{0,1},$$

such that $E^{1,0}$ and $E^{0,1}$ are the eigenspaces of the eigenvalues $+\mathfrak{i}$ and $-\mathfrak{i}$ of the action of J. For instance, recall from Example 2.18 that if X is a complex manifold with local holomorphic coordinates $(z_1 = x_1 + \mathfrak{i}y_1, \ldots, z_n = x_n + \mathfrak{i}y_n)$, then

- the real tangent bundle of X is denoted by TX and is generated by $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$,
- the complex tangent bundle of X is denoted by $\mathcal{T}X$ and is generated by $\frac{\partial}{\partial z_1} \ldots \frac{\partial}{\partial z_n}$,
- J acts on TX by
$$J\frac{\partial}{\partial x_a} = \frac{\partial}{\partial y_a} \quad \text{and} \quad J\frac{\partial}{\partial y_a} = -\frac{\partial}{\partial x_a},$$
- and
$$TX \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}X \oplus T^{0,1}X$$
such that $\mathcal{T}X = T^{1,0}X$.

In the situation above, there is a real linear isomorphism

$$E \longrightarrow E^{1,0}, \quad v \longrightarrow v^{\mathbb{C}} = \frac{1}{2}(v - \mathfrak{i}Jv), \tag{4.17}$$

that identifies the action of J on E with the complex multiplication by $\mathfrak{i}$ on $E^{1,0}$. Similarly, the real linear isomorphism

$$E \longrightarrow E^{0,1}, \quad v \longrightarrow \overline{v^{\mathbb{C}}} = \frac{1}{2}(v + \mathfrak{i}Jv),$$

identifies the action of J on E with the complex multiplication by $-\mathfrak{i}$ on $E^{1,0}$. If h is a Hermitian metric on $E^{1,0}$, the identity

$$\widetilde{h}(v_1, v_2) := \frac{1}{2}h\big(v_1^{\mathbb{C}}, \overline{v_2^{\mathbb{C}}}\big) \tag{4.18}$$

defines a real bilinear map

$$\widetilde{h}\colon E \otimes E \longrightarrow \mathbb{C}$$

satisfying

(1) (Symmetry) $\widetilde{h}(v_1, v_2) = \overline{\widetilde{h}(v_2, v_1)}$;

(2) (Positive definiteness) $\widetilde{h}(v, v) > 0$;

(3) (Compatibility with J) $\widetilde{h}(Jv_1, v_2) = -\widetilde{h}(v_1, Jv_2) = \mathfrak{i}\widetilde{h}(v_1, v_2)$.

Conversely, such $\widetilde{h}$ defines a hermitian metric h on $E^{1,0}$. If we decompose $\widetilde{h}$ as

$$\widetilde{h} = g - \mathrm{i}\omega, \tag{4.19}$$

then g is a Riemannian metric on E with respect to which J is orthogonal, and ω is the fundamental 2-form of h studied above.

HW 4.15. Suppose g is a Riemannian metric on TX such that $g(Jv_1, Jv_2) = g(v_1, v_2)$. Show that ω defined by $\omega(u, v) = g(Ju, v)$ is a 2-form and

$$\widetilde{h} = g - \mathrm{i}\omega$$

is a Hermitian metric in the sense of (1)-(3) above on (TX, J).

Now, suppose X is a complex manifold and h is a Hermitian metric on $\mathcal{T}X$ (we have not assumed that (X, ω) is Kähler yet). Then Lemma (4.11) applies to the holomorphic vector bundle $\mathcal{T}X \longrightarrow X$ and yields a unique complex linear connection ∇ satisfying $\nabla^{0,1} = \bar{\partial}$. On the other hand, h induces a Riemannian metric g on TX as above and there is a unique torsion free connection $\nabla^{\ell c}$ on TX compatible with g, called the Levi-Civita connection. By the homework above, each of h and g will uniquely specify the other one.

A natural question to ask is: *under the real isomorphism $TX \longrightarrow \mathcal{T}X$ in (4.17), how does ∇ compare to $\nabla^{\ell c}$?*

As we see below, the answer is: $\nabla = \nabla^{\ell c}$ if and only if (X, ω) is Kähler. In other words, $\nabla^{\ell c}$ is complex linear ($\nabla^{\ell c} J \equiv 0$) if and only if $\mathrm{d}\omega = 0$.

Lemma 4.16. *Let X be a complex manifold with a Hermitian metric h and the associated Riemannian metric g on TX as in (4.19). Under the real isomorphism (4.17), a Hermitian connection ∇ on $\mathcal{T}X$ induces a Riemannian connection $\nabla^{\mathbb{R}}$ on TX.*

Proof. The inverse of (4.17) is given by

$$v^{\mathbb{C}} \in \mathcal{T}X = T^{1,0}X \subset TX \otimes \mathbb{C} \longrightarrow v = 2\mathrm{Re}(v^{\mathbb{C}}) \in TX.$$

By assumption, for $\zeta_1, \zeta_2 \in \Gamma(X, \mathcal{T}X)$, we have

$$\mathrm{d}\ h(\zeta_1, \zeta_2) = h(\nabla\zeta_1, \overline{\zeta}_2) + h(\zeta_1, \overline{\nabla\zeta_2}).$$

For $\xi \in \Gamma(X, TX)$, define

$$\nabla^{\mathbb{R}}\xi = 2\mathrm{Re}\ \nabla\xi^{\mathbb{C}} = 2\mathrm{Re}\ \overline{\nabla\xi^{\mathbb{C}}}.$$

For $\xi_1, \xi_2 \in \Gamma(X, TX)$, we have

$$\begin{aligned}
\mathrm{d}\ g(\xi_1, \xi_2) &= \mathrm{d}\ \frac{1}{2}\mathrm{Re}\ h(\xi_1^{\mathbb{C}}, \overline{\xi_2^{\mathbb{C}}}) = \frac{1}{2}\mathrm{d}h(\xi_1^{\mathbb{C}}, \overline{\xi_2^{\mathbb{C}}}),\\
&= \frac{1}{2}\Big(\mathrm{Re}\ h(\nabla\xi_1^{\mathbb{C}}, \overline{\xi_2^{\mathbb{C}}}) + h(\xi_1^{\mathbb{C}}, \overline{\nabla\xi_2^{\mathbb{C}}})\Big)\\
&= g(2\mathrm{Re}\nabla\xi_1^{\mathbb{C}}, \overline{\xi_2^{\mathbb{C}}}) + g(\xi_1^{\mathbb{C}}, 2\mathrm{Re}\overline{\nabla\xi_2^{\mathbb{C}}})\\
&= g(\nabla^{\mathbb{R}}\xi_1, \xi_2) + g(\xi_1, \nabla\xi_2).
\end{aligned}$$

We conclude that $\nabla^{\mathbb{R}}$ is compatible with g. □

The converse of this statement is not always correct; i.e., a Riemannian connection $\nabla^{\mathbb{R}}$ on TX compatible with g does not necessarily define a Hermitian connection on $\mathcal{T}X$ compatible with h.

HW 4.17. Show that if $\nabla^{\mathbb{R}}$ is J-linear, then $\nabla^{\mathbb{R}}$ naturally induces a Hermitian connection on $\mathcal{T}X$ compatible with h. Here, by $\nabla^{\mathbb{R}}$ being J-linear we mean $\nabla^{\mathbb{R}}J\xi = J\nabla^{\mathbb{R}}\xi$. Since, following the product rule, the derivative $\nabla^{\mathbb{R}}J$ of the endomorphism J is defined by

$$(\nabla^{\mathbb{R}}J)\xi = \nabla^{R}(J\xi) - J\nabla^{\mathbb{R}}\xi,$$

the J-linearity of $\nabla^{\mathbb{R}}$ is the same as $\nabla^{R}J \equiv 0$. When the latter happens for a tensor, we say that tensor is parallel with respect to the connection.

Let $\nabla^{\mathbb{R}}$ be the Riemannian connection on TX induced by the Chern connection on $\mathcal{T}X$ as in Lemma 4.16. In order to compare $\nabla^{\mathbb{R}}$ with $\nabla^{\ell c}$, we need to understand the torsion of $\nabla^{\mathbb{R}}$.

Every connection D on TX induces a similarly denoted connection on T^*X satisfyig

$$\mathrm{d}(\eta(\xi)) = D\eta(\xi) + \eta(D\xi) \qquad \forall \xi \in \Gamma(X, TX),\ \eta \in \Gamma(X, T^*X).$$

This in turn induces a connection on $\Lambda^k(X, \mathbb{R})$, for all $k \geq 1$. The torsion of a connection D on TX is the skew-symmetric $(2,1)$-tensor

$$T_D(\xi_1, \xi_2) = D_{\xi_1}\xi_2 - D_{\xi_2}\xi_1 - [\xi_1, \xi_2].$$

If locally write $D = \mathrm{d} + \Theta$ as in (4.7), then

$$T_D(\xi_1, \xi_2) = \Theta(\xi_1)\xi_2 - \Theta(\xi_2)\xi_1. \tag{4.20}$$

We say D is torsion free if $T_D \equiv 0$, i.e $D_{\xi_1}\xi_2 - D_{\xi_2}\xi_1 = [\xi_1, \xi_2]$. Note that the righthand side is independent of the choice of any connection.

Lemma 4.18. *For $\eta \in \Lambda^k(X, \mathbb{R})$, if D is torsion free, we have*

$$(\mathrm{d}\eta)(\xi_0, \ldots, \xi_k) = \sum_{i=0}^{k} (-1)^i (D_{\xi_i}\eta)(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_k).$$

Proof. From the formula

$$(\mathrm{d}\eta)(\xi_0, \ldots, \xi_k) = \sum_{i=1}^{k+1} (-1)^i \xi_i \cdot \eta(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_k) + \sum_{0 \leq i \leq j \leq k} (-1)^{i+j} \eta([\xi_i, \xi_j], \xi_0, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_k)$$

we obtain that

$$\begin{aligned}(\mathrm{d}\eta)(\xi_0, \ldots, \xi_k) = &\sum_{i=0}^{k} (-1)^i (D_{\xi_i}\eta)(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_k) + \\ &\sum_{0 \leq i \leq j \leq k} (-1)^{i+j} \eta(T_D(\xi_i, \xi_j), \xi_0, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_k).\end{aligned}$$

The lemma follows. □

Corollary 4.19. *Any D-parallel differential form η with respect to a torsion free connection D is closed.*

Proposition 4.20. *Suppose X is a complex manifold, h is a Hermitian metric on $\mathcal{T}X$ with the real fundamental $(1,1)$-form ω, and ∇ is the Chern connection of (X,h). Then $\nabla^{\mathbb{R}} = \nabla^{\ell c}$ if and only if (X,ω) is Kähler (i.e. $\mathrm{d}\omega = 0$).*

Proof. If $\nabla^{\mathbb{R}} = \nabla^{\ell c}$, then $\nabla^{\mathbb{R}}$ is torsion free. The fact that ∇ is compatible with h implies $\nabla^{\mathbb{R}}\omega = 0$. Then, by Lemma 4.18, $\mathrm{d}\omega = 0$. Conversely, suppose $\mathrm{d}\omega = 0$. By (4.14), the connection matrix of Θ of ∇ in local coordinates $z = (z_1, \ldots, z_n)$ is given by

$$\Theta = \overline{H}^{-1}\partial\overline{H} = (H^T)^{-1}\partial H^T = (h_{\bar{j}i})^{-1}(\partial h_{\bar{j}i}). \tag{4.21}$$

On the other hand $\omega = \frac{\mathfrak{i}}{2}\sum h_{i\bar{j}}\mathrm{d}z_i \wedge \mathrm{d}\overline{z_j}$; therefore,

$$\mathrm{d}\omega = \partial\omega + \bar{\partial}\omega = \partial\omega + \overline{\partial\omega}, \quad \partial\omega = \frac{\mathfrak{i}}{2}\sum_{i<k,j}\Big(\frac{\partial h_{k\bar{j}}}{\partial z_i} - \frac{\partial h_{i\bar{j}}}{\partial z_k}\Big)\mathrm{d}z_i \wedge \mathrm{d}z_k \wedge \mathrm{d}\overline{z_j}.$$

Therefore, ω is closed iff

$$\frac{\partial h_{k\bar{j}}}{\partial z_i} = \frac{\partial h_{i\bar{j}}}{\partial z_k} \quad \forall\, i,j,k.$$

From these, one can conclude that the righthand side of (4.20) is zero. Just note that the Θ in (4.20) corresponds to $\nabla^{\mathbb{R}}$ and the Θ in (4.21) is for ∇. One first needs to write (4.20) in terms of the Θ in (4.21). □

From this rather long discussion, the reader should only keep in mind that the following theorem.

Theorem 4.21. *Suppose X is a complex manifold and h is a Hermitian metric on $\mathcal{T}X$. Let ∇ denote the Chern connection of h, g denote the Riemannian metric on TX induced by h, $\nabla^{\ell c}$ denote the Levi-Civita connection of g on TX, $J\colon TX \longrightarrow TX$ denote the endomorphism corresponding to multiplication by $\mathfrak{i}$, and $\nabla^{\mathbb{R}}$ denote the Riemannian connection induced by ∇ on TX. Then the following statements are equivalent.*

(1) $\nabla^{\ell c}J = 0$*;*

(2) (X,ω) *is Kähler;*

(3) $\nabla^{\mathbb{R}}$ *is torsion free;*

(4) $\nabla^{\mathbb{R}} = \nabla^{\ell c}$.

By this result, on a Kähler manifolds it is safe to denote all these connections by the same symbol ∇ and do not distinguish them.

The following arguments give a rather direct description of torsion of Kähler condition. Let X be a holomorphic manifold and h be a hermitian metric on $\mathcal{T}X$. Locally, there is an orthonormal coframe $\eta_1, \ldots, \eta_n$ for $\mathcal{T}^*X$ (depending smoothly on the coordinates z) such that

$$h = \sum_{i=1}^{n}\eta_i \otimes \overline{\eta}_i;$$

i.e. we can diagonalize the metric h. Then,

$$\omega = \frac{\mathfrak{i}}{2}\sum_{i=1}^{n}\eta_i \wedge \overline{\eta}_i$$

Lemma 4.22. *There is a unique matrix of* 1*-forms* Ψ *such that* $\Psi^T + \overline{\Psi} = 0$*,*

$$\mathrm{d}\eta_i = \sum \Psi_{ij} \wedge \eta_j + \tau_i, \tag{4.22}$$

and τ_i *are* $(2,0)$*-forms.*

Proof. Recall that while for a smooth function f we have a decomposition $\mathrm{d}f = \partial f + \bar{\partial} f$, for smooth sections of holomorphic vector bundles, such as the holomorphic cotangent bundle $\mathcal{T}^*X$, only $\bar{\partial}$ is well-defined. In order to extend the definition of ∇ to the sections we need a connection. Writing

$$\Psi = \Psi^{1,0} + \Psi^{0,1},$$

the equation (4.22) implies

$$\bar{\partial}\eta_i = \sum \Psi^{0,1}_{ij} \wedge \eta_j.$$

Since $\{\eta_i\}$ is a co-frame, $\Psi^{0,1}$ are uniquely determined. The equation $\Psi^T + \overline{\Psi} = 0$ is equivalent to

$$(\Psi^{1,0})^T = -\overline{\Psi^{0,1}}.$$

So Ψ is uniquely determined. Then, the final conclusion that

$$\mathrm{d}\eta_i = \sum \Psi_{ij} \wedge \eta_j$$

has $(2,0)$-type follows from the fact that the action of exterior derivative d and the Chern connection ∇ on sections of $\mathcal{T}^*X$ only differ at $(2,0)$-level; see HW below. □

HW 4.23. The action of the exterior derivative map d on smooth $(1,0)$-forms decomposes as

$$\mathrm{d}\colon \Lambda^{(1,0)}(X,\mathbb{C}) \longrightarrow \Lambda^{(1,1)}(X,\mathbb{C}) \oplus \Lambda^{(2,0)}(X,\mathbb{C}).$$

Also, the action of the Chern connection ∇ on smooth $(1,0)$-forms decomposes as

$$\nabla\colon \Lambda^{(1,0)}(X,\mathbb{C}) = \Gamma(X,\mathcal{T}^*X) \longrightarrow \Lambda^1(X,\mathcal{T}^*X) \cong \Lambda^{(1,1)}(X,\mathbb{C}) \oplus \Lambda^{(2,0)}(X,\mathbb{C}).$$

Show that $\nabla^{0,1} = \bar{\partial}$ implies that d and ∇ above have the same $(1,1)$-part.

The $(2,0)$-forms $\tau = (\tau_1, \ldots, \tau_n)$ in (4.22) are called the torsion. Up to a change of perspective, the vector torsion form τ is the same as $T_{\nabla^{\mathbb{R}}}$ above. In other words, (X,h) is Kähler if and only if $\tau \equiv 0$.

For instance, the Poincare metric in (4.3) on $\mathcal{H}$ can be written as

$$h = \eta \otimes \overline{\eta}, \quad \eta = \frac{\mathrm{d}z}{\mathrm{Im}(z)}.$$

Then,

$$\partial\eta = -\frac{1}{y^2}\mathrm{d}y \wedge \mathrm{d}z = -\mathrm{d}\ln(y) \wedge \eta.$$

Therefore,

$$\tau \equiv 0, \quad \Psi = -\mathrm{d}\ln(y), \quad \Psi^{1,0} = \frac{-\mathrm{d}y + \mathrm{i}\mathrm{d}x}{2} = \mathrm{i}\mathrm{d}z.$$

HW 4.24. Find the relation between Ψ and the connection matrix $\Theta = \overline{H^{-1}\bar{\partial}H}$ of ∇.

5 Dualities

In this section, we will review and further learn various forms of dualities between different cohomology groups. These dualities often arise from a pairing at a chain or cochain level that descends to cohomology.

Poincare duality. Suppose X is an oriented manifold (without boundary) or real dimension n. Let $\Lambda^k_c(X,\mathbb{R})$ denote the space of differential k-forms with compact support. The exterior map d restricts to a map between the spaces of compactly supported forms and we denote the resulting de Rham cohomology groups by $H^k_{c,\mathrm{dR}}(X,\mathbb{R})$. If X is closed, these are simply $\Lambda^k(X,\mathbb{R})$ and $H^k(X,\mathbb{R})$. By Stokes' Theorem, for each $0 \le k \le n$, the pairing

$$\langle -,-\rangle : \Lambda^k_c(M,\mathbb{R}) \times \Lambda^{n-k}(X,\mathbb{R}) \longrightarrow \mathbb{R}, \qquad (\alpha,\beta) \longrightarrow \langle \alpha,\beta\rangle := \int_X \alpha\wedge\beta. \tag{5.1}$$

descends to a bilinear map

$$\langle -,-\rangle : H^k_{c,\mathrm{dR}}(X,\mathbb{R}) \times H^{n-k}_{\mathrm{dR}}(X,\mathbb{R}) \longrightarrow \mathbb{R}. \tag{5.2}$$

Definition 5.1. An open cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ of an n-manifold X is called a *good cover* if every nonempty finite intersection $U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}$ is diffeomorphic to $\mathbb{R}^n$. A manifold which has a finite good cover is said to be of finite type.

All compact manifolds are finite type but, for instance, an open Riemann surface with infinitely many handles is not finite type.

If X is an oriented smooth n-manifold of finite type, it follows from Mayer-Vietoris long exact sequence and induction that the pairing (5.2) is non-degenerate; therefore,

$$H^k_{c,\mathrm{dR}}(X,\mathbb{R}) \cong H^{n-k}_{\mathrm{dR}}(X,\mathbb{R})^*; \tag{5.3}$$

see my Math6410 notes. If X is closed, (5.3) implies the symmetry

$$b_k = \dim_{\mathbb{R}} H^k_{\mathrm{dR}}(X,\mathbb{R}) = \dim_{\mathbb{R}} H^{n-k}_{\mathrm{dR}}(X,\mathbb{R}) = b_{n-k}. \tag{5.4}$$

For smooth manifolds, singular homology, cellular homology, and simplicial homology are identical; we denote these homology groups by $H_k(X,\mathbb{Z})$, $H_k(X,\mathbb{R})$, or $H_k(X,\mathbb{C})$ depending on what the desired coefficient ring is. Again, let X be an oriented smooth manifold. Fix a triangulation $\mathcal{K}$ of X. For each oriented k-simplex Δ in $\mathcal{K}$ and every k-form η the pairing

$$\langle \eta,\Delta\rangle \longrightarrow \int_\Delta \eta$$

is defined. Let $C_k(\mathcal{K},\mathbb{R})$ denote the vector space of formal linear sums of k-simplices with coefficients in $\mathbb{R}$. The homology group $H_k(X,\mathbb{R})$ is the degree k cohomology group of the chain complex

$$0 \longrightarrow C_n(\mathcal{K},\mathbb{R}) \longrightarrow C_{n-1}(\mathcal{K},\mathbb{R}) \longrightarrow \cdots \longrightarrow C_0(\mathcal{K},\mathbb{R}) \longrightarrow 0$$

where the boundary maps are obtained from signed formal sum of boundary of simplices. The pairing above linearly extends to a pairing

$$\langle -,-\rangle : \Lambda^k(X,\mathbb{R}) \times C_k(\mathcal{K},\mathbb{R}) \longrightarrow \mathbb{R}.$$

Again, by Stoke's theorem, this pairing descends to a pairing

$$\langle -,- \rangle : H^k(M,\mathbb{R}) \times H_k(M,\mathbb{R}) \longrightarrow \mathbb{R}. \tag{5.5}$$

If X has finite type, the pairing (5.5) is non-degenerate; therefore,

$$H^k_{\mathrm{dR}}(M,\mathbb{R}) \cong H_k(M,\mathbb{R})^*. \tag{5.6}$$

The proof of (5.6) also uses inductionsand Mayer-Vietoris long exact sequences.

If X is an oriented smooth n-manifold, combining the previous duality results over $\mathbb{R}$ and (3.4) we have

$$H^{n-k}_{c,\mathrm{dR}}(X,\mathbb{R})^* \cong H^k_{\mathrm{dR}}(X,\mathbb{R}) \cong \check{H}^k(X,\underline{\mathbb{R}}) \cong H^k_{\mathrm{sing}}(X,\mathbb{R}) \cong H_k(X,\mathbb{R})^*$$

If X is closed, the first term above is just $H^{n-k}_{\mathrm{dR}}(X,\mathbb{R})$ and we can add one more term to right to get

$$H^{n-k}_{\mathrm{dR}}(X,\mathbb{R})^* \cong H^k_{\mathrm{dR}}(X,\mathbb{R}) \cong \check{H}^k(X,\underline{\mathbb{R}}) \cong H^k_{\mathrm{sing}}(X,\mathbb{R}) \cong H_k(X,\mathbb{R})^* = H_{n-k}(M,\mathbb{R}).$$

The last isomorphism comes from the intersection pairing of k and $(n-k)$ cycles in X and is the reduced to $\mathbb{R}$ version of a stronger Poincare duality statement over $\mathbb{Z}$ in the following sense.

Theorem 5.2. *(Poincare Duality) Suppose X is a closed oriented n-manifold. There is a well-defined intersection pairing*

$$H_k(X,\mathbb{Z}) \times H_{n-k}(X,\mathbb{Z}) \longrightarrow \mathbb{Z} \tag{5.7}$$

that is unimodular; i.e., and linear functional $H_{n-k}(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$ is equal to taking intersection with some homology class $A \in H_k(X,\mathbb{Z})$, and the kernel of the surjective map

$$H_k(X,\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_{n-k}(X,\mathbb{Z}),\mathbb{Z})$$

is the subgroup $H^{\mathrm{Tor}}_k(X,\mathbb{Z})$ of torsion classes in $H_k(X,\mathbb{Z})$. Furthermore, the torsion subgroups of $H_k(X,\mathbb{Z})$ and $H_{n-k-1}(X,\mathbb{Z})$ are isomorphic.

Example 5.3. Some $K3$ surfaces Y admit a holomorphic involution ($\mathbb{Z}_2$-action) with no fixed point. The quotient $X = Y/\mathbb{Z}_2$ is called an Enriques surface. All Enriques surfaces are projective and smoothly identical. Homology groups of an Enrique surface are

$$\begin{aligned} &H_0(X,\mathbb{Z}) = \mathbb{Z}, \quad H_1(X,\mathbb{Z}) = \pi_1(X,\mathbb{Z}) = \mathbb{Z}_2, \\ &H_2(X,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}^{10}, \quad H_3(X,\mathbb{Z}) = 0, \quad H_4(X,\mathbb{Z}) = \mathbb{Z}; \end{aligned}$$

in particular

$$H^{\mathrm{Tor}}_1(X,\mathbb{Z}) \cong H^{\mathrm{Tor}}_2(X,\mathbb{Z}) \cong \mathbb{Z}_2.$$

If X is a complex manifold of complex dimension n, every analytic subvariety $Y \subset X$ of complex dimension k defines an even-degree homology/cohomology class

$$[Y] \in H_{2k}(X,\mathbb{Z}), \quad \mathrm{PD}(Y) \in H_{2(n-k)}(X,\mathbb{Z}).$$

Note that the even degree homology/cohomology groups $H_{2*}(X,\mathbb{Z})/H^{2*}(X,\mathbb{Z})$ of X form a commutative subring of the entire homology/cohomology ring. For many interesting complex manifolds, such as $\mathbb{CP}^n$ their homology/cohomology ring is entirely supported in even degrees. For example,

$$H_{2k}(\mathbb{CP}^n) \cong H^{2k}(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}, \qquad H_{\text{odd}}(\mathbb{CP}^n) = 0.$$

Furthermore, $H^*(\mathbb{CP}^n)$ is generated (as a ring) by the degree 2 cohomology class

$$h := \text{PD}(H) = c_1(\mathcal{O}(1)) = [\omega_{\text{FS}}/\pi].$$

where $H \cong \mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ is an hyperplane. For $k \geq 0$, $h^k = h \wedge \cdots \wedge k$ (k times) is the PD of $H_1 \cap \cdots \cap H_k \cong \mathbb{CP}^{n-k} \subset \mathbb{CP}^n$.

Definition 5.4. The degree of a complex m-dimensional analytic subvariety $Y \subset \mathbb{CP}^n$ is the positive integer $d > 0$ such that $[Y] = d\,[\mathbb{CP}^m] \in H_{2m}(\mathbb{CP}^n, \mathbb{Z})$, or equivalently $\text{PD}(Y) = dh^{n-m} \in H^{2(n-m)}(\mathbb{CP}^n, \mathbb{Z})$.

For instance, if $Y \subset \mathbb{CP}^n$ is a hypersurface given as the zero of a homogenous polynomial P, then $d = \deg(P)$.

Suppose X is a closed oriented manifold of real dimension m and $Y, Z \subset X$ are transverse oriented submanifolds of complementary dimension. Then, $Y \cap Z$ is a finite set $\{p_1, \ldots p_k\} \subset X$ of points and

$$T_{p_a}X \cong T_{p_a}Y \oplus T_{p_a}Z$$

Each of the three tangent spaces above is oriented via the given orientation on the corresponding manifold. We say p_a is a positive intersection point and write $\varepsilon(p_a) = +1$ iff the isomorphism above is orientation preserving. Then, the homology intersection pairing in (5.7) between $[Y]$ and $[Z]$ is equal to

$$Y \cdot Z := [Y] \cdot [Z] = \sum_{a=1}^{k} \varepsilon(p_a) \in \mathbb{Z}.$$

In the discussion above, if X is a compact complex manifold and $Y, Z \subset X$ are complex submanifolds, then they are all canonically oriented and $\varepsilon(p_a) > 0$. Therefore,

$$Y \cdot Z = k.$$

The definition of the intersection number $Y \cdot Z$ generalizes to singular analytic subvarieties. Note that $Y^{\text{sing}} \subset Y$ and $Z^{\text{sing}} \subset Z$ have complex codimension 1. We say Y and Z intersect transversely if $Y \cap Z \subset Y - Y^{\text{sing}}, Z - Z^{\text{sing}}$ and the intersection $(Y - Y^{\text{sing}}) \cap (Z - Z^{\text{sing}})$ is transverse. Then $Y \cdot Z$ is simply the number of intersection points. Furthermore, this notion is well-behaved under deformations of Y and Z among complex subvarieties because for a generic 1-parameter family $\{Y_t\}_{t \in [0,1]}$, the overall singular locus $\{Y_t^{\text{sing}}\}_{t \in [0,1]}$ has real dimension 1 less than the real dimension of Y and thus would not intersect Z. We can relax the situation even more by allowing Y and Z to intersect not transversely but still discretely in the following sense.

Suppose $p \in Y \cap Z$ is an isolated intersection point of Y and Z. We can fix a polydisk coordinate chart $U = \Delta^n \subset \mathbb{C}^n$ around p with local coordinates $z = (z_1, \ldots, z_k, z_{k+1}, \ldots, z_n)$ such that $p = 0$ and

$$Y \cap U = (z_{k+1} = 0) \cap \cdots \cap (z_n = 0) = \Delta^k \times \{0\}^{n-k}.$$

Since $Y \cap Z \cap U = p$, the restriction of the projection map

$$\pi \colon \Delta^n \longrightarrow \{0\}^k \times \Delta^{n-k}$$

to Z gives Z the structure of a finite branched cover of Δ^{n-k}. The multiplicity m_p of intersection between Y and Z at p is degree of the covering map

$$\pi|_{Z\cap U} \colon Z \cap U \longrightarrow \Delta^{n-k}.$$

As we deform Y and Z, the intersection point p may bifurcate to a collection $\{p_1, \ldots, p_\ell\}$ of intersection points but the weighted sum $m_p = \sum_i m_{p_i}$ remains the same. In conclusion, *the topological intersection number $[Y] \cdot [Z]$ of (the homology classes of) two analytic subvarieties of complementary dimension meeting in a finite set of points in a compact complex manifold X is given by*

$$Y \cdot Z = Y \cdot_X Z = \sum_{p \in Y \cap Z} m_p \geq 0.$$

Each intersection number m_p satisfies $m_p \geq 1$ with equality if and only if p is a transverse intersection point.

HW 5.5. (1) Suppose X is a complex projective variety. Prove that $b_{2k}(X) > 0$ for all $k = 0, 1, \ldots, \dim_{\mathbb{C}} X$.

(2) Prove that any analytic subvariety of $\mathbb{CP}^n$ that is homologous to a hyperplane is a hyperplane.

(3) Prove that any holomorphic automorphism of $\mathbb{CP}^n$ is induced by a linear transformation of $\mathbb{C}^{n+1}$.

(4) Describe the involutions (i.e. automorphisms of order 2) on $\mathbb{CP}^n$.

(5) Show that an involution of $\mathbb{CP}^3$ has fixed points on any complex surface (= complex dimension 2=hypersurface) $X \subset \mathbb{CP}^3$.

If X is a connected compact holomorphic manifold of complex dimension n, the pairing (5.1) breaks into a set of pairings

$$\Lambda^{p,q}(X, \mathbb{C}) \times \Lambda^{n-p,n-q}(X, \mathbb{C}) \longrightarrow \mathbb{C}, \qquad (\alpha, \beta) \longrightarrow \int_X \alpha \wedge \beta. \tag{5.8}$$

HW 5.6. Show that the pairing above descends to a pairing

$$H^{p,q}_{\bar\partial}(X, \mathbb{C}) \times H^{n-p,n-q}_{\bar\partial}(X, \mathbb{C}) \longrightarrow \mathbb{C} \tag{5.9}$$

between the corresponding Dolbeaut cohomology groups.

As we will learn in the next section, the pairing (5.8) and thus (5.9) is non-degenerate; therefore,

$$H^{p,q}_{\bar\partial}(X, \mathbb{C}) \cong H^{n-p,n-q}_{\bar\partial}(X, \mathbb{C})^*.$$

In particular,

$$h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(X, \mathbb{C}) = h^{n-p,n-q}(X) := \dim_{\mathbb{C}} H^{n-p,n-q}_{\bar\partial}(X, \mathbb{C}) \tag{5.10}$$

and

$$H^{n,n}_{\bar\partial}(X, \mathbb{C}) \cong \mathbb{C}.$$

If X is Kähler, $H^{n,n}_{\bar\partial}(X, \mathbb{C})$ is generated by the cohomology class of ω^n.

The Hodge diamond of a complex n-dimensional holomorphic manifold X is the diamond-shaped diagram

$$\begin{array}{ccccccc} & & & h^{0,0}(X,\mathbb{C}) & & & \\ & & h^{1,0}(X,\mathbb{C}) & & h^{0,1}(X,\mathbb{C}) & & \\ & h^{2,0}(X,\mathbb{C}) & & h^{1,1}(X,\mathbb{C}) & & h^{0,2}(X,\mathbb{C}) & \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ & h^{n,n-2}(X,\mathbb{C}) & & h^{n,n}(X,\mathbb{C}) & & h^{n-2,n}(X,\mathbb{C}) & \\ & & h^{n,n-1}(X,\mathbb{C}) & & h^{n-1,n}(X,\mathbb{C}) & & \\ & & & h^{n,n}(X,\mathbb{C}) & & & \end{array}$$

If X is Kähler, we will learn in Section **??** that the sum of the Dolbeault cohomology groups in each row is the complex valued de Rham comology group of the corresponding degree; i.e.,

$$H^k_{\mathrm{dR}}(X,\mathbb{C}) = \sum_{p=0}^{k} H^{p,k-p}_{\bar\partial}(X,\mathbb{C}).$$

Furthermore, by (5.4) and (5.10), the Hodge diamond is symmetric with respect to each of its axes.

6 Divisors and Line bundles

Definition 6.1. A (integral) divisor D on an analytic variety X is a locally-finite formal linear combination

$$D = \sum a_i Y_i$$

of irreducible analytic hypersurfaces $Y_i \subset X$ with the coefficients $a_i \in \mathbb{Z}$.

It is sometimes useful to also consider divisors with coefficients in $\mathbb{R}$ or $\mathbb{Q}$. Note that to every analytic hypersurface Y we can naturally associate a divisor D which is the sum of its irreducible components. However, divisors are more general in the sense that the coefficients can be negative and different components can appear with (different) multiplicities.

Definition 6.2. A divisor is called effective if $a_i \geq 0$ for all i. In this case we write $D \geq 0$.

An effective divisor will intersect all curves (complex 1-dimensional subvarieties) in X non-negatively. An irreducible hypersurafce Y can be locally reducible at certain points $x \in Y$. This happens when a hypersurface has self-intersections. If Y is locally irreducible near $x \in Y$, it is the zero set of an irreducible $g \in \mathcal{O}_{X,x}$. For every other meromorphic function f, the order of vanishing of f along Y is defined in the following way.

Definition 6.3. Let f be a meromorphic function in a neighborhood of $x \in X$. Then, the order of vanishing of f along Y (near x) is an integer $k = \mathrm{ord}_Y(f) \in \mathbb{Z}$ such that $f = g^k h$ and $h|_Y \in \mathcal{O}_{Y,x}$ is a non-trivial holomorphic function.

By Corollary 1.27, the definition of the order above is well-defined; moreover, $\mathrm{ord}_Y(f_1 f_2) = \mathrm{ord}_Y(f_1) + \mathrm{ord}_Y(f_2)$.

Definition 6.4. For every meromorphic function $f\colon X \longrightarrow \mathbb{C}$, the principal divisor of f is

$$(f) := \sum_Y \operatorname{ord}_Y(f) \cdot Y$$

where the sum is taken over all the irreducible components of $f^{-1}(0)$ and $f^{-1}(\infty)$. By separating the contributions of $f^{-1}(0)$ and $f^{-1}(\infty)$ we have

$$(f) = Z(f) - P(f)$$

where $Z(f) := \sum_{Y \subset f^{-1}(0)} \operatorname{ord}_Y(f) \cdot Y$ and $P(f) := \sum_{Y \subset f^{-1}(\infty)} -\operatorname{ord}_Y(f) \cdot Y$ are the effective zero and polar divisors of f.

Remark 6.5. If $k < 0$, then f has a pole along Y; i.e. poles should be thought of as zeros of negative order. Zeros and Poles are interchangeable under $f \longrightarrow 1/f$. In Definition 2.3.5 of Hybrecht's book, the last condition on h is stated as $h \in \mathcal{O}^*_{X,x}$. In dimensions 2 and higher, as explained in Remark 1.40, h may still be zero at x; so Hybrecht's definition is not entirely correct. In complex dimension 1, Y is a point and the last condition reads $h(Y) \neq 0$, which in turn implies $h \in \mathcal{O}_{X,x}$.

Example 6.6. In complex dimension one, i.e. when X is a Riemann surface, a divisor is simply a finite formal linear combination points $D = \sum a_i p_i$. In complex dimension 2, hypersurfaces (codimension 1) are the same as curves (dimension 1). Therefore, if X is a compact complex surface, every two divisors $D = \sum a_i Y_i$ and $D' = \sum a_i' Y_i'$ have a well-defined intersection number

$$D \cdot D' = \sum_{i,j} a_i a_j' \; Y_i \cdot Y_j'.$$

Whenever $Y \neq Y'$, the intersection number $Y \cdot Y'$ is non-negative, however, as we will see in the example of blowup below, the self-intersection number $Y \cdot Y$ can be negative. The unimodular quadratic form $H_2(X,\mathbb{Z}) \times H_2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$ plays a great role in the classification of compact complex surfaces and other smooth 4-manifolds.

HW 6.7. Find the unimodular quadratic intersection form of $X = \mathbb{CP}^1 \times \mathbb{CP}^1$.

Remark 6.8. In complex algebraic geometry, there are two different notions of divisor that are the same if the ambient space X is smooth but might be different if X is a singular variety. A Weil divisor is a formal linear combination of hypersurfaces. In a singular variety, a hypersurface, i.e. complex codimension 1 subvariety may not (locally) be the zero set of a holomorphic function. A divisor is called Cartier if every hypersurface involved in the divisor is indeed defined (locally) as the zero set of a holomorphic function. In this course, we will mostly be working with smooth target manifolds X, so the two notions are the same and we simply call them divisors.

Suppose $\mathcal{L} \longrightarrow X$ is a holomorphic line bundle and $s\colon X \longrightarrow \mathcal{L}$ is a meromorphic section. Locally around every point on X, there exists an open neighborhood U such that $\mathcal{L}|_U \cong U \times \mathbb{C}$ and $s|_U$ is defined by a meromorphic function. Moreover, different local trivializations $\mathcal{L}|_U$ and $\mathcal{L}|_{U'}$ differ by a non-zero holomorphic functions $\varphi\colon U \cap U' \longrightarrow \mathbb{C}^*$. Therefore, just as in Definition 6.4, a section s define a divisor

$$\operatorname{Div}(s) := \sum_Y \operatorname{ord}_Y(s) \cdot Y$$

where on each open set U as above $\operatorname{ord}_Y(s)$ is the order of the meromorphic function $s|_U$. Definition 6.4 is a special case where $\mathcal{L}$ is the trivial holomorphic line bundle $\mathcal{L} = X \times \mathbb{C}$. In general, if s and s' are two meromorphic sections of a holomorphic line bundle $\mathcal{L}$, the $s' = fs$ for some meromorphic function f on X, and therefore

$$\mathrm{Div}(s') - \mathrm{Div}(s) = (f). \tag{6.1}$$

If the difference of two divisors D and D' is a principal divisor, we say D and D' are linearly equivalent and write $D \sim D'$. The equations above show that $\mathrm{Div}(s')$ and $\mathrm{Div}(s)$ are linearly equivalent.

In conclusion, there is a well-defined map

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Div}(X)/\mathrm{Div}_0(X)$$

where $\mathrm{Div}(X)$ is the group of divisors on X and $\mathrm{Div}_0(X)$ is the subgroup of principal divisors. The following lemma shows that the map above is an isomorphism coming from a short exact sequence

$$0 \longrightarrow \mathrm{Div}_0(X) \longrightarrow \mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

For the surjectivity of $\mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X)$, we are assuming that every holomorphic line bundle on X has a non-trivial meromorphic section; otherwise, the sequence is not exact at right. We will prove that the assumption is true for all complex projective varieties.

Lemma 6.9. *Corresponding to every divisor $D \in \mathrm{Div}(X)$, there exists a naturally defined holomorphic line bundle $\mathcal{O}_X(D)$ and a meromorphic section s, well-defined up to scaling, such that* $\mathrm{Div}(s) = D$. *For $D, D' \in \mathrm{Div}(X)$, $\mathcal{O}(D + D') = \mathcal{O}(D) \otimes \mathcal{O}(D')$; i.e the map* $\mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X)$ *above is a group homomorphism.*

Proof. Let $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ be an atlas for X such that $D \cap U_\alpha$ is the principal divisor of some local meromorphic function $f_\alpha \colon U_\alpha \longrightarrow \mathbb{C}$. For each $\alpha, \beta \in \mathcal{I}$, let

$$\varphi_{\alpha,\beta} = f_\beta / f_\alpha \colon U_{\alpha\beta} \longrightarrow \mathbb{C}^*.$$

By definition,

$$\varphi_{\alpha,\beta} \cdot \varphi_{\beta,\gamma} \cdot \varphi_{\gamma,\alpha} = 1 \quad \forall \alpha, \beta, \gamma \in \mathcal{I}.$$

Therefore, the transition maps $\{\varphi_{\alpha\beta}\}_{\alpha,\beta \in \mathcal{I}}$ define a holomorphic line bundle $\mathcal{O}_X(D)$. Since $\varphi_{\alpha\beta} f_\alpha = f_\beta$, the meromorphic functions $\{f_\alpha\}_{\alpha \in \mathcal{I}}$ define a section of $\mathcal{O}_X(D)$. The last claim obviously follows from the construction of transition functions. □

Remark 6.10. We have $c_1(\mathcal{O}_X(D)) = \mathrm{PD}(D) \in H^2(X, \mathbb{Z})$.

HW 6.11. Suppose $\mathcal{L} \longrightarrow X$ is a holomorphic line bundle. Show that $\mathcal{L}$ has a holomorphic section if and only if $\mathcal{L} = \mathcal{O}_X(D)$ for some effective divisors D.

Definition 6.12. If D is an effective divisor, the complete linear system $|D|$ is the set of all effective divisors D' that are linearly equivalent to D.

If $\mathcal{L} = \mathcal{O}_X(D)$ for some effective divisor D, then we will also write $|\mathcal{L}|$ for $|D|$.

HW 6.13. If D is effective, show that $|D| = \mathbb{P}(\check{H}^0(X, \mathcal{O}_X(D)))$; i.e., $|D|$ is the projectivization of the vector space of non-trivial holomorphic sections of the line bundle $\mathcal{O}_X(D)$. Therefore, $\dim_{\mathbb{C}} |D| = \dim_{\mathbb{C}} \check{H}^0(X, \mathcal{O}_X(D)) - 1$. For a hyperplane divisor $H \subset \mathbb{CP}^n$, what is $|H|$?

Definition 6.14. Any linear subspace of $|D|$ is called a linear system. A linear system of complex dimension 1 (i.e. isomorphic to $\mathbb{P}^1$) is called a pencil.

HW 6.15. Show every pencil is of the form

$$\{D_\lambda = \mathrm{Div}(\lambda_0 s_0 + \lambda_1 s_1)\}_{\lambda=[\lambda_0,\lambda_1]\in\mathbb{P}^1}$$

where s_0, s_1 are two different holomorphic sections of a holomorphic line bundle $\mathcal{L}$.

For any linear system $\{D_\lambda\}_{\lambda\in\mathbb{P}^r} \subset |D|$, if $\lambda_0, \cdots, \lambda_r \in \mathbb{P}^r$ are $r+1$ linearly independent points, then

$$D_{\lambda_0} \cap \cdots \cap D_{\lambda_r} = \bigcap_{\lambda\in\mathbb{P}^r} D_\lambda.$$

The intersection above is called the base-locus of the linear system L. It is the largest analytic subvariety of X such that every holomorphic section corresponding to L vanishes on that.

Theorem 6.16. *(Bertini's Theorem) If D is an effective divisor, a generic element of any linear system $\{D_\lambda\}_{\lambda\in\mathbb{P}^r} \subset |D|$ is a hypersurface that is smooth away from the base-locus of the system.*

In particular, it is easy to see that any pencil $\{D_\lambda\}_{\lambda\in\mathbb{P}^1} \subset |D|$ with base locus $B \subset X$ gives a holomorphic map

$$X - B \longrightarrow \mathbb{P}^1. \tag{6.2}$$

Then, Bertini's Theorem is a refinement of Sard's theorem for this map.

HW 6.17. Consider the pencil $\{D_\lambda\}_{\lambda\in\mathbb{P}^1} \subset |\mathcal{O}_{\mathbb{CP}^2}(1)|$ generated by the holomorphic sections x_0 and x_1; c.f. HW 2.21. What is the base locus B of this pencil? What is the map

$$\mathbb{CP}^2 - B \longrightarrow \mathbb{CP}^1?$$

Repeat this for the pencil $\{D_\lambda\}_{\lambda\in\mathbb{CP}^1} \subset |\mathcal{O}_{\mathbb{CP}^2}(3)|$ generated by the holomorphic sections $x_0^3 + x_1^3 + x_2^3$ and $x_0x_1x_2$. For which $\lambda \in \mathbb{CP}^1$, D_λ is a smooth cubic curve?

Proof. It is enough to assume that $r = 1$ because if $r = 0$, then the base locus B is the entire single divisor in the linear system and if $r > 1$, then if a generic element of the linear system is singular away from B, then the same holds for a generic pencil contained in the system (why?). Suppose $\{D_\lambda\}_{\lambda\in\mathbb{P}^1} \subset |D|$ is pencil. By HW 6.15, locally around every point of X we have a

$$D_\lambda = \big(f_0(z_1,\ldots,z_n) + \lambda f_1(z_1,\ldots,z_n) = 0\big), \qquad \lambda \in \mathbb{P}^1,$$

where f_0, f_1 are two different (i.e. f_1 is not a constant multiple of f_0) holomorphic functions on the chosen neighborhood U. For generic $\lambda \neq 0, \infty$, if p_λ is a singular point of $D_\lambda \cap U$ in $X - B$, then

$$\begin{aligned} &f_0(p_\lambda) + \lambda f_1(p_\lambda) = 0, \\ &\frac{\partial f_0}{\partial z_i}(p_\lambda) + \lambda \frac{\partial f_1}{\partial z_i}(p_\lambda) = 0 \quad \forall\, i = 1,\ldots,n, \end{aligned}$$

but $f_0(p_\lambda), f_1(p_\lambda) \neq 0$. Substituting λ in the second equation by $\lambda = -f_0(p_\lambda)/f_1(p_\lambda)$ we obtain

$$\frac{\partial f_0}{\partial z_i}(p_\lambda) - \frac{f_0(p_\lambda)}{f_1(p_\lambda)}\frac{\partial f_1}{\partial z_i}(p_\lambda) = 0$$

which implies

$$\frac{\partial}{\partial z_i}\Big(\frac{f_0}{f_1}\Big)(p_\lambda) = 0 \qquad \forall\, \lambda \neq 0,\ i = 1,\ldots,n. \tag{6.3}$$

The union over λ of the singular points above in $U\times\mathbb{C}^*$ is itself a subset of the analytic subvariety

$$S = \{(z,\lambda) \in U \times \mathbb{C}^* \colon\ f_0(z) + \lambda f_1(z) = 0,\ \frac{\partial f_0}{\partial z_i}(z) + \lambda \frac{\partial f_1}{\partial z_i}(z) = 0\}$$

cut out by $n+1$ equations. Let $\overline{S}$ denote the image/projection of S in U; $\overline{S}$ is an analytic subvariety of U. By (6.3), the function f_0/f_1 is constant on each connected component of $\overline{S}-B$. Note that each D_λ is a level set of f_0/f_1 outside B. Therefore, since the number of connected components of $\overline{S}-B$ is finite, $\overline{S}-B$ can intersect only finitely many D_λ. □

Remark 6.18. Suppose D is an effective divisor on X. Note that the map

$$\iota_{\mathcal{O}_X(D)} \colon X \longrightarrow \mathbb{P}^N$$

in (2.4) is defined if and only if the complete linear system $|D|$ is base-point free. Otherwise, only

$$\iota_{\mathcal{O}_X(D)} \colon X - B \longrightarrow \mathbb{P}^N$$

is defined.

The next lemma that relates the normal bundle of a smooth hypersurface/divisor $D \subset X$ to the line bundle $\mathcal{O}_X(D)$ is very useful in many applications.

Lemma 6.19. *Suppose $D \subset X$ is a smooth hypersurafce. Then*

$$\mathcal{O}_X(D)|_D = \mathcal{N}_X D = \frac{\mathcal{T}X|_D}{\mathcal{T}D}.$$

Proof. The equality above is the same as

$$\mathcal{O}_X(D)|_D \otimes \mathcal{N}_X D^* \cong \mathcal{O}_D := D \times \mathbb{C}.$$

The co-normal bundle $\mathcal{N}_X D^*$ is a sub-bundle of $\mathcal{T}^*X|_D$ consisting of cotangent vectors on X along D that are zero on $\mathcal{T}D$. Suppose D is locally defined by $(f_\alpha = 0) \subset U_\alpha$. Then, the line bundle $\mathcal{O}_X(D)$ is given by transition functions $\varphi_{\alpha\beta} = f_\beta/f_\alpha$; thus, its dual $\mathcal{O}_X(-D)$ is given by transition functions $\varphi_{\alpha\beta}^{-1} = f_\alpha/f_\beta$. Since $f_\alpha|_{D\cap U_\alpha} \equiv 0$, the differential $\mathrm{d}f_\alpha$ is a section of the co-normal bundle $\mathcal{N}_X D$. Since D is smooth, $\mathrm{d}f_\alpha$ is non-zero everywhere. Finally, on $U_\alpha \cap U_\beta$,

$$\mathrm{d}f_\beta|_D = \mathrm{d}(\varphi_{\alpha\beta} f_\alpha)|_D = \big(\mathrm{d}\varphi_{\alpha,\beta}\cdot f_\alpha + \varphi_{\alpha,\beta}\mathrm{d}f_\alpha\big)|_D = \big(\varphi_{\alpha\beta}\mathrm{d}f_\alpha\big)|_D$$

Therefore, the local sections $\mathrm{d}f_\alpha$ define a nowhere-zero global section of $\mathcal{N}_X D^* \otimes \mathcal{O}_X(D)|_D$, which implies $\mathcal{N}_X D^* \otimes \mathcal{O}_X(D)|_D \cong \mathcal{O}_D$. □

Example 6.20. Let $\Sigma \subset \mathbb{CP}^2$ be a Riemann surface of degree d. Along Σ, $T\mathbb{CP}^2$ smoothly decomposes as a direct sum of complex line bundles

$$T\mathbb{CP}^2|_\Sigma = \mathcal{T}\Sigma \oplus \mathcal{N}_{\mathbb{CP}^2}\Sigma. \tag{6.4}$$

By the lemma above,

$$T\mathbb{CP}^2|_\Sigma = \mathcal{T}\Sigma \oplus \mathcal{O}_{\mathbb{CP}^2}(\Sigma)|_\Sigma.$$

We have

$$c_1(T\mathbb{P}^2) = 3h, \qquad h := \mathrm{PD}(H) \in H^2(\mathbb{CP}^2,\mathbb{Z}) \cong \mathbb{Z},$$

and

$$c_1(\mathcal{O}_{\mathbb{CP}^2}(\Sigma)) = c_1(\mathcal{O}_{\mathbb{CP}^2}(3H)) = 3h.$$

Therefore,

$$\deg(T\mathbb{CP}^2|_\Sigma) = \int_\Sigma c_1(T\mathbb{CP}^2) = c_1(T\mathbb{CP}^2) \cdot dh = 3h \cdot dh = 3d =$$
$$\deg(\mathcal{T}\Sigma) + \deg(\mathcal{O}_{\mathbb{CP}^2}(\Sigma)|_\Sigma) = \deg(\mathcal{T}\Sigma) + dh \cdot dh = \deg(\mathcal{T}\Sigma) + d^2.$$

Therefore,

$$\deg(\mathcal{T}\Sigma) = d(3-d).$$

Since $\deg(\mathcal{T}\Sigma) = \chi(\Sigma) = 2 - 2g$, we get

$$g = \frac{d(d-3)-2}{2} = \frac{(d-1)(d-2)}{2}.$$

For every complex manifold X, its canonical bundle K_X is the line bundle

$$K_X = \Lambda^{\mathrm{top}}\mathcal{T}^*X;$$

i.e. K_X is the top exterior power of the complex cotangent bundle locally generated by

$$\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n.$$

Note that for a Riemann surface Σ we simply have $K_\Sigma = \mathcal{T}^*\Sigma$ with $\deg(K_\Sigma) = 2g-2$. If $D \subset X$ is a smooth analytic hypersurface, generalizing (6.4), we have

$$TX|_D \cong \mathcal{T}D \oplus \mathcal{N}_X D.$$

Therefore,

$$K_X|_D \cong K_D \otimes \mathcal{N}_X D^* = K_D \otimes \mathcal{O}_X(-D)|_D.$$

Moving the last term to the left we get

$$K_D = (K_X \otimes \mathcal{O}_X(D))|_D. \tag{6.5}$$

The canonical bundle is the complex version of the orientation line bundle in the smooth category and plays an important role in the classification of complex manifolds. It will also appear in the Serre duality between Dolbeaut cohomology groups of holomorphic vector bundles.

In (6.2), if the Pencil $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ is generated by two holomorphic sections s_0 and s_1 of the holomorphic line bundle $\mathcal{L} = \mathcal{O}_X(D_\lambda)$, and the fibration (6.2) is simply the meromorphic function $f = s_0/s_1 \colon X - B \longrightarrow \mathbb{P}^1$. Along B, we have a 0/0 situation and the function does not extend continuously/homomorphically. Suppose, for simplicity, that B is a smooth subvariety of X, $D_0 = s_0^{-1}(0)$ and $D\infty = s_1^{-1}(0)$ are smooth along B, and they intersect transversely along B. These conditions show that there are local coordinates $(z_1, \ldots, z_n)$ along every point of B such that $D_0 = (z_1 = 0)$ and $D_\infty = (z_2 = 0)$. Then, the process described below, called blowup, allows us to change X along B to a larger complex manifold $\widetilde{X}$ such that projection $f \colon X - B \longrightarrow \mathbb{P}^1$ extends to a holomorphic map $\widetilde{f} \colon \widetilde{X} \longrightarrow \mathbb{P}^1$ whose fibers are D_λ. Moreover, there is a holomorphic map $\pi \colon \widetilde{X} \longrightarrow X$ that is the identity map over $X - B$ and $\pi \colon E = \pi^{-1}(B) \longrightarrow B$ is a $\mathbb{P}^1$-bundle.

We first describe the blowup at the origin of $\mathbb{C}^n$. Let $(z_1, \dots, z_n)$ denote the affine coordinates on $\mathbb{C}^n$ and $[x_1, \dots, x_n]$ denote the projective coordinates on $\mathbb{CP}^{n-1} = \mathbb{P}(\mathbb{C}^n)$. Then the blowup of $\mathbb{C}^n$ at the origin is the complex manifold (we will see why it is indeed a manifold)

$$\mathbb{C}^n \times \mathbb{CP}^{n-1} \supset B_0\mathbb{C}^n = \{(z,[x]) \colon z_i x_j = z_j x_i \qquad \forall\, i,j = 1,\dots,n\} \tag{6.6}$$

The equation above is homogenous in x_i and thus is well-defined. Abstractly speaking, if W is a complex vector space, then $\mathbb{P}(W)$ is the space of lines ℓ in W and

$$W \times \mathbb{P}(W) \supset B_0 W = \{(v,\ell) \colon v \in \ell\}.$$

Let

$$\pi \colon B_0W \longrightarrow W \qquad \text{and} \qquad \pi' \colon B_0W \longrightarrow \mathbb{P}(W)$$

denote the restrictions to B_0W of the projection maps to first and second factors, respectively. Then, it is clear from the definition that $\pi' \colon B_0W \longrightarrow \mathbb{P}(W)$ identifies B_0W with the total space of the tautological line bundle γ. For every $v \in W - \{0\}$, the line ℓ containing v is unique. Therefore, $\pi \colon B_0W - \pi^{-1}(0) \longrightarrow W - \{0\}$ is a biholomorphism. Furthermore, $E = \pi^{-1}(0) = \mathbb{P}(W)$ is a hypersurface in B_0W that is the space of all directions in W at 0. The hypersurface E is called the exceptional divisor.

Since blowup does not change the geometry away from the blowup locus, if X is a complex manifold and $x \in X$, then the blowup $\widetilde{X}$ of X at x can be defined as above by working in a coordinate chart around x.

The blown up local model $B_0\mathbb{C}^n$ in (6.6) can itself be covered by n charts $\widetilde{U}_i$ which are the pre-images in $\gamma = B_0\mathbb{C}^n$ of the standard n charts covering $\mathbb{CP}^{n-1}$. More precisely, let

$$B_0\mathbb{C}^n \supset \widetilde{U}_i = \{(z,[x]) \colon x_i \neq 0\} \qquad \forall\, i = 1,\dots,n, \tag{6.7}$$

and

$$u_{i,j} = x_j/x_i \qquad \forall\, j = 1,\dots,n.$$

Then, $(z_i, (u_{i,j})_{j\neq i})$ are local coordinates on $\widetilde{U}_i$ with

$$(z_1,\dots,z_n) = \pi(z_i,(u_{i,j})_{j\neq i}) = (u_{i,j}z_i)_{j=1}^n.$$

In the coordinate system above we have $E \cap \widetilde{U}_i = (z_i = 0)$.

HW 6.21. If X is an n-dimensional complex manifold and $\widetilde{X}$ is the blowup of X at x with the exceptional divisor $E \cong \mathbb{P}^{n-1}$, prove that

$$K_{\widetilde{X}} \cong \pi^* K_X \otimes \mathcal{O}_{\widetilde{X}}\big((n-1)E\big). \tag{6.8}$$

Generalizing the construction above, suppose X is a complex manifold and $Y \subset X$ is a complex manifold of complex codimension r. Then the blowup $\widetilde{X} = B_Y X$ of X along Y can also be defined and has the following properties. There is a holomorphic map $\pi \colon \widetilde{X} \longrightarrow X$ that is the identity map over $X - Y$ and $\pi \colon E = \pi^{-1}(Y) \longrightarrow Y$ is the $\mathbb{P}^{r-1}$-bundle $E = \mathbb{P}(\mathcal{N}_X Y)$. Around every point of Y, there exists an open neighborhood U with local coordinates $(z_1,\dots,z_n)$ such that

$$Y \cap U = \bigcap_{i=1}^{r} (z_i = 0).$$

Then $\pi^{-1}(U) \subset \widetilde{X}$ is the blowup manifold with respect to the first r coordinates

$$U \times \mathbb{CP}^{r-1} \supset \widetilde{U} = \{(z,[x])\colon z_i x_j = z_j x_i \qquad \forall\, i,j = 1,\ldots,r\}.$$

By the homework below, the local description of $\widetilde{X}$ is independent of the choice of the local coordinates used in the construction.

HW 6.22. Show that if U and U' are two overlapping open sets with coordinates $z = (z_1,\ldots,z_n)$ and $z' = (z'_1,\ldots,z'_n)$ as above, the change of coordinate biholomorphism $z \longrightarrow z'$ on $U \cap U'$ naturally lifts to a biholomorphic identification between $\widetilde{U}$ and $\widetilde{U'}$ along $\widetilde{U} \cap \widetilde{U'} = \pi^{-1}(U \cap U')$.

As in the case of one-point blowup, the normal line bundle $\mathcal{N}_{\widetilde{X}}E$ of E in $\widetilde{X} = B_Y X$ is the tautological line bundle of $E = \mathbb{P}(\mathcal{N}_X Y)$.

Remark 6.23. The blowup process is only non-trivial if $\dim_{\mathbb{C}} X > 1$ and $\mathrm{codim}_{\mathbb{C}} Y > 1$. In particular, if X is a complex surface, we will only have blowup at points of X.

Blow up is very useful for de-singularizing singular analytic sub-varities of a complex manifold X. In fact, by a celebrated theorem of Hironaka (1964), given a singular sub-variety $Y \subset X$, there is a sequence of blowups of X such that the "proper-transform" of Y to the final blown up space is smooth. We explain this statement through a basic example.

The irreducible complex curve

$$Y = (z_1^2 - z_2^3 = 0) \subset X = \mathbb{C}^2$$

has a cusp singularity at $0 = (0,0)$. Let

$$\widetilde{X} = B_0 X.$$

The proper transform of Y to $\widetilde{X}$ is the analytic sub-variety

$$\widetilde{Y} = \overline{\pi^{-1}(Y - \{0\})}. \tag{6.9}$$

In other words, $\widetilde{Y}$ is the closure of the pre-image of $Y - \{0\}$ in $\widetilde{X}$. Note that, $\widetilde{X} - E$ is biholomorphic to $X - \{0\}$; therefore, the pre-image of $Y - \{0\}$ in $\widetilde{X}$ is biholomorphic to $Y - \{0\}$. However, as we show below, taking closure of $Y - \{0\}$ in $\widetilde{X}$ yields a different result than Y. We describe $\widetilde{Y}$ using the local charts $\widetilde{U}_i$ in (6.7).

Here $\widetilde{X}$ is covered by $\widetilde{U}_1$ and $\widetilde{U}_2$. The local coordinates on the former are $(z_1, u_{1,2})$, the local coordinates on the latter are $(z_2, u_{2,1})$, and the change of coordinate map on the overlap is

$$(z_1, u_{1,2}) \longrightarrow (z_2 = z_1 \cdot u_{1,2}, u_{2,1} = 1/u_{1,2}).$$

The projection map to $X = \mathbb{C}^2$ on each chart is given by

$$(z_1, u_{1,2}) \longrightarrow (z_1, z_2) = (z_1, z_1 u_{1,2}) \quad \text{and} \quad (z_2, u_{2,1}) \longrightarrow (z_1, z_2) = (z_2 u_{2,1}, z_1).$$

Therefore, in terms of the coordinates on $\widetilde{U}_1$, the equation of Y can be written as

$$z_1^2 - z_2^3 = z_1^2 - z_1^3 u_{1,2}^3 = z_1^2(1 - z_1 u_{1,2}^3) = 0.$$

However, the zero set of z_1^2 is $E \cap \widetilde{U_1}$ and does not contribute to (6.9) because $(E \cap \widetilde{U_1}) \subset \pi^{-1}(0)$. Therefore,

$$\widetilde{Y} \cap \widetilde{U}_1 = (1 - z_1 u_{1,2}^3 = 0).$$

HW 6.24. Confirm that the hypersurface cutout by the equation above is smooth.

Note that $\widetilde{Y} \cap \widetilde{U}_1 \cap E = \emptyset$; i.e. $\widetilde{Y}$ has no intersection point with E inside $\widetilde{U}_1$.

Similarly, in terms of the coordinates on $\widetilde{U}_2$, the equation of Y can be written as

$$z_1^2 - z_2^3 = z_2^2 u_{2,1}^2 - z_2^3 = z_2^2(u_{2,1}^2 - z_2) = 0.$$

As before, the zero set of z_2^2 is $E \cap \widetilde{U_2}$ and does not contribute to (6.9). Therefore,

$$\widetilde{Y} \cap \widetilde{U}_2 = (z_2 - u_{2,1}^2 = 0),$$

which is a smooth complex parabola. Moreover,

$$\widetilde{Y} \cap \widetilde{U}_2 \cap E = (0,0) = [1,0] \in E \cong \mathbb{P}^1[x_0, x_1]$$

and the contact order of E and Y at this point is 2; i.e. they are tangent to each other. We conclude that $\widetilde{Y}$ is a smooth complex curve in $\widetilde{X}$ and $\widetilde{Y} \cdot E = 2$. Figure 1 illustrates this de-singularization.

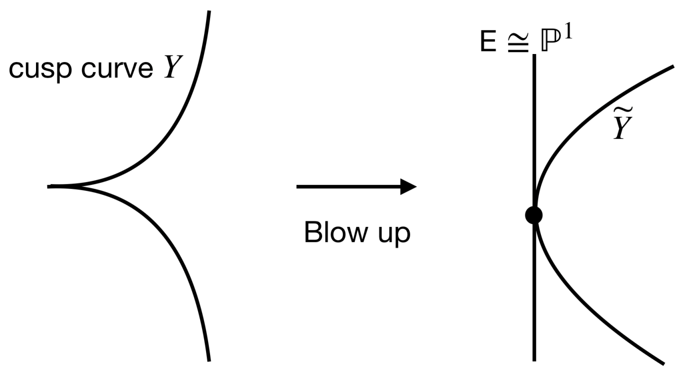

Figure 1: Resolving a cusp singularity

HW 6.25. What would happen to $\widetilde{Y}$, E, and their intersection, if further blowup $\widetilde{X}$ at the intersection point of $\widetilde{Y}$ and E?

Blowing up X at a point changes the topology of X in the following way. The Poincare dual of the exceptional divisor E and its higher exterior powers give us new generators for the even dimensional cohomology groups of $\widetilde{X}$, i.e.

$$H^{2k}(\widetilde{X}, \mathbb{Z}) = \pi^* H^{2k}(X, \mathbb{Z}) \oplus \mathbb{Z} \cdot \mathrm{PD}(E)^k \qquad \forall\ k = 1, 2, \ldots, \dim_{\mathbb{C}} X - 1.$$

For $k = 2n$, assuming X is connected, we will still have $H^{2n}(\widetilde{X}, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{PD}(E)^n = -1$. The last identity follows from

$$\mathcal{N}_{\widetilde{X}} E = \gamma = \mathcal{O}_{\mathbb{P}^{n-1}}(-1).$$

In particular, if X is a complex surface, then the self-intersection number of E is $E \cdot E = -1$. The last identity implies that E does not admit any holomorphic deformation, i.e. the only holomorphic curve in $\widetilde{X}$ homologous to E is E itself. If X is simply connected, $\widetilde{X}$ will be simply connected as well.

HW 6.26. How does blowup along a higher dimensional sub-variety $Y \subset X$ affects the cohomology groups?

Proposition 6.27. *If X is Kähler/projective any blowup $\widetilde{X}$ of X will be Kähler/projective as well.*

Proof. Here is the proof for blowup at a point $x \in X$. The general case is done similarly. We show that there is a Hermitian metric on $\mathcal{O}_{\widetilde{X}}(-E)$ whose curvature F is supported in a sufficiently small neighborhood of E, $\mathfrak{i}F$ is non-negative, and strictly positive on TE. Moreover, if ω is a Kähler form on X, for k sufficiently large, $\widetilde{\omega} = \mathfrak{i}F + k\pi^*\omega$ is a Kähler form on $\widetilde{X}$. Therefore, if X is Kähler then $\widetilde{X}$ is Kähler as well. If X is projective, then there is a positive line bundle $\mathcal{L}$ on X such that ω is the curvature of $\mathcal{L}$. Therefore, $\widetilde{\omega}$ is the curvature of $\mathcal{O}_{\widetilde{X}}(-E) \otimes \pi^*\mathcal{L}$ which shows $\widetilde{X}$ admits a positive line bundle. By Kodaira Embedding Theorem, $\widetilde{X}$ is projective.

We construct a metric h on $\mathcal{O}_{\widetilde{X}}(E)$ as follows. A neighborhood of $x \in X$ can be identified with a ball of radius ε around the origin $0 \in U_\varepsilon \subset \mathbb{C}^n$. The blowup $\widetilde{U}_\varepsilon$ of U_ε at 0 is the manifold

$$\widetilde{U}_\varepsilon = \{(\ell, z) \in \mathbb{CP}^{n-1} \times \mathbb{C}^n \colon z \in \ell, |z| < \varepsilon\}.$$

Using the projection map $\pi_1 \colon \widetilde{U}_\varepsilon \longrightarrow \mathbb{CP}^{n-1}$, we can think of $\widetilde{U}_\varepsilon$ as a disk bundle inside the tautological line bundle $\gamma \longrightarrow \mathbb{CP}^{n-1}$. Furthermore,

$$\mathcal{O}_{\widetilde{X}}(E)|_{\widetilde{U}_\varepsilon} = \pi_1^*\gamma$$

and

$$(\ell, z) \longrightarrow z$$

is a local section of $\mathcal{O}_{\widetilde{X}}(E)|_{\widetilde{U}_\varepsilon}$ that corresponds to the constant local section 1 of

$$\mathcal{O}_{\widetilde{X}}(E)|_{\widetilde{X}-E} \cong (\widetilde{X} - E) \times \mathbb{C}.$$

In other words, the local section z of $\mathcal{O}_{\widetilde{X}}(E)|_{\widetilde{U}_\varepsilon}$ and the local constant section 1 of $\mathcal{O}_{\widetilde{X}}(E)|_{\widetilde{X}-E}$ paste together to define a global section s of $\mathcal{O}_{\widetilde{X}}(E)$ with $s^{-1}(0) = E$.

The standard metric on γ, obtained from the embedding $\gamma \subset \mathbb{CP}^{n-1} \times \mathbb{C}^n$, induces a metric h_1 on $\mathcal{O}_{\widetilde{X}}(E)|_{\widetilde{U}_\varepsilon} = \pi_1^*\gamma$ for which

$$\|s(z)|_{\widetilde{U}_\varepsilon}\|^2 = |z|^2.$$

On the other hand, we have the standard product metric h_2 on $\mathcal{O}_{\widetilde{X}}(E)|_{\widetilde{X}-E} \cong (\widetilde{X} - E) \times \mathbb{C}$ for which

$$\|s(z)|_{\widetilde{X}-E}\|^2 = 1.$$

Since h_1 and h_2 are different on the overlap, we consider a partition of unity $\{\rho_1, \rho_2\}$ for the open cover $\{\widetilde{U}_\varepsilon, \widetilde{X} - \widetilde{U}_{\varepsilon/2}\}$ and define

$$h = \rho_1 h_1 + \rho_2 h_2.$$

Using the metric h above, by (4.16), we have

$$F = \bar{\partial}\partial \ln(h), \qquad F|_{\widetilde{X}-\widetilde{U}_\varepsilon} \equiv 0.$$

Therefore, F is supported in $\widetilde{U}_\varepsilon$. Moreover, restricted to $\widetilde{U}_{\varepsilon/2}$, we have

$$\mathfrak{i}F|_{\widetilde{U}_{\varepsilon/2}} = -2\pi_1^*\omega_{\mathrm{FS}}$$

where ω_{FS} is the Fubini study Kähler form on $E \cong \mathbb{CP}^{n-1}$; see Lemma 4.13. We conclude that $-\mathfrak{i}F$ is a non-negative $(1,1)$-form on $\widetilde{U}_{\varepsilon/2}$ and its restriction to TE is strictly positive.

Summing up, if we define $\omega_{-E} = -\frac{\mathfrak{i}}{2}F$, where $-F$ is the curvature of the dual metric on $\mathcal{O}_{\widetilde{X}}(-E)$, we have

$$\omega_{-E} = \begin{cases} 0 & \text{on } \widetilde{X} - \widetilde{U}_\varepsilon \\ \geq 0 & \text{on } \widetilde{U}_{\varepsilon/2} \\ > 0 & \text{on } TE \subset T\widetilde{X}|_E \,. \end{cases}$$

Now, if ω is a Kähler form on X, then $\pi^*\omega$ is a non-negative $(1,1)$-form on $\widetilde{X}$ such that

$$\pi^*\omega = \begin{cases} > 0 & \text{on } \widetilde{X} - E \\ \geq 0 & \text{on } \widetilde{X} \\ = 0 & \text{on } TE \subset T\widetilde{X}|_E \,. \end{cases}$$

Therefore, for k sufficiently large,

$$\widetilde{\omega} := \omega_{-E} + k\pi^*\omega.$$

is Kähler form on $\widetilde{X}$.

□

7 Kodaira Vanishing Theorem

The main ingredient of the proof of Kodaira Embedding Theorem and several other homological calculations is the following vanishing result for positive line bundles.

Theorem 7.1. *(Kodaira Vanishing Theorem) If $\mathcal{L} \longrightarrow X$ is a positive line bundle, then*

$$H^q(X, \Omega_X^p(\mathcal{L})) = 0 \qquad \forall\ p+q > n = \dim_{\mathbb{C}} X.$$

Remark 7.2. Recall that Ω_X^p is the sheaf of holomorphic p-forms; it is the sheaf of holomorphic sections of the tensor bundle $\Lambda_X^{p,0} := \Lambda_{\mathbb{C}}^p(\mathcal{T}^*X)$. Then, $\Omega_X^p(\mathcal{L})$ is the sheaf of holomorphic p-forms taking values in $\mathcal{L}$. The cohomology group above can be both seen as the cech cohomology group of the sheaf $\Omega_X^p(\mathcal{L})$ or the Dolbeaut cohomology group of the corresponding holomorphic vector bundle $H_{\bar\partial}^q(X, \Lambda_X^{p,0} \otimes \mathcal{L})$. When $\mathcal{L} = \mathcal{O}_X$, we simply get the (p,q) cohomology groups of X.

In order to prove this theorem, we need to briefly discuss the star operator and Harmonic forms. Suppose $\mathcal{E} \longrightarrow X$ is a holomorphic vector bundle.

Let X be a compact complex manifold with a Hermitian metric h on $\mathcal{T}X$. The metric h induces a similarly denoted metric on $\mathcal{T}^*X$ and all other tensor bundles over X. If $\eta_1, \ldots, \eta_n$ is a local orthonormal frame for $\mathcal{T}^*X$, then every

$$\eta \in \Lambda^{p,q}(X, \mathbb{C}) = \Gamma(X, \Lambda^p\mathcal{T}^*X \wedge \Lambda^q\overline{\mathcal{T}^*X})$$

can be written as

$$\eta = \sum_{\substack{I,J\\ |I|=p,|J|=q}} a_{I,J}\eta_I \wedge \overline{\eta_J}, \quad \eta_I = \eta_{i_1}\wedge\cdots\wedge\eta_{i_k} \quad \forall\, I=(i_1,\ldots,i_k),$$

with the convention

$$\eta_{\sigma(I)} = (-1)^{\mathrm{sign}(\sigma)}\eta_I \quad \forall\, \sigma\in S_k$$

The point-wise Hermitian inner product of $\eta,\eta'\in\Lambda^{p,q}(X,\mathbb{C})$ is defined by

$$\left\langle \eta,\overline{\eta'}\right\rangle = \sum_{\substack{I,J\\ |I|=p,|J|=q}} a_{I,J}\overline{a'_{I,J}}.$$

In particular, the point-wise L^2-norm of η is defined to be the function

$$|\eta|^2 = \langle \eta,\overline{\eta}\rangle = \sum_{I,J}|a_{I,J}|^2.$$

Remark 7.3. In Griffiths-Harris, $|\eta|^2$ is defined to be

$$|\eta|^2 = 2^{p+q}\sum_{I,J}|a_{I,J}|^2$$

because the norm of $\mathrm{d}z_i$ with respect to the Riemannian metric corresponding to the standard Hermitian metric on $\mathbb{C}^n$ is 2. However, by our convention in (4.18), there is no need for this extra factor.

The volume form of X is given by

$$\Omega = \frac{1}{n!}\omega^n = (-1)^{\binom{n}{2}}\left(\frac{\mathfrak{i}}{2}\right)^n \eta_1\wedge\cdots\wedge\eta_n\wedge\overline{\eta_1}\wedge\cdots\wedge\overline{\eta_n}.$$

The metric h allows us to define a complex linear (Hodge) star duality operator

$$*\colon \Lambda^{p,q}(X,\mathbb{C}) \longrightarrow \Lambda^{n-q,n-p}(X,\mathbb{C}),$$

satisfying

$$\eta\wedge *\overline{\eta'} = \left\langle \eta,\overline{\eta'}\right\rangle\Omega.$$

Note that $**\,\eta = (-1)^{\deg(\eta)}\eta$ and

$$\overline{*}\colon \Lambda^{p,q}(X,\mathbb{C}) \longrightarrow \Lambda^{n-p,n-q}(X,\mathbb{C}), \quad \overline{*}(\eta) = \overline{*\eta}, \tag{7.1}$$

is an anti $\mathbb{C}$-linear operator. Since X is compact, the integral

$$(\eta,\overline{\eta'}) = \int_X \eta\wedge\overline{*}(\eta') = \int_X \left\langle \eta,\overline{\eta'}\right\rangle\Omega$$

is defined and defines a hermitian inner product on each $\Lambda^{p,q}(X,\mathbb{C})$.

More generally, if $\mathcal{E}\longrightarrow X$ is a complex vector bundle equipped with a hermitian metric $h_{\mathcal{E}}$, every $\eta\in\Lambda^{p,q}(X,\mathcal{E})$ can be expanded as

$$\eta = \sum_{\substack{I,J\\ |I|=p,|J|=q}} a_{I,J,\alpha}\eta_I\wedge\overline{\eta_J}\otimes e_\alpha,$$

where $\{e_\alpha\}$ is a unitary frame for $\mathcal{E}$. The metric $h_\mathcal{E}$ identifies $\mathcal{E}^*$ with $\overline{\mathcal{E}}$ and we have a wedge product

$$\Lambda^{p,q}(X,\mathcal{E}) \otimes \Lambda^{p',q'}(X,\mathcal{E}^*) \longrightarrow \Lambda^{p+p',q+q'}(X,\mathbb{C}).$$

Similarly to (7.1) we have a complex linear operator

$$\bar{*}_\mathcal{E} \colon \Lambda^{p,q}(X,\mathcal{E}) \longrightarrow \Lambda^{n-p,n-q}(X,\mathcal{E}^*) = \Lambda^{n-p,n-q}(X,\overline{\mathcal{E}}) \tag{7.2}$$

and a pairing

$$(\eta,\overline{\eta'}) = \int_X \eta \wedge \bar{*}_\mathcal{E}(\eta') = \int_X \langle \eta, \overline{\eta'} \rangle\, \Omega \tag{7.3}$$

The adjoint of the $\bar\partial$-operator

$$\bar\partial \colon \Lambda^{p,q}(X,\mathcal{E}) \longrightarrow \Lambda^{p,q+1}(X,\mathcal{E})$$

with respect to the pairing above is given by

$$\bar\partial^* = -\bar{*}_\mathcal{E} \circ \bar\partial \circ \bar{*}_\mathcal{E} \colon \Lambda^{p,q}(X,\mathcal{E}) \longrightarrow \Lambda^{p,q-1}(X,\mathcal{E}).$$

Finally, the $\bar\partial$-laplacian is defined by

$$\Delta = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial \colon \Lambda^{p,q}(X,\mathcal{E}) \longrightarrow \Lambda^{p,q}(X,\mathcal{E}).$$

An $\mathcal{E}$-valued (p,q)-form η is called Harmonic if $\Delta\eta = 0$. Let $\mathcal{H}^{p,q}(X,\mathcal{E})$ denote the space of harmonic $\mathcal{E}$-valued (p,q)-forms. The main results of Hodge theory are the followings statements.

- $\mathcal{H}^{p,q}(X,\mathcal{E})$ is finite dimensional.
- Let $\mathcal{H}$ denotes the orthogonal projection $\Lambda^{p,q}(X,\mathcal{E}) \longrightarrow \mathcal{H}^{p,q}(X,\mathcal{E})$; there exists an operator $G \colon \Lambda^{p,q}(X,\mathcal{E}) \longrightarrow \Lambda^{p,q}(X,\mathcal{E})$ such that (i) $G(\mathcal{H}^{p,q}(X,\mathcal{E})) = 0$, (ii) G commutes with $\bar\partial$ and $\bar\partial^*$, and (iii) $I = \mathcal{H} + \Delta G$.
- Consequently, the natural inclusion

  $$\mathcal{H}^{p,q}(X,\mathcal{E}) \longrightarrow H^{p,q}_{\bar\partial}(X,\mathcal{E}) := H^q_{\bar\partial}(X,\Omega^p_X(\mathcal{E}))$$

  is an isomorphism; i.e. every cohomology class has a unique harmonic representative.
- The $\bar{*}$ operator gives an isomorphism

  $$H^{p,q}_{\bar\partial}(X,\mathcal{E}) \cong H^{n-p,n-q}_{\bar\partial}(X,\mathcal{E}^*)^*.$$

For $p = 0$, the last isomorphism reads

$$H^q_{\bar\partial}(X,\mathcal{E}) \cong H^{n-q}_{\bar\partial}(X,\Omega^n_X(\mathcal{E}^*))^* = H^{n-q}_{\bar\partial}(X,K_X \otimes \mathcal{E}^*)^*.$$

This isomorphism, relating the cohomology classes of $\mathcal{E}$ to that of $K_X \otimes \mathcal{E}^*$ is called Serre Duality.

Example 7.4. In Example 3.3, for $X = \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}(m)$ we found that

$$\dim_{\mathbb{C}} H^0_{\bar\partial}(\mathbb{P}^1, \mathcal{O}(m)) \cong \begin{cases} m+1 & \text{if } m \geq 0 \\ 0 & \text{if } m \leq -1 \end{cases}, \qquad \dim_{\mathbb{C}} H^1_{\bar\partial}(\mathbb{P}^1, \mathcal{O}(m)) \cong \begin{cases} 0 & \text{if } m \geq 0 \\ -(m+1) & \text{if } m \leq -1 \end{cases}.$$

As it is evident from these calculations,

$$\dim_{\mathbb{C}} H^0_{\bar\partial}(\mathbb{P}^1, \mathcal{O}(m)) = \dim_{\mathbb{C}} H^1_{\bar\partial}(\mathbb{P}^1, \mathcal{O}(2-m))$$

which follows from Serre duality because $K_{\mathbb{P}^1} = \mathcal{O}(-2)$.

For every holomorphic line bundle $\mathcal{L}$ on a genus g Riemann surface Σ we have

- $\deg(K_\Sigma) = 2g - 2$ (because Euler characteristic is $2 - 2g$) ;
- (Serre duality) $\dim_{\mathbb{C}} H^0_{\bar\partial}(\Sigma, \mathcal{L}) = \dim_{\mathbb{C}} H^1_{\bar\partial}(\Sigma, K_\Sigma \otimes \mathcal{L}^*)$;
- (Riemann Roch) $\dim_{\mathbb{C}} H^0_{\bar\partial}(\Sigma, \mathcal{L}) - \dim_{\mathbb{C}} H^1_{\bar\partial}(\Sigma, \mathcal{L}) = \deg(\mathcal{L}) + (1-g)$.

Combining the last two identities we get

$$\dim_{\mathbb{C}} H^0_{\bar\partial}(\Sigma, \mathcal{L}) - \dim_{\mathbb{C}} H^0_{\bar\partial}(\Sigma, K_\Sigma \otimes \mathcal{L}^*) = \deg(\mathcal{L}) + (1-g).$$

A holomorphic line bundle of negative degree can not have any holomorphic sections therefore (assuming $g > 0$)

- if $\deg(\mathcal{L}) > 2g - 2$, then the second term on left is zero and consequently
$$\dim_{\mathbb{C}} H^0_{\bar\partial}(\Sigma, \mathcal{L}) = \deg(\mathcal{L}) + (1-g) \geq g;$$
- if $\deg(\mathcal{L}) < 0$, then the first term on left is zero and consequently
$$\dim_{\mathbb{C}} H^1_{\bar\partial}(\Sigma, \mathcal{L}) = -\deg(\mathcal{L}) + g - 1 \geq g;$$
- if $0 \leq \deg(\mathcal{L}) \leq 2g - 2$, then depending on the choice of $\mathcal{L}$ the two cohomology groups can be both non-trivial.

With notation as before, if (X, ω) is a Kähler manifold, wedging with ω gives us the so-called Lefschetz operator

$$L \colon \Lambda^{p,q}(X, \mathcal{E}) \longrightarrow \Lambda^{p+1,q+1}(X, \mathcal{E})$$

with the adjoint operator

$$\Lambda = L^* \colon \Lambda^{p,q}(X, \mathcal{E}) \longrightarrow \Lambda^{p-1,q-1}(X, \mathcal{E}).$$

Putting $\mathcal{E} = \mathcal{O}_X$ we get these operators at the level of (p,q)-forms only, where both $\bar\partial$ and ∂ are defined and $\mathrm{d} = \partial + \bar\partial$. In the presence of a non-trivial vector bundle $\mathcal{E}$, only $\bar\partial$ is defined.

Proposition 7.5. *(Prp 1.2.6 and Prp 3.1.12 in Huybrecht: Kähler identities) Suppose X is a compact Kähler manifold of complex dimension n. For L and Λ acting on (p,q)-forms we have*

(1) $[\bar\partial, L] = [\partial, L] = 0$ *and* $[\bar\partial^*, \Lambda] = [\partial^*, \Lambda] = 0$.

(2) $[\bar\partial^*, L] = \mathrm{i}\partial,\ [\partial^*, L] = -\mathrm{i}\bar\partial$ *and* $[\Lambda, \bar\partial] = -\mathrm{i}\partial^*,\ [\Lambda, \partial] = \mathrm{i}\bar\partial^*$.

(3) $[L,\Lambda]=H$, $[H,L]=2L$, *and* $[H,\Lambda]=-2\Lambda$, *where* H *is the diagonal operator of multiplication by* $(k-n)$ *on* k*-forms.*

(4) $\Delta_{\bar\partial}=\Delta\partial=\frac{1}{2}\Delta$, *and they commute with* $*$, ∂, $\bar\partial$, ∂^*, $\bar\partial^*$, L, Λ. *In particular,* L *and* Λ *descend to maps between harmonic forms and thus to*

$$L\colon H^{p,q}(X,\mathbb{C})\longrightarrow H^{p+1,q+1}(X,\mathbb{C})\quad\text{and}\quad \Lambda\colon H^{p,q}(X,\mathbb{C})\longrightarrow H^{p-1,q-1}(X,\mathbb{C});$$

where at the cohomology level, the maps only depend on the cohomology class of ω.

Theorem 7.6. *(Hard Lefschetz Theorem) Suppose* X *is a compact Kähler manifold of complex dimension* n. *Then, for all* (p,q) *with* $k=p+q\leq n$, *the map*

$$L^{n-k}\colon H^{p,q}(X,\mathbb{C})\subset H^k(X,\mathbb{C})\longrightarrow H^{p+n-k,q+n-k}(X,\mathbb{C})\subset H^{2n-k}(X,\mathbb{C})$$

is an isomorphism and

$$\begin{aligned} H^{p,q}(X,\mathbb{C}) &= \bigoplus_{i\geq 0} L^i(H^{p-i,q-i}(X,\mathbb{C})_{\text{prim}}),\\ H^{p,q}(X,\mathbb{C})_{\text{prim}} &:= \operatorname{Ker}(\Lambda\colon H^{p,q}(X,\mathbb{C})\longrightarrow H^{p-1,q-1}(X,\mathbb{C}))\\ &= \operatorname{Ker}(L^{n-k+1}\colon H^{p,q}(X,\mathbb{C})\longrightarrow H^{p+n-k+1,q+n-k+1}(X,\mathbb{C})). \end{aligned} \tag{7.4}$$

The proof of this theorem uses representation theory. The actions of L, Λ, and H on $H^*(X,\mathbb{C})$ generate an $\mathrm{sl}(2,\mathbb{C})$ representation. The Lie algebras $\mathrm{sl}(2,\mathbb{C})$ and $\mathrm{sl}(2,\mathbb{R})$ are generated by

$$X=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad Y=\begin{bmatrix}0&0\\1&0\end{bmatrix},\quad\text{and}\quad H=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

satisfying

$$[X,Y]=XY-YX=H,\quad [H,X]=2X,\quad [H,Y]-2Y.$$

For each $n\geq 1$, there is a unique n-dimensional irreducible representations of $\mathrm{sl}(2,\mathbb{C})$ which is $\mathrm{Sym}^n(\mathbb{C}^2)$. The definition of the tensor product representation is the following. If V and W are representations of a Lie group $\mathfrak{g}$, $v\in V$, $w\in W$, and $g\in\mathfrak{g}$, then $g\cdot(v\otimes w)=g\cdot v\otimes w+v\otimes g\cdot w$. The latter naturally extends to higher tensor product and symmetric tensor product.

Example 7.7. Describe $H^*(\mathbb{CP}^n,\mathbb{C})$ as an irreducible representation of $\mathrm{sl}(2,\mathbb{C})$. Explicitly describe the actions of L and Λ. More precisely, show that $H^*(\mathbb{CP}^n,\mathbb{C})\cong\mathrm{Sym}^n(\mathbb{C}^2)$.

Proposition 7.8. *(Hodge-Riemann bilinear relations) Let* (X,ω) *be a compact Kähler manifold of complex dimension* n. *Then, for each* $0\neq\alpha\in H^{p,q}(X,\mathbb{C})_{\text{prim}}$ *we have*

$$\mathfrak{i}^{p-q}(-1)^{\binom{p+q}{2}}\int_X \alpha\wedge\overline{\alpha}\wedge\omega^{n-(p+q)}>0.$$

One of the most interesting cases of this result is when $p+q=n$ and $n=2m$ is even. Then,

$$\mathfrak{i}^{p-q}(-1)^{\binom{p+q}{2}}=(-1)^{(p-m)}(-1)^m=(-1)^p. \tag{7.5}$$

Using this, (7.4), and symmetries of hodge diamond we can show that the symmetric bilinear pairing

$$H^n(X,\mathbb{R})\otimes H^n(X,\mathbb{R})\longrightarrow\mathbb{R},\quad (\alpha,\beta)\longrightarrow\int_X\alpha\wedge\beta$$

has signature

$$\sigma(X) = \sum_{p,q=0}^{2m} (-1)^p h^{p,q}(X).$$

For instance, if $m = 1$, i.e. if (X, ω) is a compact Kähler surface, then

$$\sigma(X) = 2h^{0,0}(X) + 2h^{2,0}(X) - h^{1,1}(X) = 2h^{2,0}(X) + 2 - h^{1,1}(X).$$

More precisely, we have the following statement known as Hodge Index Theorem.

Proposition 7.9. *(Hodge Index Theorem) Suppose (X, ω) is a compact Kähler surface. Then, the intersection pairing*

$$H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \longrightarrow \mathbb{R}, \quad (\alpha, \beta) \longrightarrow \int_X \alpha \wedge \beta$$

has index $(2h^{2,0}(X) + 1, h^{1,1}(X) - 1)$. Restricted to $H^{1,1}(X, \mathbb{R})$, it is of index $(1, h^{1,1}(X) - 1)$.

Proof. We have

$$H^2(X, \mathbb{R}) = \Big((H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})) \cap H^2(X, \mathbb{R}) \Big) \oplus H^{1,1}(X, \mathbb{R}).$$

For degree reasons, any class in $H^{2,0}(X, \mathbb{C}) \oplus H^{2,0}(X, \mathbb{C})$ is primitive. Therefore, by (7.5), intersection pairing is positive definite on the first summand. By Lefschetz decomposition (7.4), the second summand further orthogonally decomposes as

$$H^{1,1}(X, \mathbb{R}) = \mathbb{R} \cdot \omega \oplus H^{1,1}(X, \mathbb{R})_{\text{prim}}.$$

Clearly, the intersection pairing is positive definite on $\mathbb{R} \cdot \omega$. By (7.5), it is negative definite on $H^{1,1}(X, \mathbb{R})_{\text{prim}}$. □

Using Hodge Index Theorem, we can exclude many compact smooth four-dimensional manifolds from the list of manifolds that admit a Kähler structure. First, if X is a closed oriented smooth manifold, then the intersection pairing

$$H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

is a symmetric unimodular form (by Poincare duality). A unimodular pairing Q is called even if $Q(x, x) \equiv 0 \bmod 2$ for all x. Otherwise, it is called odd. By Hodge Index Theorem, we know that definite forms do not arise among Kähler manifolds unless $h^{1,1}(X) = 1$. Indefinite unimodular forms Q are classified in the following way:

- If Q is odd, then $Q = I_m \oplus (-I_n)$ for some $m, n \geq 0$;
- If Q is even, then $Q = mU \oplus nE_8$ for some $m, n \geq 0$, where

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

  and

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Remark 7.10. The intersection forms E_8 and U are related by $E_8 \oplus -E_8 = 8U$.

Theorem 7.11. *For each integral symmetric bilinear unimodular form Q, there exists a closed oriented simply-connected 4-manifold X with Q as its intersection form. If Q is even, then X is C^0-unique. If Q is odd, there are two C^0-classes at least one of which is not C^1. The manifold corresponding to E_8 is not C^1. (Rokhlin:) If X is smooth then $16 \mid \sigma(X)$. (Donaldson:) If X is smooth and Q is definite then $Q = \pm I_m$.*

Remark 7.12. Changing the orientation of an oriented smooth manifold X changes its intersection Q to $-Q$. For instance, if $\overline{\mathbb{CP}^2}$ is $\mathbb{CP}^2$ with the opposite orientation, then $Q_{\overline{\mathbb{CP}^2}} = (-1)$.

Remark 7.13. If X is the connect sum of closed oriented 4-manifolds Y and Z, then $Q_X = Q_Y \oplus Q_Z$. In the smooth category, any blowup $\widetilde{X}$ of a compact complex surface X at k points is the same as the connect sum $X \# k\overline{\mathbb{CP}^2}$ of X with k copies of $\overline{\mathbb{CP}^2}$.

Examples. By the last sentence above, the intersection form of $\mathbb{CP}^2$ blown up at k points is the indefinite odd form $I_1 \oplus -I_k$. The intersection form of $\mathbb{CP}^1 \times \mathbb{CP}^1$ is U. Generalizing the last example, corresponding to the degree d holomorphic line bundle $\mathcal{O}(d) \longrightarrow \mathbb{CP}^1$, we obtain a complex surface $X_d = \mathbb{P}(\mathcal{O}(d) \oplus \mathcal{O})$ that is a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1$ with two disjoint sections of self-intersection d and $-d$. The second homology of X_d is generated by the fiber class F and either of the sections. Therefore, the intersection form of X_d is

$$Q_d = \begin{bmatrix} 0 & 1 \\ 1 & d \end{bmatrix}$$

HW 7.14. For $d \equiv d' \bmod 2$, show that there is an integral change of basis that transforms Q_d to $Q_{d'}$.

In fact, if $d \equiv d' \bmod 2$, then X_d and $X_{d'}$ are diffeomorphic.

Recall that a $K3$ surface is a simply connected compact Kähler surface X with $K_X \cong \mathcal{O}_X$. Therefore, the Hodge diamond of a $K3$ surface is of the form

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & h^{1,1}(X,\mathbb{C}) & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Since all $K3$ surfaces are diffeomorphic, by calculating the Euler characteristic of a quartic K3 surface in $\mathbb{CP}^3$ we find that $h^{1,1}(X,\mathbb{C}) = 20$. In fact, $H_2(X,\mathbb{Z}) \cong \mathbb{Z}^{22}$. By Hodge Index Theorem, the index the intersection form on $K3$ has index $(2h^{2,0}(X) + 1, h^{1,1}(X) - 1) = (3, 19)$. It turns out that the quadratic form of $K3$ surface is $-2E_8 \oplus 3U$.

Going back to the discussion of the operators L, Λ, $\bar{\partial}$, and $\bar{\partial}^*$ on a hermitian holomorphic vector bundle $\mathcal{E} \longrightarrow X$, the analogue of Proposition 7.5.(2) is the following. Let ∇ be the chern connection on $\mathcal{E}$, i.e. a complex linear hermitian connection with $\nabla^{0,1} = \bar{\partial}$.

Lemma 7.15. *(Nakano identity) Suppose $\mathcal{E} \longrightarrow X$ is a hermitian holomorphic vector bundle over a compact Kähler manifold. For L and Λ acting on $\mathcal{E}$-valued (p,q)-forms we have*

$$[\bar{\partial}^*, L] = \mathfrak{i}\nabla^{1,0} \quad \text{and} \quad [\Lambda, \bar{\partial}] = -\mathfrak{i}(\nabla^{1,0})^*.$$

We need the following lemma to prove the Kodaira Vanishing Theorem.

Lemma 7.16. *Suppose $\mathcal{E} \longrightarrow X$ is a hermitian holomorphic vector bundle over a compact Kähler manifold. Then, for the curvature $(1,1)$-form F_∇ of the Chern connection ∇ and an arbitrary harmonic form $\alpha \in \mathcal{H}^{p,q}(X,\mathcal{E})$ we have:*

$$(i)(\mathfrak{i}F_\nabla\Lambda(\alpha), \overline{\alpha}) \leq 0, \quad (ii)(\mathfrak{i}\Lambda F_\nabla(\alpha), \overline{\alpha}) \geq 0,$$

where $(-,-)$ is the inner product (7.3).

Proof. As an operator, F_∇ is given by

$$F_\nabla = \nabla \circ \nabla = (\nabla^{1,0} + \bar{\partial}) \circ (\nabla^{1,0} + \bar{\partial}) = \nabla^{1,0} \circ \bar{\partial} + \bar{\partial} \circ \nabla^{1,0}.$$

Using $\bar{\partial}\alpha = 0$, $\bar{\partial}^*\alpha = 0$, and Nakano identity, we obtain

$$\begin{aligned}
(\mathfrak{i}F_\nabla\Lambda(\alpha), \overline{\alpha}) &= \mathfrak{i}(\nabla^{1,0} \circ \bar{\partial} \circ \Lambda(\alpha), \overline{\alpha}) + \mathfrak{i}(\bar{\partial} \circ \nabla^{1,0} \circ \Lambda(\alpha), \overline{\alpha}) \\
&= \mathfrak{i}(\bar{\partial} \circ \Lambda(\alpha), \overline{(\nabla^{1,0})^*\alpha}) + \mathfrak{i}(\nabla^{1,0} \circ \Lambda(\alpha), \overline{\bar{\partial}^*\alpha}) \\
&= (\bar{\partial} \circ \Lambda(\alpha), \overline{-\mathfrak{i}(\nabla^{1,0})^*\alpha}) + 0 \\
&= (\bar{\partial}\Lambda\alpha, \overline{[\Lambda, \bar{\partial}]\alpha}) \\
&= (\bar{\partial}\Lambda\alpha, \overline{-\bar{\partial}\Lambda\alpha}) \leq 0 \,.
\end{aligned}$$

The other inequality is proved similarly. □

Proof of Kodaira Vanishing Theorem. By the positivity assumption, there is a hermitian metric on $\mathcal{E}$ such that $\omega = \mathfrak{i}F_\nabla$ is a Kähler form on X. Therefore, with respect to ω, the operator L is the same as $\mathfrak{i}F_\nabla$. Since $[L, \Lambda] = H$, for any $\alpha \in \mathcal{H}^{p,q}(X,\mathcal{E})$, by the Lemma above, we obtain

$$0 \leq ([\Lambda, \mathfrak{i}F_\nabla]\alpha, \overline{\alpha}) = ([\Lambda, L]\alpha, \overline{\alpha}) = (-H\alpha, \overline{\alpha}) = (n - (p+q))\|\alpha\|^2.$$

Therefore, $\mathcal{H}^{p,q}(X,\mathcal{E}) = 0$ whenever $p + q > n$.

Remark 7.17. A similar proof shows that if $\mathcal{L} \longrightarrow X$ is a positive line bundle and $\mathcal{E} \longrightarrow X$ is an arbitrary holomorphic vector bundle, there exists $m_0 \in \mathbb{N}$ such that

$$\text{(Serre's Vanishing Theorem)} \quad H^q(X, L^{\otimes m} \otimes \mathcal{E}) = 0 \qquad \forall\, m \geq m_0,\ q > 0. \tag{7.6}$$

HW 7.18. Use Serre's Vanishing Theorem to show that every holomorphic vector bundle on $\mathbb{CP}^1$ is a direct sum of holomorphic line bundles.

The next result is very useful in understanding the topology of hyperplane sections of a complex projective variety.

Theorem 7.19. *(Lefschetz Hyperplane Theorem/Weak Lefschetz Theorem) Suppose X is a compact kähler manifold of complex dimension n and $Y \subset X$ is a smooth complex hypersurface/divisor such that $\mathcal{O}_X(Y)$ is a positive line bundle. Then the canonical restriction map*

$$H^k(X,\mathbb{C}) \longrightarrow H^k(Y,\mathbb{C})$$

is an isomorphism for $k \leq n-2$ and it is injective for $k = n-1$.

Proof. It is enough to prove the statement for

$$H^{p,q}(X,\mathbb{C}) \longrightarrow H^{p,q}(Y,\mathbb{C}).$$

The proof below relies on two short exact sequences (of sheaves)

$$0 \longrightarrow \mathcal{O}_X(-Y) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{N}_X Y^* \longrightarrow \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0;$$

where $\Omega = \Omega^1$ denotes the sheaf of holomorphic 1-forms and $\mathcal{N}_X Y^*$ is the (sheaf of sections of the) conormal bundle of Y in X. In the first exact sequence, the first map is tensoring with a section of $\mathcal{O}_X(Y)$ vanishing along Y. Twisting/tensoring the first exact sequence with Ω^p_X yields

$$0 \longrightarrow \Omega^p_X(-Y) \longrightarrow \Omega^p_X \longrightarrow \Omega^p_X|_Y \longrightarrow 0. \tag{7.7}$$

Here, $\Omega^p_X(-Y)$ is the sheaf of $\mathcal{O}_X(-Y)$-valued p-forms.

HW 7.20. Show that any short exact sequence of holomorphic vector bundles

$$0 \longrightarrow L \longrightarrow E \longrightarrow F \longrightarrow 0$$

where L is a line bundle, induces short exact sequences of the form

$$0 \longrightarrow L \otimes \Lambda^{i-1} F \longrightarrow \Lambda^i E \longrightarrow \Lambda^i F \longrightarrow 0.$$

By the exercise above, taking the p-th exterior power of the second sequence yields

$$0 \longrightarrow \mathcal{N}_X Y^* \otimes \Omega^{p-1}_Y \longrightarrow \Omega^p_X|_Y \longrightarrow \Omega^p_Y \longrightarrow 0$$

Then, from Serre duality, the fact that

$$(\Omega^p_X)^* \otimes K_X \cong \Omega^{n-p}_X,$$

and Kodaira Vanishing we obtain that

$$H^q(X, \Omega^p_X(-Y)) \cong H^{n-q}(X, K_X \otimes \Omega^p_X(-Y)^*)^* = H^{n-q}(X, \Omega^{n-p}_X \otimes \mathcal{O}_X(Y))^* = 0$$

whenever $n - p + n - q > n \Longrightarrow p + q < n$.

The long-exact sequence corresponding to the short-exact sequence (7.7) reads

$$\cdots \longrightarrow H^q(X, \Omega^p_X(-Y)) \longrightarrow H^q(X, \Omega^p_X) \longrightarrow H^q(Y, \Omega^p_Y) \longrightarrow H^{q+1}(X, \Omega^p_X(-Y)) \longrightarrow \cdots$$

For $p + q + 1 < n$, by the vanishing statement above we have

$$H^{p,q}(X,\mathbb{C}) = H^q(X, \Omega^p_X) \cong H^q(Y, \Omega^p_X|_Y).$$

For $p + q = n$, we have

$$0 \longrightarrow H^q(X, \Omega^p_X) \longrightarrow H^q(Y, \Omega^p_X|_Y) \longrightarrow H^{q+1}(X, \Omega^p_X(-Y))$$

which shows that $H^q(X, \Omega^p_X) \longrightarrow H^q(Y, \Omega^p_X|_Y)$ is injective.

Finally, we need to compose these identifications/injections with the natural restriction map

$$H^q(Y, \Omega^p_X|_Y) \longrightarrow H^q(Y, \Omega^p_Y),$$

whose kernel and cokernel are contained are contained in the cohomology groups of $\mathcal{N}_X Y^* \otimes \Omega_Y^{p-1}$. Since the restriction of $\mathcal{O}_X(Y)$ to Y is a positive line bundle, the Kodaira Vanishing shows that

$$H^q(Y, \Omega^p_X|_Y) \longrightarrow H^q(Y, \Omega^p_Y)$$

is an isomorphism for $p+q<n$ and an injection for $p+q=n$. □

Example 7.21. Let X be a smooth degree 5 hypersurface in $\mathbb{CP}^4$. By (6.5),

$$K_X = (K_{\mathbb{CP}^4} \otimes \mathcal{O}(5))|_X = \mathcal{O}_X.$$

Therefore, X is a Calabi-Yau 3-fold. By Lefschetz Hyperplane Theorem

$$h^{p,q}(X, \mathbb{C}) = h^{p,q}(\mathbb{P}^4, \mathbb{C}) \qquad \forall\, p+q \leq 2.$$

Therefore, the upper-half of the Hodge diamond of X has the form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \end{array}$$

In order to find $h^{2,1} = h^{1,2}$, we compute the Euler characteristic of X in two ways. First, we have

$$\chi(X) = \sum_{i=0}^{6} (-1)^i \; h^i(X, \mathbb{C}) = 2 - 2h^{2,1}.$$

On the other hand,

$$\chi(X) = \int_X c_{\text{top}}(TX).$$

Using the adjunction formula

$$\mathcal{T}\mathbb{CP}^4|_X \cong \mathcal{T}X \oplus \mathcal{O}_{\mathbb{CP}^4}(5)|_X$$

and

$$c(\mathcal{T}\mathbb{CP}^4) = (1+h)^5, \quad h = \mathrm{PD}(H) = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^4, \mathbb{Z}),$$

we find that

$$c(\mathcal{T}X) = \frac{(1+h)^5}{1+5h} = \Big(1 + 5h + 10h^2 + 10h^3\Big)\Big(1 - 5h + 25h^2 - 125h^3 + \cdots\Big).$$

Therefore,

$$\chi(X) = \int_X (-125 + 125 - 50 + 10)h^3 = -40h^3 \cdot (5h) = -200.$$

It follows that

$$h^{2,1} = 101.$$

Remark 7.22. The now famous Mirror Symmetry conjecture predicts that for most Calabi-Yau 3-folds X, there is a dual Calabi-Yau 3-fold $\check{X}$ with $h^{2,1}(X) = h^{1,1}(\check{X})$ and $h^{1,1}(X) = h^{2,1}(\check{X})$. Therefore, the mirror of the quintic Calabi-Yau 3-fold above should have Hodge numbers

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \end{array}$$

i.e. it must have a very large Kähler cone and a relatively small middle cohomology. In practice, $\check{X}$ is usually constructed by first taking a discrete quotient of X and then resolving its singularities.

There is a different proof of Lefschetz Hyperplane Theorem using Morse theory which goes as follows.

By the positivity assumption, there is a hermitian metric h on $\mathcal{O}_X(Y)$ such that $\omega = \mathfrak{i}F_\nabla$ is a Kähler form on X. Here, F_∇ is the curvature $(1,1)$-form of the Chern connection associated to h. Also, by construction, restricted to $X - Y$, the holomorphic line bundle $\mathcal{O}_X(Y)|_{X-Y}$ is trivial. With respect to a trivialization

$$\mathcal{O}_X(Y)|_{X-Y} \otimes (X - Y) \times \mathbb{C}$$

we can think of $h|_{X-Y}$ as a real valued positive function $h|_{X-Y} \colon X - Y \longrightarrow \mathbb{R}_{>0}$ and

$$\omega = \mathfrak{i}\, \bar{\partial}\partial \ln(h) \tag{7.8}$$

see (4.16). The hermitian metric can be chosen in a way such that $h|_{X-Y} \colon X - Y \longrightarrow \mathbb{R}_{>0}$ is Morse with finitely many critical points. Since $\mathcal{O}_X(Y)$ has a holomorphic section that vanishes along Y, it follows that $\lim_{x\to Y} h(x) = 0$. Therefore, as $t \longrightarrow \infty$, the sub-level sets

$$X_{\leq t} = \{x \in X - Y \colon\ -\ln h(x) \leq t\}$$

give an exhaustion of $X - Y$ whose topology stabilizes after $t \geq T$, for some sufficiently large T. Now, the key observation is that, because of (7.8), for every critical point $p \in X - Y$ of the Morse function $-\ln h$, its Hessian/second-derivative matrix has at most $n = \dim_{\mathbb{C}} X$ negative eigenvalues. Therefore, by Morse theory, as far as homotopy type of X is concerned, X is obtained from Y by attaching cells of dimension at least n. In other words, $X - Y$ is build from k-handles with $k \leq n$. This yields Lefschetz Hyperplane Theorem on the stronger homotopy level and homology with $\mathbb{Z}$-coefficients.

HW 7.23. Suppose $f \colon \mathbb{C}^n \longrightarrow \mathbb{R}_{>0}$ is a smooth function with a non-degenerate critical point at the origin. Show that if $\mathfrak{i}\, \bar{\partial}\partial \ln(f)$ is positive at the origin, then its Hessian/second-derivative matrix has at least n negative eigenvalues (Hint: Relate second derivative and $\bar{\partial}\partial$).

8 Proof of Kodaira Embedding Theorem

Suppose $\mathcal{L} \longrightarrow X$ is a holomorphic line bundle. By abuse of notation, we also let $\mathcal{L}$ denote the sheaf of holomorphic sections of $\mathcal{L}$. For every $x \in X$, evaluation at x yields an exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_x(\mathcal{L}) := \mathcal{I}_x \otimes \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_x := \mathcal{L}|_x \cong \mathbb{C} \longrightarrow 0,$$

where

$$\mathcal{I}_x \subset \mathcal{O}_X$$

is the sub-sheaf of holomorphic functions vanishing at x (i.e. ideal sheaf of $x \in X$). In other words $\mathcal{I}_x \otimes \mathcal{L}$ is a sub-sheaf of $\mathcal{L}$ consisting of sections vanishing at x. There is a derivative map

$$\mathcal{I}_x(\mathcal{L}) \xrightarrow{\mathrm{d}_x} \mathcal{T}_x^* X \otimes \mathcal{L}_x$$

defined in the following way. Locally around x, a section of $\mathcal{I}_x \otimes \mathcal{L}$ has the form $f \otimes \zeta$ where f is a holomorphic function vanishing at x and ζ is a local section of $\mathcal{L}$. Then, d_x maps $f \otimes \zeta$ to $\mathrm{d}_x f \otimes \zeta$ (recall that the derivative of a section of a vector bundle is well-defined along the zero locus and does not require a connection to be defined). The kernel of the derivative map d_x above is $\mathcal{I}_x^2(\mathcal{L}) := \mathcal{I}_x^2 \otimes \mathcal{L}$. Thus, we get a short exact sequence

$$0 \longrightarrow \mathcal{I}_x^2(\mathcal{L}) \longrightarrow \mathcal{I}_x(\mathcal{L}) \xrightarrow{\mathrm{d}_x} \mathcal{T}_x^* X \otimes \mathcal{L}_x.$$

HW 8.1. Show that $\mathcal{I}_x/\mathcal{I}_x^2$ is canonically isomorphic to $\mathcal{T}_x^* X$.

Now, suppose $\mathcal{L}$ is a positive line bundle and X is compact. In order to prove the Kodaira Embedding Theorem, we need to show that for k sufficiently large,

(1) the map
$$H^0(X, \mathcal{L}^{\otimes k}) \xrightarrow{r_x \oplus r_y} \mathcal{L}_x^{\otimes k} \oplus \mathcal{L}_y^{\otimes k}$$
is surjective for all $x \neq y \in X$;

(2) and, the derivative map
$$H^0(X, \mathcal{I}_x(\mathcal{L}^{\otimes k})) \xrightarrow{d_x} \mathcal{T}_x^* X \otimes \mathcal{L}_x^{\otimes k}$$
is surjective for all $x \in X$.

The first condition above implies both that ($\star$) the map $\iota_{\mathcal{L}^{\otimes k}} \colon X \to \mathbb{P}^N$ is defined and is one-to-one. The second condition implies that $\iota_{\mathcal{L}^{\otimes k}}$ is an immersion.

If $\dim_{\mathbb{C}} X > 1$, then $\mathcal{I}_x$ is not the sheaf of sections of a holomorphic vector bundle. In order to use the cohomology theories developed for vector bundles in the earlier sections, the trick is to blowup X at x to replace x with a divisor. In this process, one needs to compare the space of global sections before and after the blowup. More precisely, the long-exact sequence of cech cohomologies corresponding to the short-exact sequence

$$0 \longrightarrow \mathcal{I}_{x,y}(\mathcal{L}^{\otimes k}) \longrightarrow \mathcal{L}^{\otimes k} \longrightarrow \mathcal{L}_x^{\otimes k} \oplus \mathcal{L}_y^{\otimes k} \longrightarrow 0$$

yields

$$\begin{aligned} 0 \longrightarrow \check{H}^0(X, \mathcal{I}_{x,y}(\mathcal{L}^{\otimes k})) \longrightarrow \check{H}^0(X, \mathcal{L}^{\otimes k}) \longrightarrow \check{H}^0(X, \mathcal{L}_x^{\otimes k} \oplus \mathcal{L}_y^{\otimes k}) \longrightarrow \\ \longrightarrow \check{H}^1(X, \mathcal{I}_{x,y}(\mathcal{L}^{\otimes k})) \longrightarrow \check{H}^1(X, \mathcal{L}^{\otimes k}) \longrightarrow 0. \end{aligned}$$

Therefore, in order to prove (1) above, we need to show that $\check{H}^1(X, \mathcal{I}_{x,y}(\mathcal{L}^{\otimes k})) = 0$ for k sufficiently large. Similarly, in order to prove (2), we need to show that $\check{H}^1(X, \mathcal{I}_x^2(\mathcal{L}^{\otimes k})) = 0$ for k sufficiently large. If $\mathcal{I}_x$ was the sheaf of section of a holomorphic line bundle, these would have been followed from Serre Vanishing Theorem (7.6). Using blowup, we will reduce the argument to a similar vanishing argument for vector bundles.

In order to prove (1), let

$$\pi\colon \widetilde{X} = B_{x,y}X \longrightarrow X$$

denote the blowup of X at both x and y with exceptional divisors E_x and E_y, respectively. Let $E = E_x + E_y$ and $\widetilde{\mathcal{L}} = \pi^*\mathcal{L}$. Pullback by π gives an injective homomorphism

$$\pi^*\colon \check{H}^0(X, \mathcal{L}^{\otimes k}) \longrightarrow \check{H}^0(\widetilde{X}, \widetilde{\mathcal{L}}^{\otimes k}).$$

On the other hand, since $\widetilde{X} - E = X - \{x, y\}$, every section of $\widetilde{\mathcal{L}}^{\otimes k}$ gives a section of $\mathcal{L}^{\otimes k}|_{X-\{x,y\}}$ which, by[2] Hartog's theorem, extends uniquely to the entire X. Therefore, the pullback homomorphism above is an isomorphism. Furthermore, by definition,

$$\widetilde{\mathcal{L}}|_{E_x} \cong E_x \times \mathcal{L}_x \quad \text{and} \quad \widetilde{\mathcal{L}}|_{E_y} \cong E_y \times \mathcal{L}_y;$$

therefore,

$$\check{H}^0(E, \widetilde{\mathcal{L}}^{\otimes k}|_E) = \mathcal{L}_x^{\otimes k} \oplus \mathcal{L}_y^{\otimes k} \cong \mathbb{C}^2.$$

In conclusion, we need to show that

$$\check{H}^0(\widetilde{X}, \widetilde{\mathcal{L}}^{\otimes k}) \longrightarrow \check{H}^0(E, \widetilde{\mathcal{L}}^{\otimes k}|_E) \tag{8.1}$$

is surjective. Now, on $\widetilde{X}$, we get an exact sequence of sheaves

$$0 \longrightarrow \widetilde{\mathcal{L}}^{\otimes k} \otimes \mathcal{O}_{\widetilde{X}}(-E) \longrightarrow \widetilde{\mathcal{L}}^{\otimes k} \longrightarrow \widetilde{\mathcal{L}}^{\otimes k}|_E \longrightarrow 0. \tag{8.2}$$

In other words, exchanging x, y for the divisor E, changes $\mathcal{I}_{x,y}$ with the sheaf of holomorphic functions on $\widetilde{X}$ vanishing along E that is isomorphic to the sheaf of holomorphic sections of $\mathcal{O}_{\widetilde{X}}(-nE)$. Recall from the proof of Proposition 6.27 that for k sufficiently large, the line bundle $\widetilde{\mathcal{L}}^{\otimes k} \otimes \mathcal{O}_{\widetilde{X}}(-E)$ is positive. Also, for k sufficiently large $\mathcal{L}^{\otimes k} \otimes K_X^*$ is positive. Therefore, $\pi^*(\mathcal{L}^{\otimes k} \otimes K_X^*)$ is non-negative and, for k_1, k_2 sufficiently large ,the line bundle

$$\widetilde{\mathcal{L}}^{\otimes k_2} \otimes \mathcal{O}_{\widetilde{X}}(-E) \otimes \pi^*(\mathcal{L}^{\otimes k_1} \otimes K_X^*)$$

is positive. For $k = k_1 + k_2$, by (6.8) and Kodaira Vanishing Theorem, we have

$$\begin{aligned}
\check{H}^1(\widetilde{X}, \widetilde{\mathcal{L}}^{\otimes k} \otimes \mathcal{O}_{\widetilde{X}}(-E)) &= \check{H}^1(\widetilde{X}, \Omega^n_{\widetilde{X}}(\widetilde{\mathcal{L}}^{\otimes k} \otimes K^*_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(-E))) \\
&= \check{H}^{n,1}(\widetilde{X}, \widetilde{\mathcal{L}}^{\otimes k} \otimes K^*_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(-E)) \\
&= \check{H}^{n,1}(\widetilde{X}, \pi^*(\mathcal{L}^{\otimes k_1} \otimes K_X) \otimes \widetilde{\mathcal{L}}^{\otimes k_2} \otimes \mathcal{O}_{\widetilde{X}}(-nE)) = 0.
\end{aligned}$$

It follows from the long-exact sequence

$$\begin{aligned}
0 \longrightarrow \check{H}^0(\widetilde{X}, \widetilde{\mathcal{L}}^{\otimes k} \otimes \mathcal{O}_{\widetilde{X}}(-E)) \longrightarrow \check{H}^0(\widetilde{X}, \widetilde{\mathcal{L}}^{\otimes k}) \longrightarrow \check{H}^0(E, \widetilde{\mathcal{L}}^{\otimes k}|_E) \longrightarrow \\
\longrightarrow \check{H}^1(\widetilde{X}, \widetilde{\mathcal{L}}^{\otimes k} \otimes \mathcal{O}_{\widetilde{X}}(-E)) \longrightarrow \cdots
\end{aligned}$$

that (8.1) is injective.

Part (2) follows from a similar argument with $\widetilde{X} = B_xX$, $E = E_x$, and

$$0 \longrightarrow \widetilde{\mathcal{L}}^{\otimes k} \otimes \mathcal{O}_{\widetilde{X}}(-2E) \longrightarrow \widetilde{\mathcal{L}}^{\otimes k} \otimes \mathcal{O}_{\widetilde{X}}(-E) \longrightarrow \Big(\widetilde{\mathcal{L}}^{\otimes k} \otimes \mathcal{O}_{\widetilde{X}}(-E)\Big)|_E \longrightarrow 0$$

instead of (8.2). Note that after the blowup, $\mathcal{T}_x^*X \otimes \mathcal{L}_x^{\otimes k}$ changes to $\mathcal{O}_{\widetilde{X}}(-E) \otimes \widetilde{\mathcal{L}}^{\otimes k}|_E$ because $\mathcal{T}_x^*X$ which is the co-normal bundle of x lifts to the co-normal bundle of E that is $\mathcal{O}_{\widetilde{X}}(-E)$.

[2]We may assume $\dim_{\mathbb{C}} X > 1$; otherwise, $\widetilde{X} = X$