Bayesian Modeling of Inhomogeneous Marked Spatial Point Patterns with Location Dependent Mark Distributions

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December 4, 2014
Introduction
Introduction

- A *spatial point pattern* describes the spatial location of events in a region \( W \)
  - location of particles in a fluid or gas
  - location of trees in a forest
  - location of bird nests on an island

- A mark (or multiple marks) may be associated with each point in the pattern
  - type of particle
  - trunk diameter
  - species of bird
Ant Nests (Harkness & Isham 1983)

- Marked spatial point pattern of \( n = 62 \) ant nests in a \( 250 \times 250 \) foot field located in Spain (subset of the original dataset)
- *Messor wasmanni* (□) and *Cataglyphis bicolor* (●)
Complete Spatial Randomness (CSR)

- Randomly generated 47 □ and 15 ● in W in right plot
- Ant nests on left, CSR on right
Ant Nests Observations

- Higher intensity in lower region
  - Modeling *spatial inhomogeneity* is no big deal
  - See Ogata & Katsura (1988), Bognar (2005), etc.

- Within species:
  - Fewer close pairs of points than under CSR
  - This regular spacing is called *spatial regularity* or *spatial inhibition*
  - Addressed in various papers

- Between species:
  - Inter-point distances are not-unlike CSR; there are close pairs and distant pairs of points

- Higher proportion of Cataglyphis bicolor (●) in lower region
  - Mark distribution appears to depend upon location
  - Ogata & Katsura (1988) addressed this for *Poisson processes* (i.e. assuming no interaction between points)
    - Spurred current work
    - Can we model interaction between points too?
Gibbs Point Processes
Setup

- Let $W$ be a bounded subset of $\mathbb{R}^2$ or $\mathbb{R}^3$
- A *spatial point process* $X$ is a finite random subset of $W$
- A realization of $X$ is called a *spatial point pattern* (SPP)
- A SPP is denoted by
  \[ x = \{x_1, \ldots, x_n\} \]
  i.e. $x$ consists of $n$ distinct points in $W$
Density Of A Spatial Point Process

- The density of a spatial point process $X$ (with respect to the unit rate Poisson process) is

$$f(x) \overset{\text{def}}{=} \frac{g(x)}{Z}$$

- $g(x)$ is the unnormalized density (or kernel)
- $Z$ is the normalization constant (or partition function)

$$Z = \sum_{n=0}^{\infty} \frac{e^{-|W|}}{n!} \int_{W} \cdots \int_{W} g(x) \, dx_1 \cdots dx_n$$

- When $n = 0$, interpret the integral to equal 1
- $Z$ is typically intractable
- Assume the unnormalized density $g$ is integrable, i.e. $Z < \infty$
- The density $f$ is normalized in the sense that

$$\sum_{n=0}^{\infty} \frac{e^{-|W|}}{n!} \int_{W} \cdots \int_{W} f(x) \, dx_1 \cdots dx_n = 1$$
Gibbs Point Process

- Hammersley-Clifford-Ripley-Kelly Theorem (Ripley & Kelly 1977, Baddeley et al. 2004, 2013): A finite Gibbs point process has probability density (with respect to the unit rate Poisson process) of the form

\[ f(x) = \frac{g(x)}{Z} \stackrel{\text{def}}{=} \frac{\exp[-U(x)]}{Z} \]

where the potential function (or energy function)

\[ U(x) = \sum_{i=1}^{n} V^{(1)}(x_i) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} V^{(2)}(x_i, x_j) + \cdots \]

- \( V^{(i)} = i^{\text{th}} \) order potential (maps sub-configurations with \( i \) points into \( \mathbb{R} \cup \{\infty\} \))

- Normalizer

\[ Z = \sum_{n=0}^{\infty} \frac{e^{-|W|}}{n!} \int_{W} \cdots \int_{W} g(x) \, dx_1 \cdots dx_n \]

is typically intractable
Special Cases

- **Poisson point processes** $\rightarrow V^{(1)}$ is the highest order potential function
  - Homogeneous: $V^{(1)}(x_i) = -\log(\kappa)$ (where $\kappa$ is intensity)
  - Inhomogeneous: $V^{(1)}(x_i) = -\log(\kappa(x_i))$
  - $Z$ is tractable (hence the popularity of the Poisson process)
  - Can not model interaction between points

- **Pairwise interaction point processes** $\rightarrow V^{(2)}$ is the highest order interaction (Ripley 1977)
  - $Z$ is intractable
  - Can model interaction between points

- **Triplets process** $\rightarrow V^{(3)}$ is the highest order interaction (Geyer 1999)
  - $Z$ is intractable
  - Can model interaction between points

- Assume $V^{(i)} \geq 0$ when $i \geq 2$ $\implies$ integrability (i.e. $Z < \infty$) (Møller et al. 2006)
  - Restricts us to repulsive or inhibitory spatial point processes
  - Better models exist for modeling clustering
Selected Literature

Marked Gibbs Point Processes
Marked Gibbs Point Processes

- Let $\mathcal{M}$ denote the mark space
- Define $\Gamma = \mathcal{W} \times \mathcal{M} = \{(\omega_1, \omega_2) : \omega_1 \in \mathcal{W}, \omega_2 \in \mathcal{M}\}$
- Let $(\mathbf{x}, \mathbf{m}) = (x_1, \ldots, x_n, m_1, \ldots, m_n)$
- Density of a marked Gibbs point process has the form
  \[
  f(\mathbf{x}, \mathbf{m}) = \frac{g(\mathbf{x}, \mathbf{m})}{Z} \overset{\text{def}}{=} \frac{\exp \left[ -U(\mathbf{x}, \mathbf{m}) \right]}{Z}
  \]

  - Energy function
    \[
    U(\mathbf{x}, \mathbf{m}) = \sum_{i=1}^{n} V^{(1)}(x_i, m_i)
    + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} V^{(2)}(x_i, x_j, m_i, m_j)
    + \cdots
    \]

  - Partition function
    \[
    Z = \sum_{n=0}^{\infty} \frac{e^{-|\mathcal{W}|}}{n!} \int_{\Gamma} \cdots \int_{\Gamma} g(\mathbf{x}, \mathbf{m}) \, d(x_1, m_1) \cdots d(x_n, m_n)
    \]
Homogeneous marked pairwise interacting point processes with mark dependent intensity

Specifiication:

\[ V^{(1)}(x_i, m_i) = -\log[\alpha(m_i)] \]
\[ V^{(2)}(x_i, x_j, m_i, m_j) = \phi(x_i, x_j, m_i, m_j) \]
\[ V^{(m)}(\cdot) \equiv 0, \quad m \geq 3 \]

- \( \alpha(\cdot) > 0 \) is the mark chemical activity function (intensity can depend upon mark)
- \( \phi(\cdot) > 0 \) is the mark pair potential function
  - Describes interaction between marked points
  - Interaction can depend upon marks

Mark distribution does not change with location

Intractable normalization constant \( Z \)

Did maximum pseudo-likelihood analysis

Bognar (2008) used a Bayesian framework
  - Computational; natural point and interval estimation
Ogata & Katsura (1988)

- Inhomogeneous marked Poisson point process with location dependent mark distributions
- Specification:
  \[ V^{(1)}(x_i, m_i) = -\log [\kappa(x_i)p(m_i|x_i)] \]
  \[ V^{(m)}(\cdot) \equiv 0, \quad m \geq 2 \]
  - \( \kappa(x) > 0 \) is the intensity at location \( x \)
  - \( p(\cdot|x) \) is the mark distribution at location \( x \)
  - \( \int_{\mathcal{M}} p(m|x) \, dm = 1 \) (summation in discrete case)
  - The normalization constant \( Z \) reduces to that of the unmarked inhomogeneous case (it is known in closed form)
- Performed a Bayesian analysis (using splines)
- Model does not allow for interaction between points
Proposed Model

- Inhomogeneous marked pairwise interacting point processes with location dependent mark distribution

- Specification:

  \[ V^{(1)}(x_i, m_i) = - \log \left[ \kappa(x_i) p(m_i | x_i) \right] \]

  \[ V^{(2)}(x_i, x_j, m_i, m_j) = \phi(x_i, x_j, m_i, m_j) \]

  \[ V^{(m)}(.) \equiv 0, \quad m \geq 3 \]

- Intractable normalization constant \( Z \)
Maximum pseudo-likelihood techniques are popular since avoid intractable $Z$

Frequentist framework provides for point estimation, but the complex distributional properties of the estimates makes interval estimation difficult
  ▶ Baddeley et al. (2005) formally explores the use of parametric bootstrap techniques


Lets use a Bayesian approach
Ant Nests Analysis
Voronoi Tessellations

- We need to model the:
  - inhomogeneous intensity
  - location dependent parameters of the mark distribution
- A Voronoi tessellation on $W$ partitions the space into tiles; tile $i$ consists of all points in $W$ closer to generating point $i$ than to any other generating point
- Associate a constant height with each tile
- Yields a (discontinuous) surface over $W$
- Certainly planes, splines, other tessellation techniques, etc. could be used to model the process intensity as well
Example of Voronoi Tessellation with 4 tiles
\( V^{(1)} \rightarrow \text{Intensity } \kappa(\cdot) \)

- \( \kappa(\cdot) \) is modeled by a Voronoi tessellation on \( W \)
  - Generating points
    \[
    C^K = (C^K_1, \ldots, C^K_{k^K})
    \]
    where \( C^K_i \in W \) for \( i = 1, \ldots, k^K \)
  - Tile heights
    \[
    H^K = (H^K_1, \ldots, H^K_{k^K})
    \]
    where \( H^K_i > 0 \) for \( i = 1, \ldots, k^K \)
  - Define
    \[
    \gamma^K = (C^K, H^K)
    \]
  - 2\( k^K \) parameters describing \( \kappa(\cdot) \)
- Let
  \[
  \kappa_{\gamma^K}(x)
  \]
  denote the height of the intensity tile at location \( x \)
  - i.e. if point \( x \) resides in tile \( j \), then \( \kappa_{\gamma^K}(x) = H^K_j \)
V^{(1)} \rightarrow \text{Mark Distribution } p(\cdot|x)

- Assume \( p(m|x) \) is indexed by \( M \) parameters
  \[
  \eta^p = (\eta^p_1, \ldots, \eta^p_M)
  \]
- Assume each parameter \( \eta^p_i \) can spatially vary
- Model \( \eta^p_i \) with a Voronoi tessellation on \( \mathcal{W} \)
  - Generating points \( C^p_i = (C^p_{i1}, \ldots, C^p_{ik_i}) \)
  - Tile heights \( H^p_i = (H^p_{i1}, \ldots, H^p_{ik_i}) \)
  - \( \gamma^p_i = (C^p_i, H^p_i) \)
  for \( i = 1, \ldots, M \)
- \( 2 \sum_{i=1}^{M} k_i \) parameters
- Planes, splines, etc. on \( \mathcal{W} \) could be used as well
$V^{(1)} \rightarrow p(\cdot | x)$ for Ant Nests

- Let
  $$m_i = \begin{cases} 0 & \text{Messor wasmanni (□)} \\ 1 & \text{Cataglyphis bicolor (●)} \end{cases}$$

- Let $\pi(x) = \text{probability of } \bullet \text{ at location } x$

- Model $\pi(x)$ with a Voronoi tessellation on $W$
  - Generating points $C^p = (C^p_1, \ldots, C^p_k)$
  - Tile heights $H^p = (H^p_1, \ldots, H^p_k)$
  - $\gamma^p = (C^p, H^p)$

- Let
  $$\pi_{\gamma^p}(x)$$
  denote the height of the tessellation tile at location $x$
  - i.e. if point $x$ resides in tile $j$, then let $\pi_{\gamma^p}(x) = H^p_j$

- Location dependent mark distribution
  $$m|x, \gamma^p \sim Bern(\pi_{\gamma^p}(x))$$

- $2k_p$ parameters
\( V^{(1)} \rightarrow \text{Specification for Ant Nests} \)

- \( V^{(1)} \) is indexed by \( \gamma = (\gamma^\kappa, \gamma^p) \)
- First order potential function is

\[
V^{(1)}_{\gamma}(x_i, m_i) = -\log \left[ \kappa_{\gamma^\kappa}(x_i) \pi_{\gamma^p}(x_i)^{m_i} (1 - \pi_{\gamma^p}(x_i))^{1-m_i} \right]
\]

for \( i = 1, \ldots, n \)
$V^{(2)}$ Specification

- Since spatial interaction will depend upon species, let (generalization of Strauss 1975)

$$V^{(2)}_\psi(x_i, x_j, m_i, m_j) =$$

$$h_{00}I(m_i = m_j = 0)I(\|x_i - x_j\| < b_{00}) +$$

$$h_{01}I(m_i \neq m_j)I(\|x_i - x_j\| < b_{01}) +$$

$$h_{11}I(m_i = m_j = 1)I(\|x_i - x_j\| < b_{11})$$

- $\|x_i - x_j\|$ = Euclidean distance between $x_i$ and $x_j$
- $\psi = (b_{00}, h_{00}, b_{01}, h_{01}, b_{11}, h_{11})$

- Straussian parameters $h_{..} \geq 0$
  - describes the strength of inhibition between pairs of points
  - $h_{..} > 0$ indicates spatial regularity
  - $h_{..} = 0$ corresponds to no spatial interaction

- Interaction distances $b_{..} > 0$
  - describes the distance at which two points cease to interact
Likelihood

- Let $\theta = (\gamma, \psi)$
- Likelihood is

$$f(x, m|\theta) = \frac{g(x, m|\theta)}{Z(\theta)} \overset{\text{def}}{=} \frac{\exp[-U(x, m|\theta)]}{Z(\theta)}$$

- Energy function

$$U(x, m|\theta) = \sum_{i=1}^{n} V_{\gamma}^{(1)}(x_i, m_i)$$

$$+ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} V_{\psi}^{(2)}(x_i, x_j, m_i, m_j)$$

- Normalizer is an intractable function of the parameters

$$Z(\theta) = \sum_{n=0}^{\infty} \frac{e^{-|W|}}{n!} \int_{\Gamma} \cdots \int_{\Gamma} g(x, m|\theta) \, d(x_1, m_1) \cdots d(x_n, m_n)$$
Priors, Posterior

- Priors:
  - Let $k_{\kappa} = k_{p} = 4$ (these need not be equal); the $k$’s are a sort of smoothing parameter
  - $V^{(1)}_{\gamma}$ parameters
    - $C_{1}^{\kappa}, \ldots, C_{4}^{\kappa} \sim \text{Unif}(W)$
    - $H_{1}^{\kappa}, \ldots, H_{4}^{\kappa} \sim \text{Unif}(0, 100)$
  - $V^{(2)}_{\psi}$ parameters
    - $C_{1}^{p}, \ldots, C_{4}^{p} \sim \text{Unif}(W)$
    - $H_{1}^{p}, \ldots, H_{4}^{p} \sim \text{Unif}(0, 1)
    - $h_{00}, h_{01}, h_{11} \sim \text{Unif}[0, 100)$
    - $b_{ij} \sim \text{Unif}(\min(\|x_{i} - x_{j}\|), 100)$ for $ij = 00, 01, 11$
    - 22 parameters
  - Let $p(\theta) = p(\gamma, \psi)$ denote the joint prior

- Posterior:
  - Bayesian inference will be based upon MCMC simulations from the full posterior distribution
    \[ p(\theta|x, m) \propto f(x, m|\theta)p(\theta) \]
Suppose the current parameter vector at iteration \( t \) is \( \theta(t) \).

A single iteration of the MH algorithm proceeds as follows:

1. Generate \( \theta^* \) from some proposal distribution, \( q(\theta^* | \theta(t)) \)
2. Set \( \theta(t+1) = \theta^* \) with probability

\[
\alpha(\theta(t), \theta^*) = \min \left[ 1, \frac{f(x, m | \theta^*) p(\theta^*) q(\theta(t) | \theta^*)}{f(x, m | \theta(t)) p(\theta(t)) q(\theta^* | \theta(t))} \right]
\]

\[= \min \left[ 1, \frac{g(x, m | \theta^*) Z(\theta(t)) p(\theta^*) q(\theta(t) | \theta^*)}{g(x, m | \theta(t)) Z(\theta^*) p(\theta^*) q(\theta^* | \theta(t))} \right]
\]

otherwise set \( \theta(t+1) = \theta(t) \)

The acceptance probability contains a ratio of intractable normalizing constants

\[Z(\theta(t)) / Z(\theta^*)\]

thus we can *not* do a standard MH algorithm
Back when Matt had beautiful, luscious hair...

- Estimated the intractable ratio using importance sampling, plugged-in the estimate into the acceptance probability, and accepted/rejected
  - obtaining a new importance sampling estimate within each iteration was computationally costly
- The limiting distribution *resembles* $p(\theta|x, m)$ provided importance sampling estimates are good
- However, limiting distribution is *not* $p(\theta|x, m)$
- A similar method is suggested by Liang & Jin (2013)
Exchange Algorithm (Murray et al. 2006)

- Exchange algorithm can produce samples from the exact posterior, not an approximation of it
- Suppose the current parameter vector at iteration $t$ is $\theta^{(t)}$
- A single iteration of the exchange algorithm proceeds as follows:
  1. Generate $\theta^*$ from some proposal distribution, $q(\theta^*|\theta^{(t)})$
  2. Generate a marked spatial point pattern, say $(x_w, m_w)$, from $f(\cdot|\theta^*)$
  3. Set $\theta^{(t+1)} = \theta^*$ with probability

$$\alpha(\theta^{(t)}, \theta^*) = \min \left[ 1, \frac{f(x, m|\theta^*)p(\theta^*)q(\theta^{(t)}|\theta^*)f(x_w, m_w|\theta^{(t)})}{f(x, m|\theta^{(t)})p(\theta^{(t)})q(\theta^*|\theta^{(t)})f(x_w, m_w|\theta^*)} \right]$$

$$= \min \left[ 1, \frac{g(x, m|\theta^*)p(\theta^*)q(\theta^{(t)}|\theta^*)g(x_w, m_w|\theta^{(t)})}{g(x, m|\theta^{(t)})p(\theta^{(t)})q(\theta^*|\theta^{(t)})g(x_w, m_w|\theta^*)} \right]$$

otherwise set $\theta^{(t+1)} = \theta^{(t)}$

- Exchange algorithm can be viewed as a more direct version of the Møller et al. (2006) auxiliary variable algorithm
Simulating \((x_w, m_w)\)

- Marked spatial point patterns can be generated using:
  - reversible jump MCMC (RJMCMC) algorithm of Green (1995) (the birth/death algorithm, which preceded RJMCMC, is a special case)

- Neither algorithm needs the normalization constant \(Z(\cdot)\); because we are generating spatial point patterns at a fixed \(\theta^*\), the intractable normalizers \(Z(\cdot)\) cancel in the acceptance probability

- Once the algorithm converges, we could obtain a marked spatial point pattern \((x_w, m_w)\) from \(f(\cdot|\theta^*)\) and use this to perform a single update in the exchange algorithm

- Doing updates in this way would yield samples from the exact posterior \(p(\theta|x, m)\)
Simulating \((x_w, m_w)\): The Catch

- Both the birth/death and RJMCMC algorithms require *burn-in*
- Insufficient burn-in would:
  1. yield a marked spatial point pattern \((x_w, m_w)\) whose distribution is *not* \(f(\cdot | \theta^*)\)
  2. subvert the cancelation of the intractable \(Z(\cdot)\)'s
  3. and yield the wrong limiting distribution
Simulating \((x_w, m_w)\): Burn-in Period

- Lets use a very conservative (long) burn-in period
- Trace plot of the total energy \(U\) during the burn-in period of the birth/death algorithm for generating \((x_w, m_w)\) from \(f(\cdot|\theta^*)\) (starting from a Poisson spatial point pattern)

![Trace plot of total energy during burn-in period](image)

- Used a 5,000 iteration burn-in period for the birth/death algorithm
- Very computational
  - Every iteration of the exchange algorithm requires running a new birth/death (or RJMCMC) algorithm, with sufficient burn-in, to obtain \((x_w, m_w)\) from \(f(\cdot|\theta^*)\)
Simulating \((x_w, m_w)\): Notes

- Tracking the total energy \(U(\cdot)\) only provides *rough guidance* on “convergence”
- Insufficient burn-in of the birth/death (RJMCMC) algorithm typically yields an over-dispersed posterior distribution (and incorrect point estimates)
  - Burn-in period of 50 iterations yielded over-dispersed posterior
  - Burn-in period of 500 iterations yielded posterior inferences similar to the 5,000 iteration burn-in period
- What about perfect (exact) sampling?
  - Appears much too slow to use within the exchange algorithm
  - Appears much more difficult to implement
  - Plus Matt is too lazy to code this
Proposals

- Within each iteration of the exchange algorithm, a parameter was randomly selected (random scan algorithm, one-at-a-time)
  - Uniform random walk proposals
  - Variance of the proposals was tuned to obtain acceptance rates from 20-50%
Computational Details

- Used UI Neon Cluster (LT node)
- Coding in C++ (delicate, tedious, but readable)
- Largest computational burden: the \( \binom{n}{2} \) pairwise Euclidean distances needed to evaluate \( V^{(2)}_{\psi} \)
  - Must be computed (or updated) for every iteration of the birth/death (or RJMCMC) algorithm that simulates \((x_w, m_w)\) from \( f(\cdot | \theta^*) \) (could do a subset of distances if we smartly update; more prone to coding errors)
- Used OpenMP and the Intel MKL library to parallelize across 16 Intel Xeon processors and an Intel Phi co-processor (with 240 cores)
  - Parallelization reduced computing time only 50% or so
    - Lots of inter-processor communication
    - Small dataset \((n = 62)\) and small simulated datasets \((x_w, m_w)\) limit speed gains
- The exchange algorithm was run for 50,000 iterations (following a 5,000 iteration burn-in period)
- Animation
Results: $\kappa_{\gamma\kappa}(x)$

- Marginal posterior mean of intensity $\kappa_{\gamma\kappa}(x)$ (50 × 50 grid)
- Unlike a Poisson process, the intensity is not proportional to the density of points
- Voronoi tessellation is crude; posterior mean is surprisingly smooth
Results: \( \pi_{\gamma p}(x) \)

- Marginal posterior mean of \( \pi_{\gamma p}(x) \) (probability of Cataglyphis bicolor ●) under pairwise interaction model
Results: $V^{(2)}_\psi$ Parameters

- Posterior mean and 95% credible set for the $V^{(2)}_\psi$ parameters

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<tr>
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<th>Mean</th>
<th>95% Cred. Set</th>
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<th>Mean</th>
<th>95% Cred. Set</th>
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<tr>
<td>$b_{00}$</td>
<td>47.13</td>
<td>(37.06, 58.76)</td>
<td>$h_{00}$</td>
<td>0.556</td>
<td>(0.256, 0.920)</td>
</tr>
<tr>
<td>$b_{01}$</td>
<td>37.51</td>
<td>(8.70, 71.93)</td>
<td>$h_{01}$</td>
<td>0.285</td>
<td>(0.011, 1.245)</td>
</tr>
<tr>
<td>$b_{11}$</td>
<td>41.44</td>
<td>(32.04, 49.03)</td>
<td>$h_{11}$</td>
<td>2.628</td>
<td>(1.060, 5.914)</td>
</tr>
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- Messor wasmanni (□) exhibit moderate regularity ($\hat{h}_{00} = 0.556$)
- Cataglyphis bicolor (●) exhibit strong repulsion ($\hat{h}_{11} = 2.628$)
- Repulsion of the between-species nest locations is quite weak (not unlike CSR) ($\hat{h}_{01} = 0.285$)
- Other authors fixed $b_{..} = 45$
- Results agree with Takacs & Fiksel (1986) and Särkkä (1993) (used fixed $b$’s)
Big Question: Does Modeling the Interaction Matter?

- Marginal posterior mean of $\pi_{\gamma p}(x)$ (probability of Cataglyphis bicolor •) under non-interaction (Poisson) model ($V(i) \equiv 0$ for $i \geq 2$)
Why the difference?

- Cataglyphis (●) are strongly inhibitory
- Messor (□) are better able to exist near each other
- Thus, process must push harder to yield a Cataglyphis nest (●) near the bottom of \( W \)
- Hence, the underlying probability of a Cataglyphis nest (●), \( \pi_{\gamma}^p (\cdot) \), must be higher near the bottom of \( W \)
Mini Simulation Study
Simulation Study: $V^{(1)}$ Specification

- Dataset was simulated on the $500 \times 500$ square region
- First order potential:

\[
\begin{align*}
\kappa(x) &= 0.001 \\
p(m|x) &= N(0.5, 0.1) \\
\kappa(x) &= 0.001 \\
p(m|x) &= N(1.5, 0.1) \\
\kappa(x) &= 0.002 \\
p(m|x) &= N(0.5, 0.1) \\
\kappa(x) &= 0.002 \\
p(m|x) &= N(1.5, 0.1)
\end{align*}
\]
Simulation Study: $V^{(2)}$ Specification

- The second order potential function was

$$V^{(2)}_{\psi}(x_i, x_j, m_i, m_j) = \begin{cases} h & \|x_i - x_j\|s_{ij} \leq b \\ 0 & \text{otherwise} \end{cases}$$

where $s_{ij} = (0.5(m_i + m_j))^{-1}$ and $\psi = (b, h)$

- The interaction distance scales according to the mark
  - For two points with small marks (near 0.5), $s_{ij} \approx 2 \implies$ interaction distance $\approx b/2$
  - For two points with large marks (near 1.5), $s_{ij} \approx 2/3 \implies$ interaction distance $\approx 3b/2$
  - One large and one small mark will have interaction distance $\approx b$

- Used $b = 20$ and $h = 1$
Simulated Dataset

- Simulated dataset had 198 points (circle is proportional to mark)
Results: $V^{(1)}$

- The $k = 2$ Voronoi tiles were fixed at "true" locations
- Uniform priors, RW proposals, 10,000 updates
- Posterior means and 95% credible intervals
Results: $V^{(2)}_{\psi}$

- Second order potential parameters:

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<th>Post. Mean</th>
<th>95% Cred. Int.</th>
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<tbody>
<tr>
<td>$b$</td>
<td>20</td>
<td>19.91</td>
<td>(18.76, 21.12)</td>
</tr>
<tr>
<td>$h$</td>
<td>1</td>
<td>0.979</td>
<td>(0.596, 1.284)</td>
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Summary
Discussion

- It is important to model interaction in marked spatial point patterns to better understand underlying process
- Bayesian approach:
  - provides inference for all parameters; Frequentist approaches typically fix interaction distance(s) (e.g. $b_{00}, b_{01}, b_{11}$)
  - provides natural interval estimates (no bootstrap)
  - seems to allow much more complex modeling than Frequentist approaches
  - Can easily get credible sets for intensity and parameters of the mark distribution at any location $x \in W$ (not shown)
- Gibbs model is more flexible/general than I thought
Open Questions

- Multiple marks
- Allow number of tiles $k$ to vary in MCMC sampler
  - Not clear if Exchange algorithm can be modified to handle a dimension change
  - Not clear if RJMCMC can be modified to handle an intractable normalizer
- Use Pseudo-Marginal algorithm (Andrieu & Roberts 2009) instead of Exchange algorithm
  - Needs more work; mixes poorly in current setup
  - Worked great with diffusions (Stramer & Bognar 2011)
- Edge effects (literature suggests it is minimal for ants dataset)
- We assumed interaction parameters $\psi$ didn’t change across the space $W$
  - Want interaction parameters to depend upon location, i.e.

$$\psi(x_i, x_j)$$
Thank You
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Diggle, P. J. (2003), Statistical Analysis of Spatial Point Patterns, second edn, Arnold.

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