Homework (Linear Combination of Normal Distributions)
Prob. and Stat. for Eng. (STAT:2020; Bognar)

1. Suppose X_1, X_2, and X_3 are independent random variables where

$$X_1 \sim N(\mu_1 = 1, \sigma_1^2 = 1^2)$$
$$X_2 \sim N(\mu_2 = 2, \sigma_2^2 = 2^2)$$
$$X_3 \sim N(\mu_3 = 3, \sigma_3^2 = 3^2)$$

   (a) Let $W = X_1 - X_2$. What is the distribution of W? Be sure to state all parameters. *Show your work using clear notation.*

   (b) Using your answer in (1a), find $P(X_1 > X_2)$. *Show your work using clear notation.*

   (c) Let $W = X_1 - 6X_2 + 2X_3$. What is the distribution of W? Be sure to state all parameters. *Show your work using clear notation.*

   (d) Using your answer in (1c), find $P(X_1 - 6X_2 > 5 - 2X_3)$. *Show your work using clear notation.*

2. An artist makes pottery. There are two major steps: wheel throwing and firing. The time (in minutes) for wheel throwing can be modeled by a $X_1 \sim N(\mu = 40, \sigma^2 = 2^2)$ distribution and the time for firing can be modeled by a $X_2 \sim N(\mu = 60, \sigma^2 = 3^2)$ distribution. Assume independence.

   (a) Determine the probability that a piece of pottery will be completed in less than 95 minutes.

   (b) Determine the probability that a piece of pottery will take longer than 110 minutes.

   (c) Determine the probability that $2X_1 > 1.5X_2$.

   (d) Determine the probability that $2X_1 > X_2 + 15$.

   (e) Suppose 10 pieces of pottery are randomly selected. Determine the probability that the mean firing time $\bar{X}$ is between 58 and 61 minutes.

   (f) Suppose 10 pieces of pottery are randomly selected. Let $\bar{X}$ denote the sample mean firing time. Determine the 10th percentile of $\bar{X}$.

   (g) Determine the probability that 2 pieces of pottery will take less than 210 minutes using a linear combination. *Think carefully when doing this problem. Note that we can not simply find $P(2X_1 + 2X_2 < 210)$. Hint: Let Y_1 denote the completion time for the first piece, and let Y_2 denote the completion time for the second piece.*

3. Suppose $X_1, \ldots, X_{25}$ are independent and identically distributed normal random variables with mean $\mu = 100$ and standard deviation $\sigma = 20$, i.e.

$$X_i \overset{iid}{\sim} N(\mu = 100, \sigma^2 = 20^2 = 400)$$

   for $i = 1, \ldots, 25$. Let the sample mean $\bar{X} = \frac{1}{25}\sum_{i=1}^{25} X_i$.

   (a) Find $P(98 < \bar{X} < 105)$.

   (b) Find the 10th percentile of $\bar{X}$.

4. Suppose X_1, X_2, and X_3 are independent random variables where

$$X_1 \sim N(\mu_1 = \mu, \sigma_1^2 = 1^2)$$
$$X_2 \sim N(\mu_2 = \mu, \sigma_2^2 = 2^2)$$
$$X_3 \sim N(\mu_3 = \mu, \sigma_3^2 = 3^2)$$

   If $0.9 = P(5X_1 + 2X_2 - 4X_3 > 10)$, find μ.