22M174/22C174: Optimization techniques.

Homework 5. Due 03/13/13.

1. Apply Algorithm I.4.2.5 (BFGS on quadratics with exact line-search, p. I.25) by hand to the quadratic

$$q(x_1, x_2) = x_1 - \frac{3}{4}x_2 + \frac{4}{9}x_1^2 - 2x_1x_2 + 3x_2^2$$

starting with

$$x_0 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

Compute the updates x_k and B_k and stop when the algorithm has found the minimizer or $k = 4$. Modify Algorithm I.4.2.5 for the DFP update and repeat the calculations for this case. Are the updates B_k for BFGS and DFP close to each other and to the exact Hessian? If yes, at which iteration?

2. Given a set of $m+1$ linearly independent vectors $\{d_0, \ldots, d_m\}$ in $\mathbb{R}^n$ $(m+1 \leq n)$ and a symmetric positive definite matrix H, show that the following *(generalized) Gram-Schmidt procedure*

$$p_0 := d_0, \quad p_j := d_j - \sum_{i=0}^{j-1} \frac{d_j^T H p_i}{p_i^T H p_i} p_i \quad \text{for } j = 1, \ldots, m$$

produces a set of vectors $\{p_0, \ldots, p_m\}$ conjugate with respect to matrix H. Hint: prove this result by induction on j.

3. For conjugate direction methods (Algorithm I.4.3.2 on p. I.28 with H replaced by A or Algorithm 3.5.1 on p.72 of the typed notes) show that the "quasi-Newton" condition

$$As_k = y_k$$

holds where $s_k := x_{k+1} - x_k, y_k := g_{k+1} - g_k, g_k := Ax_k - b$. Then show that the conjugacy condition $p_i^T A p_j = 0$ for $i \neq j$ is equivalent to the orthogonality condition $y_i^T p_j = 0$ for $i \neq j$ as long as $\alpha_i \neq 0$. Hint: use the relation $g_{k+1} = g_k + \alpha_k A p_k$.

4. Show by induction on k that for conjugate direction methods (Algorithm I.4.3.2 on p. I.28 with H replaced by A or Algorithm 3.5.1 on p.72 of the typed notes) we have $g_k^T p_i = 0$ for $i = 0, \ldots, k-1$. Hint: use the relation $g_{k+1} = g_k + \alpha_k A p_k$.

5. Show that for the linear conjugate gradient (CG) method (Algorithm I.4.3.4 on p. I.30 or Algorithm 3.5.2 on p.74 of the typed notes) the coefficients α_k and β_k satisfy

$$\alpha_k = \frac{g_k^T g_k}{p_k^T A p_k}, \qquad \beta_k = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k}.$$

6. We consider the system of linear equations $Ax = b$ given by the following symmetric and positive definite matrix

$$A = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}$$

of dimension $n = 15$ and right-hand-side b given by $b := Ax^\star$ where $x^\star := [1, -2, 3, -4, \ldots, -14, 15]^T$. Starting from $x_0 := b$ (apply the conjugate gradient (CG) method (Algorithm I.4.3.4 on p. I.30 or Algorithm 3.5.2 on p.74 of the typed notes). Print the forward relative errors

$$\frac{\|x^\star - x_k\|_A}{\|x^\star\|_A}$$

for $k = 0, \ldots, 15$ where $\|v\|_A := \sqrt{v^T A v}$.