# 22M174/22C174: Optimization techniques.

# Homework 4. Due 03/01/13.

1. Find the two zeros of the system of nonlinear equations

$$F(x_1, x_2) := \begin{bmatrix} x_1^2 + x_2^2 - 9 \\ x_1 + x_2 - 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let $x_0 := [5, 1]^T$ and $B_0 := F'(x_0)$. Carry out two iterations of Broyden's method. Show that for $k \geq 0$ we have $(B_k)_{21} = 1, (B_k)_{22} = 1$. Show that the equation $(x_k)_1 + (x_k)_2 - 3 = 0$ is satisfied for $k \geq 1$, hence $[1, 1]s_k = 0$ for $k \geq 1$. Then show that $(B_{k+1} - B_k)[1, 1]^T = 0$ for $k \geq 1$, hence $(B_k)_{11} + (B_k)_{12} = 9$ for $k \geq 1$. Assuming $F(x_k) \neq 0$ and $\lim_{k\to\infty} x_k = [3, 0]^T$, from the quasi-Newton equation show that

$$\lim_{k\to\infty} ((B_k)_{11} - (B_k)_{12}) = 6.$$

Do we have $\lim_{k\to\infty} B_k = F'(x^*)$ where $x^*$ is one of the zeros?

2. Consider the system of nonlinear equations

$$F(x_1, x_2) := \begin{bmatrix} \sin(x_1 e^{3x_2} - 1) \\ x_1^3 x_2 + x_1^3 - 7x_2 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has a zero at $x^* = [1, 0]^T$.

   (a) Starting from $x_0 = [1.3, -0.15]^T$ apply Newton's method until $\|x^* - x_k\|_2 \leq 10^{-14}$. At each iteration print the two components of $x_k$ and the error $\|x^* - x_k\|_2$.

   (b) Starting from $x_0 = [1.3, -0.15]^T$ with $B_0 := F'(x_0)$ (or $C_0 := (F'(x_0))^{-1}$), apply Broyden's method until $\|x^* - x_k\|_2 \leq 10^{-14}$. At each iteration print the two components of $x_k$ and the error $\|x^* - x_k\|_2$.

3. Consider

$$B_k = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad s_k = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad y_k = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

Observe that $B_k$ is symmetric. Show that $B_k$ is positive definite. Compute Broyden's update $B_{k+1}$ (II.4.3.3) and show that it is neither symmetric nor positive definite. Compute the SR1 update $B_{k+1}^{SR1}$ (on p. I.22) and show that it is symmetric, but not positive definite. Compute the BFGS update $B_{k+1}^{BFGS}$ (I.4.2.4 on p. I.23) and the DFP update $B_{k+1}^{DFP}$ (I.4.2.6 on p. I.24) and show that both updates are symmetric and positive definite.

4. Prove that for
$$\phi := \frac{y_k^T s_k}{y_k^T s_k - s_k^T B_k s_k}$$
in the Broyden class of updates (I.4.2.8 on p. I.24) we obtain the SR1 update (on p. I.22).