22M174/22C174: Optimization techniques.

Homework 3. Due 02/20/13.

1. Consider a matrix $M \in \mathbb{R}^{n\times n}$. Show that there exists a unique symmetric matrix $S \in \mathbb{R}^{n\times n}$ ($S^T = S$) such that
$$x^T M x = x^T S x \quad \forall x \in \mathbb{R}^n.$$
Give explicitly the matrix S.

2. Prove that the inverse M^{-1} of a positive definite matrix $M \in \mathbb{R}^{n\times n}$ (not necessarily symmetric) is positive definite. Hint: do not try to prove these results using eigenvalues, because you cannot assume that the eigenvalues of a nonsymmetric positive definite matrix M are real, for example
$$M = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$
is nonsymmetric, positive definite, but its eigenvalues are complex ($\lambda_{1,2}(M) = 2 \pm i$). Even if a nonsymmetric matrix has real positive eigenvalues, it does not imply that the matrix is positive definite. For example the eigenvalues of
$$M = \begin{bmatrix} 9 & 4 \\ -4 & -1 \end{bmatrix}$$
are $\lambda_1(M) = 1, \lambda_2(M) = 7$. However, this matrix is not positive definite, since $p^T M p = 9p_1^2 - p_2^2$.

3. Does the following quadratic objective function
$$q(x_1, x_2) = 3 - 3x_1 - 3x_2 + x_1^2 + 2x_1x_2 + x_2^2$$
have a global minimum value? If yes, find all the global minimizer(s) and give the minimum value of q.

4. Consider the nonlinear system $F(x) = 0$ where $F : \mathbb{R}^n \to \mathbb{R}^n$.

   (a) Show that Newton's method is *affine invariant*, i.e., Newton's method applied to $F(x) = 0$ starting at x_0 and Newton's method applied to $G(y) := MF(Ny + v) = 0$ starting at y_0 satisfying $Ny_0 + v = x_0$ where M and N are two invertible matrices in $\mathbb{R}^{n\times n}$ and $v \in \mathbb{R}^n$, lead to iterates x_k and y_k satisfying $Ny_k + v = x_k$ for $k = 0, 1, 2, \ldots$

(b) Show that Newton's method is in general not invariant when $F(x) = 0$ is transformed to $G(x) := M(x)F(x) = 0$ where $M(x) \in \mathbb{R}^{n\times n}$ is an invertible matrix function. Write down the linear models (used for Newton's method around a point x_k) $L_{F,k}(x) \approx F(x)$ and $L_{G,k}(x) \approx G(x)$. Is in general the zero x_{k+1} of $L_{F,k}(x) = 0$ also the zero of $L_{G,k}(x) = 0$? Remark:

$$DM(x)(\cdot,\cdot)$$

is a bilinear application

$$(DM(x)(u,v))_i = \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{\partial M_{ij}}{\partial x_k}(x) u_j v_k.$$

5. *Sherman-Morrison-Woodbury formula.* Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix, show that for $U, V \in \mathbb{R}^{n\times m}$ we have $A + UV^T \in \mathbb{R}^{n\times n}$ is invertible $\iff I_m + V^T A^{-1} U \in \mathbb{R}^{m\times m}$ is invertible. Hint: in this situation, show that we have

$$\begin{aligned}(A+UV^T)^{-1} &= A^{-1} - A^{-1}U(I_m + V^T A^{-1}U)^{-1}V^T A^{-1},\\ (I_m + V^T A^{-1}U)^{-1} &= I_m - V^T(A+UV^T)^{-1}U.\end{aligned}$$