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Unitary representations and Osterwalder-Schrader duality. (English. English summary)

The mathematical legacy of Harish-Chandra (*Baltimore*, *MD*, 1998), 333–401, *Proc. Sympos. Pure Math.*, 68, *Amer. Math. Soc.*, *Providence*, *RI*, 2000.

FEATURED REVIEW.

The paper successfully glues various topics together and develops a beautiful duality principle in representation theory. The topics are as follows: (1) reflection positivity (which was originally related to a reflection in the time variable in quantum field theory), (2) causal symmetric spaces, (3) functional integration, (4) the spectral theory of one-parameter unitary groups in Hilbert space, (5) the Segal-Bargmann transform, (6) unitary highest weight modules for semisimple Lie groups. The main idea is to construct a duality operation for certain unitary representations of semisimple Lie groups; at the group level, this operation is connected with the c-duality of causal symmetric spaces that relates "compactly causal" spaces to "noncompactly causal" ones.

Recall the main definitions. A symmetric Lie algebra is a pair $(\mathfrak{g},\tau)$, where $\mathfrak{g}$ is a finite-dimensional real Lie algebra and τ is an involutive automorphism of $\mathfrak{g}$. Let $\mathfrak{h} := \{X \in \mathfrak{g}\colon\ \tau(X) = X\}$ and $\mathfrak{q} := \{X \in \mathfrak{g}\colon\ \tau(X) = -X\}$. A symmetric simple Lie algebra is said to be compactly causal (or Hermitian symmetric) if $\mathfrak{q}$ contains a regular H-invariant cone C such that the interior C^0 of C contains a point of $\mathfrak{k}$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition related to a Cartan involution commuting with τ. The c-dual pair of $(\mathfrak{g},\tau)$ is formed by the Lie algebra $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q} \subset \mathfrak{g}_{\mathbf{C}}$ with the involution $\tau^c := \tau_{\mathbf{C}}|\mathfrak{g}^c$. A unitary representation π of G in a Hilbert space H is said to be reflection symmetric if there is a unitary operator J on H such that $J^2 = \mathrm{id}$ and $J\pi(g) = \pi(\tau(g))J$, $g \in G$.

At the level of representations, for a given unitary representation of a group of the form G^c, one can use an involution on the representation space satisfying a certain positivity condition on a subspace to produce a contraction representation of a semigroup of the form $H\exp(C)$, where $H = (G^c)^{\tau^c}$, and C is an H-invariant convex cone lying in the space of τ^c-fixed points in the Lie algebra G^c. Applying the Luscher-Mack theorem [M. Luscher and G. Mack, Comm. Math. Phys. 41 (1975), 203–234; MR 51#7503] (a kind of analytic continuation of a representation from a group to its c-dual) or the Jorgensen theorem on the analytic continuation of local representations [Pacific J. Math.

125 (1986), no. 2, 397–408; MR 88m:22030], the authors construct a reflection symmetric unitary representation of the group G in the same space and completely identify this representation provided that the associated Riemannian symmetric space for the c-dual pair $(\mathfrak{g}^c, \tau^c)$ is a Cayley-type symmetric space (i.e., is related to one of the Lie algebras $\mathfrak{sp}(n, \mathbf{R})$, $\mathfrak{su}(n,n)$, $\mathfrak{so}^*(4n)$, $\mathfrak{so}(2,k)$, and $\mathfrak{e}_{7,-25}$). The duality thus obtained works well for highest weight modules, and this can be extended at the group level for the generalized principal series representations and for complementary series representations. When obtaining these results, the authors were forced to develop an additional technique related to bounded symmetric domains.

The authors explain the relationship of the above duality with the approach to the quantum field theory developed by K. Osterwalder and R. Schrader [Comm. Math. Phys. 31 (1973), 83–112; MR 48#7834] that is based on the reflection positivity (and the c-duality between the Euclidean motion group E_n, which is a semidirect product of $\mathrm{SO}(n)$ and $\mathbf{R}^n$, and the universal covering of the Poincare group). The technique involves the reflection positivity phenomenon for a path space measure.

The main technical step which is heavily used in the proof of the main duality result is the following two-step operator-theoretic lemma, which is of independent interest. (1) Let J be a period-two unitary operator in a Hilbert space H, and let $K_0 \subset H$ be a closed subspace such that $\langle v|J(v)\rangle \geq 0$ for $v \in K_0$. Let γ be an invertible operator on H such that $J\gamma = \gamma^{-1}J$ which leaves K_0 invariant and for which $(\gamma^{-1})^*\gamma$ is bounded on H. Then γ induces a bounded operator $\widetilde{\gamma}$ on the (Gelfand-Naimark-Segal) Hilbert space $K = (\widetilde{K_0/N})$, where $N = \{v \in K_0|\ \langle v|Jv\rangle = 0\}$, and the norm $\|\widetilde{\gamma}\|$ with respect to the J-inner product does not exceed the spectral radius of $(\gamma^{-1})^*\gamma$. (2) If S is a semigroup of operators satisfying the conditions in (1), then $\widetilde{(\gamma_1\gamma_2)} = \widetilde{\gamma_1}\widetilde{\gamma_2}$ for $\gamma_1, \gamma_2 \in S$. This result is used when constructing contractive self-adjoint representations of semigroups associated with cones in the corresponding Lie groups (and for the semigroup $\mathbf{R}^+$ in the one-parameter case). In the reviewer's opinion, it should be noted that, despite clear differences in the setting, the situation treated in the lemma resembles that occurring in the Tomita-Takesaki theory for von Neumann algebras.

The exposition contains a lot of examples and direct calculations (the $(ax+b)$-group, one-parameter group, $\mathrm{SU}(1,1)$, the Heisenberg group, the Poincare group, etc.) which help one to understand the entire web of relationships. In particular, the generalized Bargmann transform is also included in the context of the above duality.

Unfortunately, it is impossible to discuss the details of the exposition because the volume of such a discussion would be comparable to that of the original text. Some problems treated in the paper were also clarified in [P. E. T. Jorgensen and G. Olafsson, J. Funct. Anal. 158 (1998), no. 1, 26–88; MR 99m:22013] and developed in [J. Hilgert and B. Krotz, J. Funct. Anal. 169 (1999), no. 2, 357–390; MR 2001d:22010].

The geometry and the representations in use are closely related to those studied in Harish-Chandra's papers on bounded symmetric domains and holomorphic discrete series, and the approach is significantly influenced by Segal's works on causality and invariant cones. This completely explains the dedication of this paper (published in the Harish-Chandra memorial volume) to the memories of these two outstanding mathematicians.

{For the entire collection see MR 2001b:22001.}

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