Introduction

The workshop was centered around two important topics in modern harmonic analysis: Wavelets and frames, as well as the related topics Gabor analysis and operator algebras. The fundamental concepts involved rely on operator theory, as well as on the theory of representations of groups and algebras. The applications include such areas as PDE, numerical analysis, and physics. Partial travel funding was provided by grants in Austria, Germany, and the USA. The three organizers from the US, are part of a Focused Research Group (FRG), funded by the US National Science Foundation (NSF), and two other participants are in this FRG group, Professors Chris Heil, GATECH, USA, and Akram Aldroubi, Vanderbilt University, USA. The organizers thank the US NSF for partial support.

One of the first thing that a student in a linear algebra class is confronted with is the notion of a basis. A basis $\{v_j\}_j$ allows us to write each element $u \in V$ in a unique way as linear combination $u = \sum_j c_j u_j$. For many applications and practical problems, the uniqueness is not necessarily desirable, and often one prefer a generating set of a special form. This leads naturally to the definition of a frame, that we would like to recall here. Let V be a (complex or real) Hilbert space. A sequence $\{v_j\}$ in V is called a *frame* if there exists positive numbers $0 < A \leq B$ such that for each $v \in V$, we have:

$$A\|v\|^2 \leq \sum_j |(v, v_j)|^2 \leq B\|v\|^2 \,.$$

The numbers A and B are called the *frame bounds*. This definition is equivalent to the requirement, that the operator $S : V \to \ell^2$, $S(v) = \{(v, v_j)\}_j$ is continuous with continuous inverse $S^{-1} : \mathrm{Im}(S) \to V$. Gabor frames, are frames in $L^2(\mathbb{R}^d)$ constructed by *modulations* and *translations*, i.e., given a square-integrable function g our sequence is $g_{nm}(x) = e^{2\pi i(m,x)}g(x-n)$, $(n, m) \in \Lambda$, where Λ is a discrete subset of $\mathbb{R}^d$. On the other hand, the wavelet frames are constructed using dilations and translations. Thus given a set $\Delta \subset \mathrm{GL}(d, \mathbb{R})$ and $\Gamma \subset \mathbb{R}^d$, as well as a suitable square integrable function ψ, we set $\psi_{D,\gamma}(x) = |\det D|^{1/2}\psi(Dx + \gamma)$. The reader can find several interesting questions and problems related to those concepts in the following abstracts.

We would like to describe two simple problems here. If the density of the points in Λ is too small, then a Gabor frame cannot be constructed, and if the density is too large, then one can construct a frame, but not a basis. In particular the sequence $\{g_{nm}\}_{(n,m)\in\Lambda}$ is always linearly independent. The tutorial by C. Heil discussed the simple question: what about the case where Λ is finite, is the sequence $\{g_{mn}\}$ then linearly independent or dependent? The answer to this simple question is unknown, except in some trivial cases. In the wavelet case, an important question has been the construction of *wavelet sets*. Given the set Δ and Λ, find the measurable subsets $\Omega \subset \mathbb{R}^d$ of positive, and finite measure, such that, with $\hat{\psi} = \chi_\Omega$, the sequence $\{\psi_{D,\gamma}\}$, is an orthogonal basis for $L^2(\mathbb{R}^d)$. Such a set is called a *wavelet set*. This line of work includes both geometry (tilings of $\mathbb{R}^d$) and analysis (the Fuglede conjecture).

Rather than formal presentations of recent advances in the field, this workshop tried instead to aim at outlining the important problems and directions, as we see it, for future research, and to discuss the impact of the current main trends. In particular, the speakers were constantly interupted with questions and comments.

A special problem session was organized by D. Larson one afternoon. Another afternoon session was devoted to an informal discussion of further open problems, new directions, and trends. The topics that emerged in these discussions included the following general areas:

(1) Functional equations and approximation theory: wavelet approximation in numerical analysis, PDE, and mathematical physics. At the meeting, we discussed some operator theoretic methods that resonate with what numerical analysts want, and questions about localizing wavelets. Two workshop lectures covered connections to numerical analysis and PD.

(2) Gabor frames: We had many discussion, much activity, and several talks on aspects of this. H. Feichtinger explained some important results and open problems involving frames and Gelfand triples, K-H Gröchenig gave a lecture on new formulations and results Wiener's type Lemmas in particular for twisted convolution algebras and Gabor frames. They are striking for applications; in that they yield sharper frame bounds. And they involve non-commutative geometry, and other operator algebraic tools.

(3) Continuous vs discrete wavelet transforms: we had several talks at the Oberwolfach workshop where the various operations, translation, scaling, phase modulation, and rotation, get incorporated into a single group. Links to representations of Lie groups! This viewpoint seems to hold promise for new directions, and for unifying a number of current wavelet constructions, tomography, scale-angle representations, parabolic scaling, wavelet packets, curvelets, ridgelets, de-noising ... Wavelets are usually thought of as frames in function spaces constructed by translations and dilations. Much less is understood in the case of compact manifolds like the n-dimensional sphere, where both "dilations" and "translations" are not obviously defined. The talk by Ilgewska-Nowak explained some joint work with M. Holschneider on construction of discrete wavelet transform on the sphere.

(4) Harmonic analysis of Iterated Function Systems (IFS). Several of the participants have worked on problems in the area, and Jorgensen spoke about past work, and directions for the future. The iterated function systems he discussed are closely related to the study of spectral pairs, and the Fuglede problem. Recent work by Terence Tao makes the subject especially current.

(5) Multiplicity theory, spectral functions, Grammians, generators for translation invariant subspaces, and approximation rates. We had joint activity at the workshop on problems in the general area, and we anticipate joint papers emerging from it. A. Aldroubi lectured on the engineering motivations. In particular he discussed translation invariant subspaces of $L^2(\mathbb{R})$ where two lattice-scales are involved, and issues about localizing the corresponding generating functions for such subspaces.

(6) Decompositions of operators and construction of frames: D. Larson discussed the problem of when is a positive operator a sum of finitely many orthogonal projections, and related it to frame theory. Problems and some recent results and techniques of D. Larson and K. Kornelson were discussed in this context, involving other related types of targeted decompositions of operators. In response, H. Feichtinger and K-H Gröchenig pointed out that similar techniques just may lead to progress on a certain problem in modulation space theory. There are plans to follow up on this lead.

The organizers:

H. Feichtinger, P. Jorgensen, D. Larson, and G. Ólafsson