HW 2.1: 2, 4, 9, 11 and 2.2: 1, 2, 3-7, 12, 14, 17,

Defn: A graph is a pair of sets (V, E) where the elements of V are called vertices and the elements of E are unordered pairs of elements of V. The elements of E are called edges.

Defn: A directed graph or digraph is a pair of sets (V, E) where the elements of V are called vertices and the elements of E are ordered pairs of elements of V. The elements of E are called arcs.

Defn: A digraph is said to have multiple arcs if there exists more than one arc from u to v.

Hence a digraph that contains two arcs from u to v has multiple arcs, but a digraph that contains exactly one arc from u to v and one arc from v to u does NOT have multiple arcs.

Defn: A graph is said to have multiple edges if there exists more than one edge between u and v.

Note the above definitions mean that we are assuming (unless otherwise stated) that graphs do not contain multiple edges and digraphs do not contain multiple arcs.

In chapters 2 and 3, we will also assume (unless otherwise stated) that graphs and digraphs have a finite number of vertices and do not contain loops.

2.2: Connectedness

table 2.1

v reachable from u if there exists a path from u to v.

Thm 2.1: If v is reachable from u, then there is a simple path from u to v.

Proof 1 (ugly version): If v is reachable from u, there exists a path $u = u_1, u_2, ..., u_{n-1}, u_n = v$

If $u_1 = u_2$ let $p_1 =$ the path $u_2, ..., u_n$
If $u_1 \neq u_2$ let $p_1 =$ the path $u_1, u_2, ..., u_n$

Suppose the path $p_{k-1} = u_{m_1}, ..., u_{m_j}, u_k, u_{k+1}, ..., u_n$ is defined such that

1.) $m_i < m_r$ if $i < r$.
2.) $m_i < k$ for $i = 1, ..., j$.
3.) $u_k \neq u_{m_i}$ for $i = 1, ..., j$.

If $u_k = u_{m_i}$ for some $i = 1, ..., j$
let $p_k =$ the path $u_{m_1}, ..., u_{m_i}, u_{k+1}, ..., u_n$
If $u_k \neq u_{m_i}$ for some $i = 1, ..., j$
let $p_k =$ the path $u_{m_1}, ..., u_{m_j}, u_k, u_{k+1}, ..., u_n$

The path p_{n-1} is a simple path from u to v.

v joined to u if there exists a semipath between u and v.

2.2.2 Distance

The length of a path is the number of arcs in the path

If v is reachable from u, $d(u, v) =$ the distance between u and $v =$ the length of a shortest path from u to v

Note

0.) $d(u, v)$ is not defined if v is NOT reachable from u.

1.) $d(u, u) = 0$.

2.) $d(u, v) \neq d(v, u)$.

3.) If w is reachable from u and v is reachable from w, then v is reachable from u and $d(u, v) \leq d(u, w) + d(w, v)$.

2.2.4 Connectedness Categories

A digraph is strongly connected or strong if for every pair of vertices u and v, u is reachable from v AND v is reachable from u.

A digraph is unilaterally connected or unilateral if for every pair of vertices u and v, u is reachable from v OR v is reachable from u.

A digraph is weakly connected or weak if every pair of vertices u and v is joined.

A digraph is disconnected if it is not weakly connected.

A digraph is connected of

degree 0 if it is disconnected

degree 1 if it is weak, but not unilateral

degree 2 if it is unilateral, but no strong

degree 3 if it is strong

2.2.5 Criteria for Connectedness

Thm 2.3: A digraph is strongly connected if and only if it has a complete closed path.

Lemma: In any set of vertices of a unilateral digraph D, there is a vertex which can reach all others in the set.

Thm 2.4: A digraph is unilaterally connected if and only if it has a complete path.

Thm 2.5: A digraph is weakly connected if and only if it has a complete semipath.

2.2.6: A graph is connected if between any pair of vertices, there is a chain. Thus a graph is connected if and only if it has a complete chain.