(1)

$$y''' - 4y'' + y' + 6y = 8e^{4t}$$

~~§3.5: Guess $y = Ae^{4t}$ (since not homog)~~ easier for this example

~~§3.6: $\Psi = u_1\phi_1 + u_2\phi_2 + u_3\phi_3$~~

Step 1: Solve homog

Guess $y = e^{rt}$

$$r^3 - 4r^2 + r + 6 = 0$$

Check $r = \pm 1, \pm 2, \pm 3, \pm 6$

$$(-1)^3 - 4(-1)^2 - 1 + 6 = -1 - 4 - 1 + 6 = 0$$

$$(r+1)(r^2 + \underline{-5}r + 6) = 0$$

Check: $r^3 - 5r^2 + 6r + r^2 - 5r + 6$
$= r^3 - 4r^2 + r + 6$ ✓

$$(r+1)(r^2 - 5r + 6) = 0$$
$$(r+1)(r-2)(r-3) = 0$$

$$\Rightarrow r = -1, 2, 3$$

$\Rightarrow$ homog gen soln: $y = c_1e^{-t} + c_2e^{2t} + c_3e^{3t}$

Step 2: Find a non homog soln

§3.5: Guess $y = Ae^{4t}$ (since not homog) & easier for this example

§3.6: $\Psi = u_1\phi_1 + u_2\phi_2 + u_3\phi_3$

$$w(e^{-t}, e^{2t}, e^{3t})$$

$$= \begin{vmatrix} e^{-t} & e^{2t} & e^{3t} \\ -e^{-t} & 2e^{2t} & 3e^{3t} \\ e^{-t} & 4e^{2t} & 9e^{3t} \end{vmatrix}$$

$R_2 + R_3$, $R_3 - R_1$

$$= \begin{vmatrix} e^{-t} & e^{2t} & e^{3t} \\ 0 & 3e^{2t} & 4e^{3t} \\ 0 & 3e^{2t} & 8e^{3t} \end{vmatrix}$$

$R_3 - R_2$

$$= \begin{vmatrix} e^{-t} & e^{2t} & e^{3t} \\ 0 & 3e^{2t} & 4e^{3t} \\ 0 & 0 & 4e^{3t} \end{vmatrix}$$

$$= 12e^{(-1+2+3)t} = 12e^{4t}$$

Using Abel's thm

$$W(e^{-t}, e^{2t}, e^{3t}) = c\, e^{-\int p_1(t)\,dt}$$

$$= c\, e^{-\int(-4)\,dt} = c e^{\int 4\,dt}$$

$$= c e^{4t}$$

Evaluate at 0

$$W(0) = c e^{0} = c$$

$$W(0) = \begin{vmatrix} e^{0} & e^{0} & e^{0} \\ -e^{0} & 2e^{0} & 3e^{0} \\ e^{0} & 4e^{0} & 9e^{0} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 3 & 4 \\ 1 & 3 & 8 \end{vmatrix}$$

$$= 1(24-12)$$

$$= 12$$

$$\Rightarrow c = 12$$

$$W(e^{-t}, e^{2t}, e^{3t}) = 12 e^{4t}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & e^{2t} & e^{3t} \\ 0 & 2e^{2t} & 3e^{3t} \\ 1 & 4e^{2t} & 9e^{3t} \end{vmatrix}$$

$$= +1\left(3e^{5t} - 2e^{5t}\right) = e^{5t}$$

$$W_2 = \begin{vmatrix} e^{-t} & 0 & e^{3t} \\ -e^{-t} & 0 & 3e^{3t} \\ e^{-t} & 1 & 9e^{3t} \end{vmatrix}$$

$$= -1 \begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & 3e^{3t} \end{vmatrix} = -\left(3e^{2t} + e^{2t}\right)$$

$$= -4e^{2t}$$

$$W_3 = \begin{vmatrix} e^{-t} & e^{2t} & 0 \\ -e^{-t} & 2e^{2t} & 0 \\ e^{-t} & 4e^{2t} & 1 \end{vmatrix}$$

$$= +1 \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 2e^{t} + e^{t} = 3e^{t}$$

$$u_1 = \int \frac{g}{a}\,\frac{W_1}{W}\,dt = \int \frac{8e^{4t}\cdot e^{5t}}{12e^{4t}}\,dt$$

$$= \int \frac{2}{3} e^{5t}\,dt = \frac{2}{15} e^{5t}$$

(Don't need +C since only need 1 anti-derivative)

$$u_2 = \int \frac{g}{a}\,\frac{W_2}{W}\,dt = \int \frac{8e^{4t}\cdot(-4e^{2t})}{12e^{4t}}\,dt$$

$$= -\int \frac{8e^{2t}}{3}\,dt = -\frac{4}{3} e^{2t}$$

$$u_3 = \int \frac{g}{a}\,\frac{W_3}{W}\,dt = \int \frac{8e^{4t}(3e^{t})}{12e^{4t}}\,dt$$

$$= \int 2e^{t}\,dt = 2e^{t}$$

$$y = \left(\frac{2}{15}e^{5t}\right)e^{-t} + \left(-\frac{4}{3}e^{2t}\right)e^{2t} + \left(2e^{t}\right)e^{3t}$$

$$= \left(\frac{2}{15} - \frac{4}{3} + 2\right)e^{4t} = \frac{2-20+30}{15}e^{4t}$$

$$= \frac{12}{15}e^{4t}$$

$$= \frac{4}{5}e^{4t}$$

Thus general soln

$$y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{3t} + \frac{4}{5} e^{4t}$$